SOA Sample Exam Solutions

Solution 1

A Chapter 1, Put-Call Parity

We can use put-call parity to solve this problem:

$$C_{Eur}(K,T) + Ke^{-rT} = S_0 + P_{Eur}(K,T)$$
$$\left[C_{Eur}(K,T) - P_{Eur}(K,T)\right] - S_0 = -Ke^{-rT}$$
$$0.15 - 60 = -70e^{-4r}$$
$$-59.85 = -70e^{-4r}$$
$$\ln\left(\frac{59.85}{70}\right) = -4r$$
$$r = 0.03916$$

Solution 2

D Chapter 2, Arbitrage

Let X be the number of calls with a strike price of \$55 that are purchased for Mary's portfolio. If we assume that the net cost of establishing the portfolio is zero, then we can solve for X:

$$11 - 3\times 6 + 1 + 3X = 0$$
$$X = 2$$

The table below shows that regardless of the stock price at time T, Mary's payoff is positive. Therefore, Mary is correct. This implies that John is incorrect.

Mary's Portfolio

		Time T			
Transaction	**Time 0**	$S_T < 40$	$40 \le S_T \le 50$	$50 \le S_T \le 55$	$55 < S_T$
Buy 1 of $C(40)$	–11.00	0.00	$S_T - 40$	$S_T - 40$	$S_T - 40$
Sell 3 of $C(50)$	3(6.00)	0.00	0.00	$-3(S_T - 50)$	$-3(S_T - 50)$
Buy 2 of $C(55)$	–2(3.00)	0.00	0.00	0.00	$2(S_T - 55)$
Lend \$1	-1.00	e^{rT}	e^{rT}	e^{rT}	e^{rT}
Total	0.00	e^{rT}	$e^{rT} + S_T - 40$	$e^{rT} + 110 - 2S_T$	e^{rT}

Let Y be the number of calls and puts with a strike price of \$50 that are sold for Peter. If we assume that the net cost of establishing the portfolio is zero, then we can solve for Y:

$$2\times 3-2\times 11+11-3+2-Y\times 6+Y\times 8=0$$
$$2Y=6$$
$$Y=3$$

In evaluating Peter's portfolio, we can make use of the fact that purchasing a call option and selling a put option is equivalent to purchasing a prepaid forward on the stock and borrowing the present value of the strike price. We can see this by writing put-call parity as:

$$C_{Eur}(K,T)-P_{Eur}(K,T)=F_{0,T}^{P}(S)-Ke^{-rT}$$

Therefore, purchasing a call option and selling a put option results in a payoff of:

$$S_T-K$$

Since Peter purchases offsetting amounts of puts and calls for any given strike price, we can use this result to evaluate his payoffs.

Peter's Portfolio

Transaction	**Time 0**	**Time T**
Buy 2 of $C(55)$ & sell 2 of $P(55)$	$2(11.00-3.00)$	$2(S_T-55)$
Buy 1 of $C(40)$ & sell 1 of $P(40)$	$3.00-11.00$	S_T-40
Sell 3 of $C(50)$ & buy 3 of $P(50)$	$3(6.00-8.00)$	$-3(S_T-50)$
Lend \$2	-2.00	$2e^{rT}$
Total	0.00	$2e^{rT}$

Peter's portfolio is certain to have a positive payoff at time T, so Peter is correct.

Solution 3

13.202% Chapter 2, Application of Option Pricing Concepts

We are told that the price of a European put option with a strike price of \$103 has a value of \$15.21. The payoff of the put option at time 1 is:

$$Max\left[0,103-S(1)\right]$$

If we can express the payoff of the single premium deferred annuity in terms of the expression above, then we will be able to obtain the price of the annuity.

The payoff at time 1 is:

$$\text{Time 1 Payoff} = \pi(1-y\%)Max\left[\frac{S(1)}{100}, 1.03\right]$$
$$= \pi(1-y\%)\times\frac{1}{100}\times Max\left[S(1), 103\right]$$
$$= \pi(1-y\%)\times\frac{1}{100}\times\left(Max\left[0, 103-S(1)\right]+S(1)\right)$$

The current value of a payoff of $Max\left[0, 103-S(1)\right]$ at time 1 is \$15.21, and the current value of a payoff of $S(1)$ at time 1 is 100. Therefore, the current value of the payoff is:

$$\text{Current value of payoff} = \pi(1-y\%)\times\frac{1}{100}(15.21+100)$$

For the company to break even on the contract, the current value of the payoff must be equal to the single premium of π:

$$\pi(1-y\%)\times\frac{1}{100}(15.21+100) = \pi$$
$$(1-y\%)\times\frac{1}{100}(15.21+100) = 1$$
$$y\% = 0.13202$$

Solution 4

C Chapter 4, Two-Period Binomial Model

Since the stock does not pay dividends, the price of the American call option is equal to the price of an otherwise equivalent European call option.

The stock price tree and the associated option payoffs at the end of 2 years are:

			Call Payoff
		32.9731	10.9731
	25.6800		
20.0000		22.1028	0.1028
	17.2140		
		14.8161	0.0000

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{e^{(0.05-0.00)1}-0.8607}{1.2840-0.8607} = 0.4502$$

The value of the call option is:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{i=0}^{n} \left[\binom{n}{i} (p^*)^{(n-i)} (1-p^*)^i V(S_0 u^{n-i} d^i, K, hn) \right]$$
$$= e^{-0.05(2)} \left[(0.4502)^2 (10.9731) + 2(0.4502)(1-0.4502)(0.1028) + 0 \right]$$
$$= 2.0585$$

Solution 5

0.2255 Chapter 4, Three-Period Binomial Model for Currency

The values of u and d are:

$$u = e^{(r-r_f)h+\sigma\sqrt{h}} = e^{(0.08-0.09)0.25+0.30\sqrt{0.25}} = 1.15893$$
$$d = e^{(r-r_f)h-\sigma\sqrt{h}} = e^{(0.08-0.09)0.25-0.30\sqrt{0.25}} = 0.85856$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-r_f)h} - d}{u-d} = \frac{e^{(0.08-0.09)(0.25)} - 0.85856}{1.15893 - 0.85856} = 0.46257$$

The tree of pound prices and the tree of option prices are below:

Pound			2.2259	American Put			0.0000
		1.9207				0.0000	
	1.6573		1.6490		0.0939		0.0000
1.4300		1.4229		0.2255		0.1783	
	1.2277		1.2216		0.3473		0.3384
		1.0541				**0.5059**	
			0.9050				0.6550

The tree of prices for the American put option is found by working from right to left. The rightmost column is found as follows:

$$Max\left[0,\ 1.5600 - 2.2259\right] = 0.0000$$
$$Max\left[0,\ 1.5600 - 1.6490\right] = 0.0000$$
$$Max\left[0,\ 1.5600 - 1.2216\right] = 0.3384$$
$$Max\left[0,\ 1.5600 - 0.9050\right] = 0.6550$$

The prices after 6 months are found using the risk-neutral probabilities. If the exchange rate falls to 1.0541 at the end of 6 months, then early exercise is optimal:

$$Max\left\{e^{-0.08(0.25)}\left[(0.46257)(0.0000)+(1-0.46257)(0.0000)\right],1.5600-1.9207\right\}$$
$$=Max\left\{0.0000,-0.3607\right\}=0.0000$$
$$Max\left\{e^{-0.08(0.25)}\left[(0.46257)(0.0000)+(1-0.46257)(0.3384)\right],1.5600-1.4229\right\}$$
$$=Max\left\{0.1783,0.1371\right\}=0.1783$$
$$Max\left\{e^{-0.08(0.25)}\left[(0.46257)(0.3384)+(1-0.46257)(0.6550)\right],1.5600-1.0541\right\}$$
$$=Max\left\{0.4985,0.5059\right\}=0.5059$$

The prices after 3 months are:

$$Max\left\{e^{-0.08(0.25)}\left[(0.46257)(0.0000)+(1-0.46257)(0.1783)\right],1.5600-1.6573\right\}$$
$$=Max\left\{0.0939,-0.0973\right\}=0.0939$$
$$Max\left\{e^{-0.08(0.25)}\left[(0.46257)(0.1783)+(1-0.46257)(0.5059)\right],1.5600-1.2277\right\}$$
$$=Max\left\{0.3473,0.3323\right\}=0.3473$$

The current price of the option is:

$$Max\left\{e^{-0.08(0.25)}\left[(0.46257)(0.0939)+(1-0.46257)(0.3473)\right],1.5600-1.4300\right\}$$
$$=Max\left\{0.2255,0.1300\right\}=0.2255$$

The value of the American put option at time 0 is 0.2255.

Solution 6

C Chapter 7, Black-Scholes Call Price

The first step is to calculate d_1 and d_2:

$$d_1=\frac{\ln(S/K)+(r-\delta+0.5\sigma^2)T}{\sigma\sqrt{T}}=\frac{\ln(20/25)+(0.05-0.03+0.5\times 0.24^2)\times 0.25}{0.24\sqrt{0.25}}$$
$$=-1.75786$$
$$d_2=d_1-\sigma\sqrt{T}=-1.75786-0.24\sqrt{0.25}=-1.87786$$

We have:

$$N(d_1)=N(-1.75786)=0.03939$$
$$N(d_2)=N(-1.87786)=0.03020$$

The value of one European call option is:

$$C_{Eur} = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$
$$= 20e^{-0.03(0.25)} \times 0.03939 - 25e^{-0.05(0.25)} \times 0.03020 = 0.03629$$

The value of 100 of the European call options is:

$$100 \times 0.03629 = 3.629$$

Solution 7

$7.62 million Chapter 7, Currency Options and Black-Scholes

We can draw a couple of conclusions from Statement (vi):

- *Since the logarithm of the dollar per yen exchange rate is an arithmetic Brownian motion, the dollar per yen exchange rate follows geometric Brownian motion.*
- *Since the dollar per yen exchange rate follows geometric Brownian motion, the Black-Scholes framework applies. This means that put and call options on yen can be priced using the Black-Scholes formula.*

Company A has decided to buy a dollar-denominated put option with yen as the underlying asset. The domestic currency is dollars, and the current value of the one yen is:

$$x_0 = \frac{1}{120} = 0.0083333 \text{ dollars}$$

Since the option is at-the-money, the strike price is equal to the value of one yen:

$$K = \frac{1}{120} = 0.0083333 \text{ dollars}$$

The domestic interest rate is 3.5%, and the foreign interest rate is 1.5%:

$$r = 3.5\%$$
$$r_f = 1.5\%$$

The volatility of the yen per dollar exchange rate is equal to the volatility of the dollar per yen exchange rate. We convert the daily volatility into annual volatility:

$$\sigma = \frac{\sigma_h}{\sqrt{h}} = \frac{0.00261712}{\sqrt{\frac{1}{365}}} = 0.00261712 \times \sqrt{365} = 0.05$$

The values of d_1 and d_2 are:

$$d_1 = \frac{\ln(x_0/K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1) + (0.035 - 0.015 + 0.5 \times 0.05^2)0.25}{0.05\sqrt{0.25}}$$
$$= 0.21250$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.21250 - 0.05\sqrt{0.25} = 0.18750$$

We have:

$$N(-d_1) = N(-0.21250) = 0.41586$$
$$N(-d_2) = N(-0.18750) = 0.42563$$

The value of a put option on one yen is:

$$P_{Eur}(S, K, \sigma, r, T, \delta) = Ke^{-rT}N(-d_2) - Se^{-r_fT}N(-d_1)$$
$$= \frac{1}{120}e^{-0.035(0.25)} \times 0.42563 - \frac{1}{120}e^{-0.015(0.25)} \times 0.41586$$
$$= 0.0000634878$$

Since the option is for ¥120,000,000,000 the value of the put option in dollars is:

$$120{,}000{,}000{,}000 \times 0.0000634878 = 7{,}618{,}538$$

Solution 8

D Chapter 8, Black-Scholes and Delta

Since the stock does not pay dividends, the price of the American call option is equal to the price of the otherwise equivalent European call option.

Since the stock does not pay dividends, delta is equal to $N(d_1)$:

$$\Delta_{Call} = e^{-\delta T}N(d_1)$$
$$0.50 = e^{0 \times 0.25}N(d_1)$$
$$N(d_1) = 0.50 \qquad \Rightarrow \qquad d_1 = 0.00$$

The formula for d_1 can be solved for the risk-free rate of return:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$
$$0.00 = \frac{\ln(40/41.50) + (r - 0 + 0.5 \times 0.30^2)0.25}{0.30\sqrt{0.25}}$$
$$r = 0.102256$$

The value of d_2 is:

$$d_2 = d_1 - \sigma\sqrt{T} = 0.00 - 0.30\sqrt{0.25} = -0.15$$

The value of the call option is:

$$\begin{aligned}C_{Eur}(S,K,\sigma,r,T,\delta) &= Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)\\&= 40(0.5) - 41.5e^{-0.102256\times 0.25}N(-0.15)\\&= 20 - 40.453\left[1 - N(0.15)\right]\\&= -20.453 + 40.453N(0.15)\\&= 40.453N(0.15) - 20.453\\&= 40.453\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.15} e^{-0.5x^2}dx - 20.453\\&= 16.138\int_{-\infty}^{0.15} e^{-0.5x^2}dx - 20.453\end{aligned}$$

The correct answer is D.

Solution 9

B Chapter 9, Market-Maker Profit

The market-maker's profit is zero if the stock price movement is one standard deviation:

$$\text{One standard deviation move} = \sigma S_t\sqrt{h}$$

To answer the question, we must determine the value of σ. We can determine the value of $N(d_1)$:

$$\begin{aligned}\Delta_{Call} &= e^{-\delta T}N(d_1)\\0.61791 &= e^{-0.0(0.25)}N(d_1)\\0.61791 &= N(d_1)\end{aligned}$$

Using the cumulative normal distribution calculator, this implies that:

$$d_1 = 0.3$$

We can use the formula for d_1 to solve for the value of σ:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$0.30 = \frac{\ln(50/50) + (0.10 - 0 + 0.5\sigma^2)0.25}{\sigma\sqrt{0.25}}$$

$$0.15\sigma = 0.025 + 0.125\sigma^2$$

$$0.125\sigma^2 - 0.15\sigma + 0.025 = 0$$

We make use of the quadratic formula:

$$\sigma = \frac{0.15 \pm \sqrt{(-0.15)^2 - 4(0.125)(0.025)}}{2(0.125)}$$

$$\sigma = 0.20 \qquad \text{or} \qquad \sigma = 1.0$$

The first of these volatility values, 0.20, seems more reasonable, and indeed, it produces one of the answer choices:

$$\sigma S_t \sqrt{h} = 0.20 \times 50 \times \sqrt{\frac{1}{365}} = 0.5234$$

If we use $\sigma = 1.0$, then we do not obtain one of the answer choices:

$$\sigma S_t \sqrt{h} = 1.0 \times 50 \times \sqrt{\frac{1}{365}} = 2.6171$$

Therefore, we conclude that $\sigma = 0.20$, and the stock price moves either up or down by 0.5234.

Solution 10

E Chapter 14, Black-Scholes Framework

Statement (i) is true, because when the Black-Scholes framework applies, the natural log of stock prices follows arithmetic Brownian motion.

Statement (ii) is true, because when the Black-Scholes framework applies, stock prices are lognormally distributed:

$$\ln S(t+h) - \ln S(t) \sim N\left((\alpha - 0.5\sigma^2)h,\ \sigma^2 h\right)$$

Since the stock prices are lognormally distributed, we have:

$$Var[X(t+h) - X(t)] = Var[\ln S(t+h) - \ln S(t)] = \sigma^2 h$$

The quickest way to analyze statement (iii) is to convert the expression into its continuous form:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left[X\left(jT/n\right) - X\left((j-1)T/n\right) \right]^2 = \int_0^T \left[dX(t)\right]^2$$

The natural log of the stock prices follows arithmetic Brownian motion:

$$dX(t) = d\left[\ln S(t)\right] = \left(\alpha - \delta - 0.5\sigma^2\right)dt + \sigma dZ(t)$$

Substituting into the stochastic integral above, we have:

$$\int_0^T [dX(t)]^2 = \int_0^T [d\ln S(t)]^2 = \int_0^T \Big[(\alpha-\delta-0.5\sigma^2)dt + \sigma dZ(t)\Big]^2$$
$$= \int_0^T \Big[(\alpha-\delta-0.5\sigma^2)^2 dt^2 + 2(\alpha-\delta-0.5\sigma^2)dt \times \sigma dZ(t) + \sigma^2 \left(dZ(t)\right)^2\Big]$$

We make use of the following multiplication rules to simplify the expression above:

$dt^2 = 0$ and $dt \times dZ(t) = 0$ and $[dZ(t)]^2 = 0$

The stochastic integral can now be simplified:

$$\int_0^T \Big[(\alpha-\delta-0.5\sigma^2)^2 dt^2 + 2(\alpha-\delta-0.5\sigma^2)dt \times \sigma dZ(t) + \sigma^2 \left(dZ(t)\right)^2\Big]$$
$$= \int_0^T \Big[0 + 0 + \sigma^2 dt\Big] = \int_0^T \sigma^2 dt = \sigma^2 T$$

Therefore Statement (iii) is true. The correct answer is Choice E.

Alternate solution for Statement (iii)

Statement (iii) is true, because when the Black-Scholes framework applies, the natural log of the stock prices follows arithmetic Brownian motion:

$$dX(t) = d\big[\ln S(t)\big] = \left(\alpha - \delta - 0.5\sigma^2\right)dt + \sigma dZ(t)$$

Therefore:

$$X(t) = \left(\alpha - \delta - 0.5\sigma^2\right)t + \sigma Z(t)$$

Recall that we use $Y(t)$ to denote a random draw from a binomial approximation. For small values of h, we have:

$$Z(t+h) - Z(t) = Y(t+h)\sqrt{h}$$

We therefore can write:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left[X(jT/n)-X((j-1)T/n)\right]^2$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left[\left(\alpha-\delta-0.5\sigma^2\right)jT/n+\sigma Z(jT/n)-\left(\alpha-\delta-0.5\sigma^2\right)(j-1)T/n-\sigma Z((j-1)T/n)\right]^2$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left[\left(\alpha-\delta-0.5\sigma^2\right)T/n+\sigma\left[Z(jT/n)-Z((j-1)T/n)\right]\right]^2$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left[\left(\alpha-\delta-0.5\sigma^2\right)T/n+\sigma Y\left(\frac{jT}{n}\right)\sqrt{\frac{T}{n}}\right]^2$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left[\left(\alpha-\delta-0.5\sigma^2\right)^2\left(\frac{T}{n}\right)^2+2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{\frac{3}{2}}\sigma Y\left(\frac{jT}{n}\right)+\frac{\sigma^2 T}{n}\right]$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left[\left(\alpha-\delta-0.5\sigma^2\right)^2\left(\frac{T}{n}\right)^2\right]+\lim_{n\to\infty}\sum_{j=1}^{n}\left[2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{\frac{3}{2}}\sigma Y\left(\frac{jT}{n}\right)\right]+\lim_{n\to\infty}\sum_{j=1}^{n}\left[\frac{\sigma^2 T}{n}\right]$$

We now consider each of the three terms separately.

The first of these terms is zero because:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left[\left(\alpha-\delta-0.5\sigma^2\right)^2\left(\frac{T}{n}\right)^2\right]=\lim_{n\to\infty}\left[n\times\left(\alpha-\delta-0.5\sigma^2\right)^2\left(\frac{T}{n}\right)^2\right]$$

$$=\left(\alpha-\delta-0.5\sigma^2\right)^2(T)^2\lim_{n\to\infty}\left[\frac{1}{n}\right]=\left(\alpha-\delta-0.5\sigma^2\right)^2(T)^2\times 0=0$$

The second term is also zero. This is because both the maximum possible value and the minimum possible value of the second term are zero. Consider the value that is obtained if the binomial random variable $Y(jT/n)$ is always equal to 1:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left[2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{3/2}\sigma Y(jT/n)\right]$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left[2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{3/2}\sigma\right]$$

$$=\lim_{n\to\infty}\left[n\times 2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{3/2}\sigma\right]=2\left(\alpha-\delta-0.5\sigma^2\right)T\lim_{n\to\infty}\frac{1}{\sqrt{n}}$$

$$=2\left(\alpha-\delta-0.5\sigma^2\right)T\times 0=0$$

If the binomial variable $Y(jT/n)$ is always -1, then the second term is:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{j=1}^{n}\left[2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{3/2}\sigma Y\left(jT/n\right)\right] \\
&\quad=\lim_{n\to\infty}\sum_{j=1}^{n}\left[2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{3/2}\sigma\times(-1)\right] \\
&\quad=-\lim_{n\to\infty}\left[n\times 2\left(\alpha-\delta-0.5\sigma^2\right)\left(\frac{T}{n}\right)^{3/2}\sigma\right]=-2\left(\alpha-\delta-0.5\sigma^2\right)T\lim_{n\to\infty}\frac{1}{\sqrt{n}} \\
&\quad=-2\left(\alpha-\delta-0.5\sigma^2\right)T\times 0=0
\end{aligned}$$

Since the maximum possible value of the second term approaches zero, and the minimum possible value of the second term approaches zero, we conclude that the second term approaches zero as n becomes very large.

Finally, the third term is:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left[\sigma^2\frac{T}{n}\right]=\lim_{n\to\infty}\sigma^2\frac{T}{n}\sum_{j=1}^{n}[1]=\lim_{n\to\infty}\sigma^2\frac{T}{n}n=\lim_{n\to\infty}\sigma^2 T=\sigma^2 T$$

Therefore, we have:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left[X\left(jT/n\right)-X\left((j-1)T/n\right)\right]^2=\sigma^2 T$$

The solution using the multiplication rules is much quicker than this alternate solution!

Solution 11

E Chapter 14, Black-Scholes Framework

Statement (i) is true, because when the Black-Scholes framework applies, the stock prices are lognormally distributed:

$$\ln S(t+h)-\ln S(t)\ \sim\ N\left((\alpha-\delta-0.5\sigma^2)h,\ \sigma^2 h\right)$$

If we are given $S(t)$, this implies:

$$\ln S(t+h)\ \sim\ N\left(\ln S(t)+(\alpha-\delta-0.5\sigma^2)h,\ \sigma^2 h\right)$$

$$Var\left[\ln(S(t+h)\middle|S(t)\right]=\sigma^2 h$$

Statement (ii) is true, because when the Black-Scholes framework applies, the stock prices follow geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)$$

The variance is:

$$Var\left[\frac{dS(t)}{S(t)}\middle|S(t)\right] = Var\left[(\alpha - \delta)dt + \sigma dZ(t)\right] = 0 + \sigma^2 Var\left[dZ(t)\right] = \sigma^2 dt$$

Statement (iii) is true, because when the Black-Scholes framework applies, the stock prices follow geometric Brownian motion:

$$dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t)$$

Therefore, we have:

$$\begin{aligned} Var\left[S(t+dt)\middle|S(t)\right] &= Var\left[S(t) + dS(t)\middle|S(t)\right] = 0 + Var\left[dS(t)\middle|S(t)\right] \\ &= Var\left[(\alpha - \delta)S(t)dt + \sigma S(t)dZ(t)\middle|S(t)\right] = 0 + \sigma^2\left[S(t)\right]^2 Var\left[dZ(t)\right] \\ &= \sigma^2\left[S(t)\right]^2 dt \end{aligned}$$

Solution 12

B Chapter 15, Sharpe Ratio

When the price follows geometric Brownian motion, the natural log of the price follows arithmetic Brownian motion:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t) \quad \Leftrightarrow \quad d\left[\ln S(t)\right] = \left(\alpha - 0.5\sigma^2\right)dt + \sigma dZ(t)$$

Therefore:

$$\frac{dY(t)}{Y(t)} = Adt + BdZ(t) \quad \Leftrightarrow \quad d\left[\ln Y(t)\right] = \left(A - 0.5B^2\right)dt + BdZ(t)$$

The arithmetic Brownian motion provided in the question for $d\left[\ln Y(t)\right]$ allows us to solve for B:

$$d\left[\ln Y(t)\right] = \mu dt + 0.085dZ(t) \quad \text{and} \quad d\left[\ln Y(t)\right] = \left(A - 0.5B^2\right)dt + BdZ(t)$$

$$A - 0.5B^2 = \mu$$
$$B = 0.085$$

Since X and Y have the same underlying source of risk, they must have the same Sharpe ratio. This allows us to solve for A:

$$\frac{\alpha_X - r}{\sigma_X} = \frac{\alpha_Y - r}{\sigma_Y}$$
$$\frac{0.07 - 0.04}{0.12} = \frac{A - 0.04}{0.085}$$
$$A = 0.06125$$

Solution 13

E Chapter 15, Itô's Lemma

The drift is the expected change in the asset price per unit of time.

(i) For the first equation, the partial derivatives are:

$$U_Z = 2 \qquad U_{ZZ} = 0 \qquad U_t = 0$$

This results in:

$$dU(t) = U_Z dZ + \frac{1}{2}U_{ZZ}(dZ)^2 + U_t dt = 2dZ + 0 + 0 = 2dZ$$

Since there is no dt term, the drift is zero for dU.

(ii) For the second equation, the partial derivatives are:

$$V_Z = 2Z \qquad V_{ZZ} = 2 \qquad V_t = -1$$

This results in:

$$dV(t) = V_Z dZ + \frac{1}{2}V_{ZZ}(dZ)^2 + V_t dt = 2Z(t)dZ + \frac{1}{2}\times 2(dZ)^2 - 1dt$$
$$= 2Z(t)dZ + 1dt - 1dt = 2Z(t)dZ$$

Since there is no dt term, the drift is zero for dV.

(iii) We must find:

$$dW(t) = d\left[t^2 Z(t)\right] - 2tZ(t)dt$$

For the first term, we must use Itô's Lemma. Let's define:

$$X(t) = t^2 Z(t)$$

The partial derivatives are:

$$X_Z = t^2 \qquad X_{ZZ} = 0 \qquad X_t = 2tZ$$

This results in:

$$dX(t) = X_Z dZ + \frac{1}{2} X_{ZZ}(dZ)^2 + X_t dt = t^2 dZ + \frac{1}{2} \times 0 \times (dZ)^2 + 2tZdt$$
$$= t^2 dZ(t) + 2tZ(t)dt$$

Therefore:

$$dW(t) = d\left[t^2 Z(t)\right] - 2tZ(t)dt = dX(t) - 2tZ(t)dt = t^2 dZ(t) + 2tZ(t)dt - 2tZ(t)dt$$
$$= t^2 dZ(t)$$

Since there is no dt term, the drift is zero for dW.

Solution 14

0.0517 Chapter 19, Vasicek Model

The Vasicek model of short-term interest rates is:

$$dr = a(b - r)dt + \sigma dZ$$

Therefore, we can determine the value of a:

$$dr = 0.6(b - r)dt + \sigma dZ \qquad \Rightarrow \qquad a = 0.6$$

In the Vasicek model, the Sharpe ratio is constant:

$$\phi(r,t) = \phi$$

Therefore, for any r, t, and T, we have:

$$\phi = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

Since the Sharpe ratio is constant:

$$\frac{\alpha(0.04,0,2) - 0.04}{q(0.04,0,2)} = \frac{\alpha(0.05,1,4) - 0.05}{q(0.05,1,4)}$$

We now make use of the following formula for $q(r,t,T)$ in the Vasicek model:

$$q(r,t,T) = B(t,T)\sigma(r) = B(t,T)\sigma$$

Substituting this expression for $q(r,t,T)$ into the preceding equality allows us to solve for $\alpha(0.05,1,4)$:

$$\frac{\alpha(0.04,0,2)-0.04}{q(0.04,0,2)}=\frac{\alpha(0.05,1,4)-0.05}{q(0.05,1,4)}$$

$$\frac{\alpha(0.04,0,2)-0.04}{B(0,2)\sigma}=\frac{\alpha(0.05,1,4)-0.05}{B(1,4)\sigma}$$

$$\frac{0.04139761-0.04}{\frac{1-e^{-0.6(2-0)}}{0.6}\sigma}=\frac{\alpha(0.05,1,4)-0.05}{\frac{1-e^{-0.6(4-1)}}{0.6}\sigma}$$

$$\frac{0.00139761}{1-e^{-0.6(2-0)}}=\frac{\alpha(0.05,1,4)-0.05}{1-e^{-0.6(4-1)}}$$

$$\frac{0.00139761}{0.6988}=\frac{\alpha(0.05,1,4)-0.05}{0.8347}$$

$$\alpha(0.05,1,4)=0.0516694$$

Solution 15

1.3264 Chapter 18, Black-Derman-Toy Model

In each column of rates, each rate is greater than the rate below it by a factor of:

$$e^{2\sigma_i\sqrt{h}}$$

Therefore, the missing rate in the third column is:

$$r_{dd}=0.135e^{-2\sigma_2\sqrt{1}}=0.135\times\frac{0.135}{0.172}=0.106$$

The missing rates in the fourth column are:

$$r_{duu}=0.168e^{-2\sigma_3\sqrt{1}}=0.168\div\sqrt{\frac{0.168}{0.110}}=0.1359$$

$$r_{ddd}=0.11e^{-2\sigma_3\sqrt{1}}=0.110\div\sqrt{\frac{0.168}{0.110}}=0.0890$$

The tree of short-term rates is:

			16.80%
		17.20%	
	12.60%		13.59%
9.00%		13.50%	
	9.30%		11.00%
		10.60%	
			8.90%

The caplet pays off only if the interest rate at the end of the third year is greater than 10.5%. The payoff table is:

Time 0	Time 1	Time 2	Time 3
			5.3938
		0.0000	
	0.0000		2.7238
0.0000		0.0000	
	0.0000		0.4505
		0.0000	
			0.0000

The payments have been converted to their equivalents payable at the end of 3 years. The calculations for the tree above are shown below:

$$\frac{100\times(0.1680-0.105)}{1.1680}=5.3938$$
$$\frac{100\times(0.1359-0.105)}{1.1359}=2.7238$$
$$\frac{100\times(0.1100-0.105)}{1.1100}=0.4505$$

We work recursively to calculate the value of the caplet. For example, if the interest rate increases at the end of the first year and at the end of the second year, then its value at the end of the second year is:

$$\frac{0.5\times5.3938+0.5\times2.7238}{1.1720}=3.4632$$

The completed tree is:

Time 0	Time 1	Time 2	Time 3
			5.3938
		3.4632	
	2.1588		2.7238
1.3264		1.3984	
	0.7329		0.4505
		0.2036	
			0.0000

The value of the year-4 caplet is \$1.3264.

Solution 16

B Chapter 15, Prepaid Forward Price of S^a

We make use of the following expression for the prepaid forward price of the derivative:

$$F_{t,T}^{P}\left[S(T)^x\right] = e^{-r(T-t)}\left[S(t)\right]^x e^{\left[x(r-\delta)+0.5x(x-1)\sigma^2\right](T-t)}$$

We can substitute this expression for the prepaid forward price in the equation provided in the question:

$$F_{t,T}^{P}\left[S(T)^x\right] = S(t)^x$$

$$e^{-r(T-t)}\left[S(t)\right]^x e^{\left[x(r-\delta)+0.5x(x-1)\sigma^2\right](T-t)} = S(t)^x$$

$$e^{\left[-r+x(r-\delta)+0.5x(x-1)\sigma^2\right](T-t)} = 1$$

$$\left[-r + x(r-\delta) + 0.5x(x-1)\sigma^2\right](T-t) = 0$$

$$-r + x(r-\delta) + 0.5x(x-1)\sigma^2 = 0$$

Putting in the risk-free interest rate of 4%, the volatility of 20% and the dividend yield of 0%, we have:

$$-r + x(r-\delta) + 0.5x(x-1)\sigma^2 = 0$$

$$-0.04 + x(0.04-0.00) + 0.5x^2(0.20)^2 - 0.5x(0.20)^2 = 0$$

$$0.02x^2 + 0.02x - 0.04 = 0$$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$x = 1 \quad \text{or} \quad x = -2$$

We were told in the question that 1 is a solution. The other solution is -2.

Solution 17

A Chapter 13, Estimating Volatility

Notice that the price increases by 25% 4 times and it decreases by 25% 4 times. This means that the mean return is zero, and each of the deviations is the same size. This simplifies the calculations below.

Since we have 9 months of data, we can calculate 8 monthly returns, and $k = 8$. Each monthly return is calculated as a continuously compounded rate:

$$r_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

The next step is to calculate the average of the returns:

$$\bar{r} = \frac{\sum_{i=1}^{k} r_i}{k}$$

The returns and their average are shown in the third column below:

Month	Price	$r_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$	$(r_i - \bar{r})^2$
1	80		
2	64	–0.22314	0.049793
3	80	0.22314	0.049793
4	64	–0.22314	0.049793
5	80	0.22314	0.049793
6	100	0.22314	0.049793
7	80	–0.22314	0.049793
8	64	–0.22314	0.049793
9	80	0.22314	0.049793
		$\bar{r} = 0.00000$	$\sum_{i=1}^{8} (r_i - \bar{r})^2 = 0.398344$

The fourth column shows the squared deviations and the sum of squares.

The estimate for the standard deviation of the monthly returns is:

$$\sigma_h = \sqrt{\frac{\sum_{i=1}^{k} (r_i - \bar{r})^2}{k-1}}$$

$$\sigma_{\frac{1}{12}} = \sqrt{\frac{0.398344}{7}} = 0.23855$$

We adjust the monthly volatility to obtain the annual volatility. Since $h = \frac{1}{12}$:

$$\sigma = \sigma_h \sqrt{\frac{1}{h}} = 0.23855\sqrt{12} = 0.82636$$

This problem isn't very difficult if you are familiar with the statistical function of your calculator.

On the TI-30X IIS, the steps are:

[2nd][STAT] (Select 1-VAR) [ENTER]

[DATA]

X1= $\ln\left(\frac{64}{80}\right)$ [ENTER] ↓↓ (Hit the down arrow twice)

X2 = $\ln\left(\frac{80}{64}\right)$ [ENTER] ↓↓

X3 = $\ln\left(\frac{64}{80}\right)$ [ENTER] ↓↓

X4 = $\ln\left(\frac{80}{64}\right)$ [ENTER] ↓↓

X5 = $\ln\left(\frac{100}{80}\right)$ [ENTER] ↓↓

X6 = $\ln\left(\frac{80}{100}\right)$ [ENTER] ↓↓

X7 = $\ln\left(\frac{64}{80}\right)$ [ENTER] ↓↓

X8 = $\ln\left(\frac{80}{64}\right)$ [ENTER]

[STATVAR] $\rightarrow$ $\rightarrow$ (Arrow over to *Sx*)

$\times \sqrt{(12)}$ [ENTER]

The result is: 0.826363140

To exit the statistics mode:

[2nd] [EXITSTAT] [ENTER]

On the BA II Plus calculator, the steps are:

[2nd][DATA] [2nd][CLR WORK]

64/80 = LN [ENTER] ↓↓ (Hit the down arrow twice)

80/64 = LN [ENTER] ↓↓

64/80 = LN [ENTER] ↓↓

80/64 = LN [ENTER] ↓↓

100/80 = LN [ENTER] ↓↓

80/100 = LN [ENTER] ↓↓

64/80 = LN [ENTER] ↓↓

80/64 = LN [ENTER]

[2nd][STAT] ↓↓↓ $\times \sqrt{(12)}$ =

The result is: 0.82636314

To exit the statistics mode: [2nd][QUIT]

Solution 18

A Chapter 10, Delta-Hedging Gap Call Options

The price of the gap call option is:

$$C_{GapCall} = Se^{-\delta T}N(d_1) - K_1 e^{-rT}N(d_2)$$
$$C_{GapCall} = SN(d_1) - 130N(d_2)$$

The delta of the option is the derivative of the option price with respect to the stock's price:

$$\begin{aligned}\Delta_{GapCall} &= \frac{\partial C_{GapCall}}{\partial S} \\ &= N(d_1) + S\frac{\partial N(d_1)}{\partial S} - 130\frac{\partial N(d_2)}{\partial S} \\ &= N(d_1) + S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - 130\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S} \\ &= N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - 130N'(d_2)\frac{\partial d_2}{\partial S}\end{aligned}$$

Note that:

$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}dx \qquad \Rightarrow \qquad N'(d) = \frac{1}{\sqrt{2\pi}} e^{-0.5d^2}$$

The derivatives of d_1 and d_2 with respect to the stock's price are shown below:

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{K_2 e^{-rT}}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \frac{\ln(S) + \ln\left(\frac{e^{-\delta T}}{K_2 e^{-rT}}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \qquad \Rightarrow \qquad \frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}} = \frac{1}{S}$$

$$d_2 = d_1 - \sigma\sqrt{T} \Rightarrow \qquad \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T}} = \frac{1}{S}$$

Let's calculate the values of d_1 and d_2 that we'll need for the formula for delta:

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{K_2 e^{-rT}}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100e^{-0\times 1}}{100e^{-0\times 1}}\right) + \frac{1^2}{2}\times 1}{1\sqrt{1}} = 0.5 \qquad \Rightarrow \qquad N(d_1) = 0.69146$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.5 - 1\sqrt{1} = -0.5$$

Now we can calculate the delta of the gap call option:

$$\begin{aligned}\Delta_{GapCall} &= N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - 130N'(d_2)\frac{\partial d_2}{\partial S} \\ &= 0.69146 + S\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2}\frac{1}{S} - 130\frac{1}{\sqrt{2\pi}}e^{-0.5d_2^2}\frac{1}{S} \\ &= 0.69146 + \frac{1}{\sqrt{2\pi}}e^{-0.5\times 0.5^2} - 130\frac{1}{\sqrt{2\pi}}e^{-0.5(-0.5)^2}\frac{1}{100} \\ &= 0.69146 + \frac{1}{\sqrt{2\pi}}e^{-0.5\times 0.5^2}(1-1.3) \\ &= 0.58584\end{aligned}$$

Since the market-maker sells 1,000 of the gap call options, the market-maker multiplies the delta of one call option by 1,000 to determine the number of shares that must be purchased in order to delta-hedge the position:

$$1{,}000 \times 0.58584 = 585.84$$

Solution 19

C Chapter 11, Forward Start Option

In one year, the value of the call option will be:

$$\begin{aligned}C_{Eur}(S_1) &= S_1e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \\ &= S_1N(d_1) - S_1e^{-0.08}N(d_2)\end{aligned}$$

In one year, the values of d_1 and d_2 will be:

$$\begin{aligned}d_1 &= \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(S_1/S_1) + (0.08 - 0.00 + 0.5\times 0.30^2)(1)}{0.30\sqrt{1}} \\ &= 0.41667 \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.41667 - 0.30\sqrt{1} = 0.11667\end{aligned}$$

From the normal distribution table:

$$\begin{aligned}N(d_1) &= N(0.41667) \approx 0.66154 \\ N(d_2) &= N(0.11667) \approx 0.54644\end{aligned}$$

In one year, the value of the call option can be expressed in terms of the stock price at that time:

$$\begin{aligned}C_{Eur}(S_1) &= S_1N(d_1) - S_1e^{-0.08}N(d_2) \\ &= S_1 \times 0.66154 - S_1e^{-0.08} \times 0.54644 \\ &= 0.15711 \times S_1\end{aligned}$$

In one year, the call option will be worth 0.15711 shares of stock.

The prepaid forward price of one share of stock is:

$$F_{0,T}^{P}(S) = e^{-rT}F_{0,T}(S)$$
$$F_{0,1}^{P}(S) = e^{-0.08}F_{0,1}(S)$$
$$F_{0,1}^{P}(S) = e^{-0.08}\times 100$$
$$F_{0,1}^{P}(S) = 92.31163$$

The value today of 0.15711 shares of stock in one year is:

$$0.15711\times 92.31163 = 14.5033$$

Solution 20

D Chapter 8, Portfolio Delta and Elasticity

The elasticity of portfolio A is the weighted average of the elasticities of its components:

$$\Omega_{Port} = \sum_{i=1}^{n}\omega_i\Omega_i$$
$$5 = \frac{2\times 4.45}{2\times 4.45+1.90}\Omega_{Call} + \frac{1.90}{2\times 4.45+1.90}\Omega_{Put}$$
$$54 = 8.90\Omega_{Call} + 1.90\Omega_{Put}$$

We can express the delta of an option in terms of the elasticity of the option:

$$\Omega = \frac{S\Delta}{V} \qquad \Rightarrow \qquad \Delta = \Omega\frac{V}{S}$$

The delta of portfolio B is the sum of the deltas of its components:

$$\Delta_{Port} = \sum_{i=1}^{n}q_i\Delta_i$$
$$3.4 = 2\Delta_{Call} - 3\Delta_{Put}$$
$$3.4 = 2\Omega_{Call}\frac{4.45}{45} - 3\Omega_{Put}\frac{1.90}{45}$$
$$153 = 8.90\Omega_{Call} - 5.70\Omega_{Put}$$

We now have system of 2 equations with 2 unknowns:

$$54 = 8.90\Omega_{Call} + 1.90\Omega_{Put}$$
$$153 = 8.90\Omega_{Call} - 5.70\Omega_{Put}$$

Let's subtract the second equation from the first equation:

$$-99 = 7.60\Omega_{Put}$$
$$\Omega_{Put} = -13.0263$$

Solution 21

C Chapter 19, Cox-Ingersoll-Ross Model

We begin with the Sharpe ratio and parameterize it for the CIR model:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\bar{\phi}\frac{\sqrt{r}}{\sigma} = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma(r)}$$

$$\bar{\phi} r = \frac{\alpha(r,t,T) - r}{B(t,T)}$$

We use the value of $\alpha(0.05,7,9)$ provided in the question:

$$\bar{\phi} \times 0.05 = \frac{\alpha(0.05,7,9) - 0.05}{B(7,9)}$$

$$\bar{\phi} \times 0.05 = \frac{0.06 - 0.05}{B(7,9)}$$

$$0.01 = B(7,9)\bar{\phi}(0.05)$$

$$B(7,9)\bar{\phi} = 0.20$$

Making use of the fact that $B(7,9) = B(11,13)$, we have:

$$\bar{\phi} \times 0.04 = \frac{\alpha(0.04,11,13) - 0.04}{B(11,13)}$$

$$\begin{aligned} \alpha(0.04,11,13) &= 0.04 + B(11,13)\bar{\phi}(0.04) \\ &= 0.04 + 0.20(0.04) \\ &= 0.048 \end{aligned}$$

Solution 22

D Chapter 19, Risk-Neutral Process & Sharpe Ratio

The realistic process for the short rate follows:

$$dr = a(r)dt + \sigma(r)dZ \qquad \text{where:} \qquad a(r) = 0.09 - 0.5r \qquad \& \qquad \sigma(r) = 0.3$$

The risk-neutral process follows:

$$dr = \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z}$$

We can use the coefficient of the first term of the risk-neutral process to solve for the Sharpe ratio, $\phi(r,t)$:

$$0.15 - 0.5r = a(r) + \sigma(r)\phi(r,t)$$

$$0.15 - 0.5r = 0.09 - 0.5r + 0.3\phi(r,t)$$

$$\phi(r,t) = 0.20$$

The derivative, like all interest-rate dependent assets in this model, must have a Sharpe ratio of 0.20. Let's rearrange the differential equation for g, so that we can more easily observe its Sharpe ratio:

$$\frac{dg}{g} = \frac{\mu}{g}dt - 0.4dZ \qquad \Rightarrow \qquad \phi(r,t) = \frac{\frac{\mu}{g} - r}{0.4}$$

Since the Sharpe ratio is 0.20:

$$0.20 = \frac{\frac{\mu}{g} - r}{0.4}$$

$$0.08 = \frac{\mu}{g} - r$$

$$\mu = (r + 0.08)g$$

Solution 23

C Chapter 16, Sharpe Ratio

The stock and the call option must have the same Sharpe ratio:

$$\frac{0.10 - 0.04}{\sigma} = \frac{\gamma - 0.04}{\sigma_C}$$

The cost of the shares required to delta-hedge the call option is the number of shares required, $\Delta = V_S$, times the cost of each share, S. Therefore, based on statement (iv) in the question, we have:

$$V_S \times S = 9$$

We can find the volatility of the call option in terms of the volatility of the underlying stock:

$$\sigma_C = \frac{SV_S}{V}\sigma = \frac{9}{6}\sigma = 1.5\sigma$$

We can now solve for γ:

$$\frac{0.10 - 0.04}{\sigma} = \frac{\gamma - 0.04}{1.5\sigma} \qquad \Rightarrow \qquad \gamma = 0.13$$

Solution 24

E Chapter 14, Ornstein-Uhlenbeck Process

Let's find the differential form of each choice.

Choice A does not contain a random variable, so its differential form is easy to find:

$$dX(t) = \left[-\lambda X(0)e^{-\lambda t} + \alpha\lambda e^{-\lambda t}\right]dt$$

Choice B is easily recognized as an arithmetic Brownian motion:

$$dX(t) = \alpha dt + \sigma dZ(t)$$

Choice C is easily recognized as a geometric Brownian motion:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t)$$

Choice D is:

$$dX(t) = \alpha\lambda e^{\lambda t}dt + \sigma e^{\lambda t}dZ(t)$$

Choice E contains 3 terms. The first two terms do not contain random variables, so their differentials are easy to find. The third term will be more difficult:

$$dX(t) = -\lambda X(0)e^{-\lambda t}dt + \lambda\alpha e^{-\lambda t}dt + d\left[\int_0^t \sigma e^{-\lambda(t-s)}dZ(s)\right]$$

The third term has a function of t in the integral. We can pull the t-dependent portion out of the integral, so that we are finding the differential of a product. We then use the following version of the product rule to find the differential:

$$d\left[U(t)V(t)\right] = dU(t)V(t) + U(t)dV(t)$$

The differential of the third term is:

$$\begin{aligned}
d\left[\sigma\int_0^t e^{-\lambda(t-s)}dZ(s)\right] &= \sigma d\left[e^{-\lambda t}\int_0^t e^{\lambda s}dZ(s)\right] \\
&= \sigma d\left[e^{-\lambda t}\right]\int_0^t e^{\lambda s}dZ(s) + \sigma e^{-\lambda t}d\left[\int_0^t e^{\lambda s}dZ(s)\right] \\
&= -\lambda\sigma e^{-\lambda t}\left(\int_0^t e^{\lambda s}dZ(s)\right)dt + \sigma e^{-\lambda t}e^{\lambda t}dZ(t) \\
&= -\lambda\sigma\left(\int_0^t e^{-\lambda(t-s)}dZ(s)\right)dt + \sigma dZ(t)
\end{aligned}$$

Putting the three differentials together, we have:

$$\begin{aligned}
dX(t) &= -\lambda X(0)e^{-\lambda t}dt + \lambda\alpha e^{-\lambda t}dt - \lambda\sigma\left(\int_0^t e^{-\lambda(t-s)}dZ(s)\right)dt + \sigma dZ(t) \\
&= -\lambda\left[X(0)e^{-\lambda t} - \alpha e^{-\lambda t} + \sigma\left(\int_0^t e^{-\lambda(t-s)}dZ(s)\right)\right]dt + \sigma dZ(t) \\
&= -\lambda\left[X(0)e^{-\lambda t} + \alpha - \alpha e^{-\lambda t} + \sigma\left(\int_0^t e^{-\lambda(t-s)}dZ(s)\right) - \alpha\right]dt + \sigma dZ(t) \\
&= -\lambda\left[X(0)e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \sigma\left(\int_0^t e^{-\lambda(t-s)}dZ(s)\right) - \alpha\right]dt + \sigma dZ(t) \\
&= -\lambda\left[X(t) - \alpha\right]dt + \sigma dZ(t) \\
&= \lambda\left[\alpha - X(t)\right]dt + \sigma dZ(t)
\end{aligned}$$

Solution 25

B Chapter 11, Chooser Options

The price of the chooser option can be expressed in terms of a call option and put option:

$$\text{Price of Chooser Option} = C_{Eur}\left(S_0, K, T\right) + e^{-\delta(T-t_1)}P_{Eur}\left(S_0, Ke^{-(r-\delta)(T-t_1)}, t_1\right)$$

The risk-free interest rate and the dividend yield are both zero, so the put option has the same strike price as the call option:

$$\text{Price of Chooser Option} = C_{Eur}\left(S_0, K, T\right) + e^{-\delta(T-t_1)}P_{Eur}\left(S_0, Ke^{-(r-\delta)(T-t_1)}, t_1\right)$$

$$\text{Price of Chooser Option} = C_{Eur}(95, 100, 3) + P_{Eur}(95, 100, 1)$$

We can use put-call parity to find the value of the 1-year call option:

$$\begin{aligned}
&C_{Eur}(95, 100, 1) + K = S_0 + P_{Eur}(95, 100, 1) \\
&4 + 100 = 95 + P_{Eur}(95, 100, 1) \\
&P_{Eur}(95, 100, 1) = 9
\end{aligned}$$

We can now solve for the price of the 3-year call option:

$$\begin{aligned}
&\text{Price of Chooser Option} = C_{Eur}(95, 100, 3) + P_{Eur}(95, 100, 1) \\
&20 = C_{Eur}(95, 100, 3) + 9 \\
&C_{Eur}(95, 100, 3) = 11
\end{aligned}$$

Solution 26

D Chapter 2, Bounds on Option Prices

Let's begin by noting the prepaid forward price of the stock and the present value of the strike price:

$$F_{t,T}^{P}(S) = e^{-\delta(T-t)}S_t = e^{-0\times(0.5)}S_t = S_t$$
$$Ke^{-r(T-t)} = 100e^{-0.10(0.5)} = 95.12$$

Since the stock does not pay dividends, the American call option has the same price as the European call option, so both call options correspond to the same graph. The value of a call increases with the stock price, so the call options must correspond to either Graph I or Graph II. For call options we have:

$$Max\left[0, F_{t,T}^{P}(S) - Ke^{-r(T-t)}\right] \le C_{Eur}(S_t, K, T-t) \le C_{Amer}(S_t, K, T-t) \le S_t$$
$$Max\left[0, S_t - 95.12\right] \le C_{Eur}(S_t, K, T-t) \le C_{Amer}(S_t, K, T-t) \le S_t$$

The left portion of the inequality above describes the lower boundary of Graph II. The right portion of the inequality above describes the upper boundary of Graph II. Therefore, the call options correspond to Graph II, which narrows the answer choices to (D) and (E).

For the put options, we have:

$$Max\left[0, Ke^{-r(T-t)} - F_{t,T}^{P}(S)\right] \le P_{Eur}(S_t, K, T-t) \le P_{Amer}(S_t, K, T-t) \le K$$
$$Max\left[0, 95.12 - S_t\right] \le P_{Eur}(S_t, K, T-t) \le P_{Amer}(S_t, K, T-t) \le 100$$

For the European put option, the left portion of the inequality above describes the lower boundary of Graph IV. From the inequality above, we can see that it is possible for the European put option to have a tighter upper bound than the American put option. In fact, a European put option cannot have a price greater than the present value of the strike price. Therefore, the upper boundary is \$95.12, and we can conclude that Graph IV corresponds to the European put option, which still leaves answer choices (D) and (E).

The American put option could have a price as high as the strike price of \$100 if the stock price falls to zero. This is described by the upper boundary of Graph III. From the inequality above, we can see that it is possible for the American put option to have a tighter lower bound than the European put option. In fact, since the American put option can be exercised at any time, the lower boundary is the maximum of zero and the exercise value, as shown in Graph III. Therefore, Graph III corresponds to the American put option, and choice (D) is correct.

Solution 27

A Chapter 3, Replication

The end-of-year payoffs of the call and put options in each scenario are shown in the table below. The rightmost column is the payoff resulting from buying the put option and selling the call option.

	End of Year Price of Stock X	End of Year Price of Stock Y	C_X Payoff	P_Y Payoff	$P_Y - C_X$ Payoff
Outcome 1	\$200	\$0	105	95	–10
Outcome 2	\$50	\$0	0	95	95
Outcome 3	\$0	\$300	0	0	0

We need to determine the cost of replicating the payoffs in the rightmost column above. We can replicate those payoffs by determining the proper amount of Stock X, Stock Y, and the risk-free asset to purchase.

Let's define the following variables:

$X =$ Number of shares of Stock X to purchase

$Y =$ Number of shares of Stock Y to purchase

$Z =$ Amount to lend at the risk-free rate

We have 3 equations and 3 unknown variables:

Outcome 1: $200X + 0Y + Ze^{0.10} = -10$

Outcome 2: $50X + 0Y + Ze^{0.10} = 95$

Outcome 3: $0X + 300Y + Ze^{0.10} = 0$

Subtracting the second equation from the first equation allows us to find X:

$$150X = -105$$
$$X = -0.7$$

We can put this value of X into the first equation to find the value of Z:

$$200(-0.7) + Ze^{0.10} = -10$$
$$Ze^{0.10} = 130$$
$$Z = 117.6289$$

We can use this value of Z in the third equation to find the value of Y:

$$300Y + 117.629e^{0.10} = 0$$
$$Y = -0.4333$$

The cost now of replicating the payoffs resulting from buying the put and selling the call is equal to the cost of establishing a position consisting of X shares of Stock X, Y shares of Stock Y, and Z invested at the risk-free rate:

$$100X + 100Y + Z = 100 \times (-0.7) + 100 \times (-0.4333) + 117.6289 = 4.30$$

Solution 28

A Chapter 11, All-or-Nothing Options

For the square of the final stock price to be greater than 100, the final stock price must be greater than 10:

$$[S(1)]^2 > 100 \qquad \Leftrightarrow \qquad S(1) > 10$$

Therefore, the option described in the question is 100 cash-or-nothing call options that have a strike price of 10. The current value of the option is:

$$100 \times \text{CashCall}(S, K, T) = 100 \times e^{-rT} N(d_2) = 100 \times e^{-0.02} N(d_2)$$

To find the delta of the option, we must find the derivative of the price with respect to the stock price:

$$\frac{\partial\left(100 \times e^{-0.02} N(d_2)\right)}{\partial S} = 100e^{-0.02} \frac{\partial\left(N(d_2)\right)}{\partial S} = 100e^{-0.02} N'(d_2) \frac{\partial d_2}{\partial S}$$

The derivative of d_2 with respect to the stock price is:

$$\frac{\partial d_2}{\partial S} = \frac{\partial\left(d_1 - \sigma\sqrt{T}\right)}{\partial S} = \frac{\partial\left(\dfrac{\ln\left(\dfrac{Se^{-\delta T}}{Ke^{-rT}}\right) + \dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right)}{\partial S} = \frac{\dfrac{Ke^{-rT}}{Se^{-\delta T}} \times \dfrac{e^{-\delta T}}{Ke^{-rT}}}{\sigma\sqrt{T}} = \frac{1}{S\sigma\sqrt{T}}$$

$$= \frac{1}{10 \times 0.20 \times 1} = 0.5$$

The current value of d_2 is:

$$d_2 = \frac{\ln\left(\dfrac{Se^{-\delta T}}{Ke^{-rT}}\right) - \dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \frac{\ln\left(\dfrac{10}{10e^{-0.02}}\right) - \dfrac{0.2^2}{2}}{0.20} = 0$$

The density function for the standard normal random variable is:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$$

We can now calculate the delta of the option:

$$100e^{-0.02}N'(d_2)\frac{\partial d_2}{\partial S}=100e^{-0.02}\frac{1}{\sqrt{2\pi}}e^{-0.5d_2^2}\times 0.5=100e^{-0.02}\frac{1}{\sqrt{2\pi}}\times e^0\times 0.5$$
$$=19.552$$

Therefore, 19.552 shares must be purchased to delta-hedge the option.

Solution 29

D Chapter 18, Black-Derman-Toy Model

Each node is $e^{2\sigma_i\sqrt{h}}$ times the one below it. This means that 6% is $e^{4\sigma_2}$ times as large as 2%:

$$0.06=0.02e^{4\sigma_2}$$
$$\sigma_2=\ln\left(\frac{0.06}{0.02}\right)\times\frac{1}{4}=0.27465$$

The interest rate for the node above 2% is:

$$0.02e^{2\sigma_2}=0.02e^{2\times 0.27465}=0.034641$$

The tree of prices for the 3-year bond is:

		0.9434
	0.9095	
?		0.9665
	0.9451	
		0.9804

The calculations to obtain these prices are:

$$P(2,3,r_{uu})=\frac{1}{1.06}=0.9434$$
$$P(2,3,r_{ud})=P(2,3,r_{du})=\frac{1}{1.034641}=0.9665$$
$$P(2,3,r_{dd})=\frac{1}{1.02}=0.9804$$
$$P(1,3,r_u)=\frac{0.5(0.9434+0.9665)}{1.05}=0.9095$$
$$P(1,3,r_d)=\frac{0.5(0.9665+0.9804)}{1.03}=0.9451$$

We cannot calculate the current price of the bond, P(0,3) because we do not know the value of r_0, but we do not need P(0,3) to answer this question.

The formula for the one-year yield volatility for a T-year zero-coupon bond is:

$$\text{Yield volatility} = 0.5 \times \ln\left[\frac{P(1,T,r_u)^{-1/(T-1)} - 1}{P(1,T,r_d)^{-1/(T-1)} - 1}\right]$$

The yield volatility of the 3-year bond is:

$$\text{Yield volatility} = 0.5 \times \ln\left[\frac{0.9095^{-1/(3-1)} - 1}{0.9451^{-1/(3-1)} - 1}\right] = 0.26435$$

Solution 30

A Chapter 18, Black-Derman-Toy Model

The value of r_0 is the same as the yield on the 1-year zero-coupon bond, and the yield volatility of the 2-year bond is σ_1:

$$\frac{1}{1+r_0} = 0.9434 \qquad \Rightarrow \qquad r_0 = 6\%$$

$$\sigma_1 = 10\%$$

The first part of the interest rate tree is:

$$\begin{array}{ccccc} & r_u = R_1 e^{2\sigma_1} & & & r_d e^{0.20} \\ r_0 & & \Rightarrow & 6\% & \\ & r_d = R_1 & & & r_d \end{array}$$

We need to determine the value of r_d. The tree must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.06}(0.5)\left(\frac{1}{1+r_d e^{0.20}} + \frac{1}{1+r_d}\right)$$

$$0.8850 = 0.9434(0.5)\left(\frac{1}{1+r_d e^{0.20}} + \frac{1}{1+r_d}\right)$$

The quickest way to answer this question is to use trial and error.

Let's try Choice C first:

$$0.9434(0.5)\left(\frac{1}{1+(0.07)e^{0.20}} + \frac{1}{1+0.07}\right) = 0.8754$$

Since the 0.8754 is less than 0.8850, let's try a lower short rate.

Let's try Choice A:

$$0.9434(0.5)\left(\frac{1}{1+(0.0594)e^{0.20}}+\frac{1}{1+0.0594}\right)=0.8850$$

Choice A is the correct answer.

Solution 31

B Chapter 8, Delta

The delta of a bull spread is the same regardless of whether it is constructed of calls or puts. Let's assume that the bull spread consists of calls.

Since the bull spread consists of purchasing the lower-strike call and selling the higher-strike call, the delta of the bull spread is:

$$\Delta_{50}-\Delta_{60}$$

We can find the values of delta using:

$$\Delta_{Call}=e^{-\delta T}N(d_1) \quad \text{and} \quad d_1=\frac{\ln(S/K)+(r-\delta+0.5\sigma^2)T}{\sigma\sqrt{T}}$$

With $T=0.25$ and $K=50$, we have:

$$d_1=\frac{\ln(50/50)+\left(0.05+0.5(0.20)^2\right)(0.25)}{0.20\sqrt{0.25}}=0.17500$$

$$N(0.17500)=0.56946$$

$$\Delta_{50}=e^{-0\times 0.25}(0.56946)=0.56946$$

With $T=0.25$ and $K=60$, we have:

$$d_1=\frac{\ln(50/60)+\left(0.05+0.5(0.20)^2\right)(0.25)}{0.20\sqrt{0.25}}=-1.64822$$

$$N(-1.64822)=0.04965$$

$$\Delta_{60}=e^{-0\times 0.25}(0.04965)=0.04965$$

Therefore, the delta of the bull spread is initially:

$$\Delta_{50}-\Delta_{60}=0.56946-0.04965=0.51981$$

With $T = 2/12$ and $K = 50$, we have:

$$d_1 = \frac{\ln(50/50) + \left(0.05 + 0.5(0.20)^2\right)\frac{2}{12}}{0.20\sqrt{\frac{2}{12}}} = 0.14289$$

$$N(0.14289) = 0.55681$$

$$\Delta_{50} = e^{-0\times 2/12}(0.55681) = 0.55681$$

With $T = 2/12$ and $K = 60$, we have:

$$d_1 = \frac{\ln(50/60) + \left(0.05 + 0.5(0.20)^2\right)\frac{2}{12}}{0.20\sqrt{\frac{2}{12}}} = -2.09009$$

$$N(-2.09009) = 0.01830$$

$$\Delta_{60} = e^{-0\times 2/12}(0.01830) = 0.01830$$

Therefore, the delta of the bull spread after 1 month is:

$$\Delta_{50} - \Delta_{60} = 0.55681 - 0.01830 = 0.53851$$

The change in the delta of the bull spread is:

$$0.53851 - 0.51981 = 0.01870$$

Solution 32

E Chapter 14, Geometric Brownian Motion

The instantaneous return on the mutual fund is the weighted average of the return on the stock and the return on the risk-free asset:

$$\begin{aligned}\frac{dW(t)}{W(t)} &= \varphi\frac{dS(t)}{S(t)} + (1-\varphi)rdt \\ &= \varphi\left[\alpha dt + \sigma dZ(t)\right] + (1-\varphi)rdt \\ &= \left[\varphi\alpha + (1-\varphi)r\right]dt + \varphi\sigma dZ(t)\end{aligned}$$

The expression above does not match Choice (A), because the final term in Choice (A) does not include the factor φ. Therefore, we can rule out Choice (A).

As written above, we see that $W(t)$ is a geometric Brownian motion. Therefore, the price can be expressed as:

$$W(t) = W(0)e^{\left[\varphi\alpha + (1-\varphi)r - 0.5\varphi^2\sigma^2\right]t + \varphi\sigma Z(t)}$$

Although the expression above is close to Choices (B) and (C), it is slightly different from both choices. Therefore, we turn to Choices (D) and (E).

Since Choices (D) and (E) contain $[S(t)/S(0)]^{\varphi}$, let's find the value of this expression:

$$S(t) = S(0)e^{(\alpha-0.5\sigma^2)t+\sigma Z(t)}$$

$$\left[\frac{S(t)}{S(0)}\right]^{\varphi} = e^{\varphi(\alpha-0.5\sigma^2)t+\varphi\sigma Z(t)}$$

$$\left[\frac{S(t)}{S(0)}\right]^{\varphi} = e^{(\varphi\alpha-0.5\varphi\sigma^2)t+\varphi\sigma Z(t)}$$

We can now describe the process of the mutual fund with:

$$\begin{aligned}
W(t) &= W(0)e^{\left[\varphi\alpha+(1-\varphi)r-0.5\varphi^2\sigma^2\right]t+\varphi\sigma Z(t)} \\
&= W(0)e^{\left[\varphi\alpha-0.5\varphi\sigma^2\right]t+\varphi\sigma Z(t)}e^{\left[(1-\varphi)r-0.5\varphi^2\sigma^2+0.5\varphi\sigma^2\right]t} \\
&= W(0)\left[\frac{S(t)}{S(0)}\right]^{\varphi}e^{\left[(1-\varphi)r-0.5\varphi^2\sigma^2+0.5\varphi\sigma^2\right]t} \\
&= W(0)\left[\frac{S(t)}{S(0)}\right]^{\varphi}e^{\left[(1-\varphi)r+0.5\varphi\sigma^2(-\varphi+1)\right]t} \\
&= W(0)\left[\frac{S(t)}{S(0)}\right]^{\varphi}e^{(1-\varphi)\left[r+0.5\varphi\sigma^2\right]t}
\end{aligned}$$

Solution 33

C Chapter 11, Forward Start Options

The values of d_1 and d_2 for a 3-month put option with a strike price that is 90% of the current stock price are:

$$\begin{aligned}
d_1 &= \frac{\ln(S/0.90S)+(r-\delta+0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{-\ln(0.90)+(0.08-0.00+0.5\times 0.3^2)(0.25)}{0.3\sqrt{0.25}} \\
&= 0.91074 \\
d_2 &= d_1-\sigma\sqrt{T} = 0.9107-0.3\sqrt{0.25} = 0.76074
\end{aligned}$$

From the cumulative normal distribution calculator:

$$N(-d_1) = N(-0.91074) = 0.18122$$
$$N(-d_2) = N(-0.76074) = 0.22341$$

The cost of a 3-month put option that has a strike price of 90% of the current stock price is a function of the then-current stock price:

$$\begin{aligned} P_{Eur}(S, 0.90S, 0.25) &= 0.90Se^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1) \\ &= 0.90Se^{-0.25\times 0.08}N(-d_2) - Se^{-0.00}N(-d_1) \\ &= S\left[0.90e^{-0.02}(0.22341) - 0.18122\right] \\ &= 0.01587S \end{aligned}$$

Therefore, each of the 3-month put options can be purchased with 0.01587 shares of stock. The cost of purchasing all 4 of the options now is equal to the cost of acquiring $4\times 0.01587 = 0.06347$ shares of stock. Since the current stock price is \$45, this cost is:

$$0.06347\times 45 = 2.8562$$

Solution 34

A Chapter 14, Multiplication Rules

As n goes to infinity, we can replace the summation with an integral:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left\{Z[jh] - Z[(j-1)h]\right\}^3 = \int_0^T [dZ(t)]^3$$

We can now use the multiplication rules to evaluate the integral:

$$\int_0^T [dZ(t)]^3 = \int_0^T [dZ(t)]^2 \times dZ(t) = \int_0^T dt\times dZ(t) = \int_0^T 0 = 0$$

The expected value and variance of zero are both zero:

$$N(0,0)$$

Solution 35

B Chapter 14, Itô's Lemma

Since $X = R^2$, the partial derivatives of $X(t)$ are:

$$\begin{aligned} X_R &= 2R \\ X_{RR} &= 2 \\ X_t &= 0 \end{aligned}$$

We can use Itô's Lemma to find an expression for the differential of $X(t)$:

$$dX(t) = X_R dR(t) + \frac{1}{2}X_{RR}[dR(t)]^2 + X_t dt$$

Let's find $dR(t)$. The first two terms of $R(t)$ do not contain random variables, so their differentials are easy to find. The third term will be more difficult:

$$dR(t) = -R(0)e^{-t}dt + 0.05e^{-t}dt + d\left[0.1\int_0^t e^{s-t}\sqrt{R(s)}dZ(s)\right]$$

The third term has a function of t in the integral. We can pull the t-dependent portion out of the integral, so that we are finding the differential of a product. We then use the following version of the product rule to find the differential:

$$d\left[U(t)V(t)\right] = dU(t)V(t) + U(t)dV(t)$$

The differential of the third term is:

$$\begin{aligned} d\left[0.1\int_0^t e^{s-t}\sqrt{R(s)}dZ(s)\right] &= d\left[0.1e^{-t}\int_0^t e^{s}\sqrt{R(s)}dZ(s)\right] \\ &= -0.1e^{-t}dt\int_0^t e^{s}\sqrt{R(s)}dZ(s) + 0.1e^{-t}e^{t}\sqrt{R(t)}dZ(t) \\ &= -0.1e^{-t}dt\int_0^t e^{s}\sqrt{R(s)}dZ(s) + 0.1\sqrt{R(t)}dZ(t) \end{aligned}$$

Putting all three terms together, we have:

$$\begin{aligned} dR(t) &= -R(0)e^{-t}dt + 0.05e^{-t}dt - 0.1e^{-t}dt\int_0^t e^{s}\sqrt{R(s)}dZ(s) + 0.1\sqrt{R(t)}dZ(t) \\ &= -\left[R(0)e^{-t} - 0.05e^{-t} + 0.1e^{-t}\int_0^t e^{s}\sqrt{R(s)}dZ(s)\right]dt + 0.1\sqrt{R(t)}dZ(t) \\ &= -\left[R(0)e^{-t} + 0.05 - 0.05e^{-t} + 0.1\int_0^t e^{s-t}\sqrt{R(s)}dZ(s)\right]dt + 0.05dt + 0.1\sqrt{R(t)}dZ(t) \\ &= -\left[R(0)e^{-t} + 0.05(1-e^{-t}) + 0.1\int_0^t e^{s-t}\sqrt{R(s)}dZ(s)\right]dt + 0.05dt + 0.1\sqrt{R(t)}dZ(t) \\ &= -R(t)dt + 0.05dt + 0.1\sqrt{R(t)}dZ(t) \end{aligned}$$

Making use of the multiplication rules, we see that:

$$[dR(t)]^2 = 0.01R(t)dt$$

We can now find the differential of $X(t)$. To simplify the notation, we use R for $R(t)$ and X for $X(t)$ below:

$$\begin{aligned}
dX &= X_R dR + \frac{1}{2} X_{RR}(dR)^2 + X_t dt \\
&= 2RdR + \frac{1}{2} \times 2(dR)^2 + 0dt \\
&= 2RdR + (dR)^2 \\
&= 2R[-Rdt + 0.05dt + 0.1\sqrt{R}dZ] + 0.01Rdt \\
&= -2R^2 dt + 0.10Rdt + 0.2R^{3/2}dZ + 0.01Rdt \\
&= \left[0.11R - 2R^2\right]dt + 0.2R^{3/2}dZ
\end{aligned}$$

We are given that X is equal to R^2:

$$X = R^2 \qquad \Rightarrow \qquad \sqrt{X} = R$$

Substituting for R, we have:

$$\begin{aligned}
dX &= \left[0.11R - 2R^2\right]dt + 0.2R^{3/2}dZ \\
&= \left[0.11\sqrt{X} - 2X\right]dt + 0.2X^{3/4}dZ
\end{aligned}$$

Solution 36

E Chapter 16, Black-Scholes Equation

To simplify the notation, let's set the exponent in the price of the derivative security equal to a:

$$V = S^{-k/\sigma^2} = S^a$$

The partial derivatives are shown below:

$$\begin{aligned}
V_S &= aS^{a-1} \\
V_{SS} &= a(a-1)S^{a-2} \\
V_t &= 0
\end{aligned}$$

The Black-Scholes Equation gives us:

$$0.5\sigma^2 S^2 V_{SS} + (r-\delta)SV_S + V_t + D(t) = rV$$
$$0.5\sigma^2 S^2 a(a-1)S^{a-2} + (0.04-0)SaS^{a-1} + 0 + 0 = 0.04S^a$$
$$0.5\sigma^2 a(a-1) + (0.04-0)a = 0.04$$
$$0.5\sigma^2 a(a-1) + 0.04a - 0.04 = 0$$
$$0.5\sigma^2 a(a-1) + 0.04(a-1) = 0$$
$$(a-1)(0.5\sigma^2 a + 0.04) = 0$$
$$a-1=0 \quad \text{or} \quad 0.5\sigma^2 a + 0.04 = 0$$
$$a = 1 \quad \text{or} \quad a = -\frac{0.08}{\sigma^2}$$

The question specifies that k is positive, so a must be negative, and therefore a cannot be equal to 1:

$$a = -\frac{0.08}{\sigma^2}$$
$$\frac{-k}{\sigma^2} = \frac{-0.08}{\sigma^2}$$
$$k = 0.08$$

Solution 37

D Chapter 14, Brownian Motion

We make use of the following equivalency:

$$\frac{dS(t)}{S(t)} = (\alpha-\delta)dt + \sigma dZ(t) \quad \Leftrightarrow \quad \ln\left[\frac{S(t+h)}{S(t)}\right] \sim N\left((\alpha-\delta-0.5\sigma^2)h,\ \sigma^2 h\right)$$
$$\frac{dS(t)}{S(t)} = 0.03dt + 0.2dZ(t) \quad \Leftrightarrow \quad \ln\left[\frac{S(t+h)}{S(t)}\right] \sim N\left(0.01h,\ 0.04h\right)$$

To avoid a cluttered appearance, we use S_t to represent $S(t)$ in the solution below.

We can rewrite the expression for G, so that it is the product of independent random variables:

$$G = \left[S_1 \times S_2 \times S_3\right]^{1/3} = \left[\left(S_0 \frac{S_1}{S_0}\right)\left(S_0 \frac{S_1}{S_0}\frac{S_2}{S_1}\right)\left(S_0 \frac{S_1}{S_0}\frac{S_2}{S_1}\frac{S_3}{S_2}\right)\right]^{1/3}$$
$$= \left[S_0{}^3 \times \frac{S_1{}^3}{S_0{}^3} \times \frac{S_2{}^2}{S_1{}^2} \times \frac{S_3}{S_2}\right]^{1/3} = S_0 \times \left(\frac{S_1}{S_0}\right) \times \left(\frac{S_2}{S_1}\right)^{\frac{2}{3}} \times \left(\frac{S_3}{S_2}\right)^{\frac{1}{3}}$$

Taking the natural log, we have a sum of independent random variables:

$$\ln G = \ln(S_0) + \ln\left(\frac{S_1}{S_0}\right) + \frac{2}{3}\ln\left(\frac{S_2}{S_1}\right) + \frac{1}{3}\ln\left(\frac{S_3}{S_2}\right)$$

The variance is:

$$Var[\ln G] = 0 + 0.04 + \left(\frac{2}{3}\right)^2(0.04) + \left(\frac{1}{3}\right)^2(0.04) = 0.06222$$

Solution 38

B Chapter 19, Delta-Gamma Approximation for Bonds

In the model described in the question, we have:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$
$$P_r(r,t,T) = -B(t,T)A(t,T)e^{-B(t,T)r} = -B(t,T)P(r,t,T)$$
$$P_{rr}(r,t,T) = [B(t,T)]^2 A(t,T)e^{-B(t,T)r} = [B(t,T)]^2 P(r,t,T)$$

The delta-gamma-theta approximation is:

$$P[r(t+h),t+h,T] - P[r(t),t,T] \approx [r(t+h)-r(t)]P_r + 0.5[r(t+h)-r(t)]^2 P_{rr} + P_t h$$

This question asks us to use only the delta and gamma portions of the approximation, so we remove the theta term, $P_t h$, from the expression above:

$$P[r(t+h),t+h,T] - P[r(t),t,T] \approx [r(t+h)-r(t)]P_r + 0.5[r(t+h)-r(t)]^2 P_{rr}$$

We replace $r(t)$ with 0.05, and we replace $r(t+h)$ with 0.03.

The delta-gamma approximation is therefore:

$$P_{Est}(0.03,0,3) - P(0.05,0,3) \approx [-0.02]P_r(0.05,0,3) + 0.5[-0.02]^2 P_{rr}(0.05,0,3)$$
$$P_{Est}(0.03,0,3) \approx P(0.05,0,3) + [-0.02]P_r(0.05,0,3) + 0.5[-0.02]^2 P_{rr}(0.05,0,3)$$
$$P_{Est}(0.03,0,3) \approx P(0.05,0,3)$$
$$-[-0.02]B(0,3)P(0.05,0,3) + 0.5[-0.02]^2[B(0,3)]^2 P(0.05,0,3)$$
$$P_{Est}(0.03,0,3) \approx P(0.05,0,3)\left[1 + 0.02\times 2 + 0.5\times(-0.02)^2\times 2^2\right]$$
$$P_{Est}(0.03,0,3) \approx P(0.05,0,3)\times 1.0408$$

Therefore, we have:

$$\frac{P_{Est}(0.03,0,3)}{P(0.05,0,3)} \approx \frac{P(0.05,0,3)\times 1.0408}{P(0.05,0,3)} = 1.0408$$

Solution 39

B Chapters 3 & 18, Risk-Neutral Probability

Let's use $P(0,2)$ to denote the price of a 2-year zero-coupon bond that matures for \$1.

We can make use of put-call parity:

$$C(108)+108\times P(0,2)=S_0+P(108)$$
$$P(108)-C(108)=108\times P(0,2)-100$$

We can use the stock prices to determine the risk-neutral probability that the up state of the world occurs:

$$p^*=\frac{e^{(r-\delta)h}-d}{u-d}=\frac{(1+r_0)^h-d}{u-d}=\frac{(1.05)^1-0.95}{1.10-0.95}=\frac{0.10}{0.15}=\frac{2}{3}$$

If the up state occurs, then the zero-coupon bond will have a value of $\frac{1}{1.06}$ at time 1, and if the down state occurs, then the zero-coupon bond will have a value of $\frac{1}{1.04}$ at time 1. The time 0 value is found using the risk-neutral probabilities and the risk-free rate at time 0:

$$P(0,2)=\frac{1}{1.05}\left[\frac{2}{3}\times\frac{1}{1.06}+\frac{1}{3}\times\frac{1}{1.04}\right]=0.90423$$

We can now use the equation for put-call parity described above to find the solution:

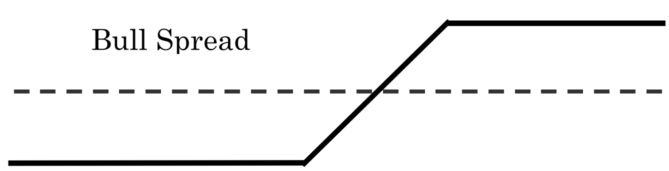
$$P(108)-C(108)=108\times P(0,2)-100=108\times 0.90423-100=-2.3429$$

Solution 40

D Chapter 2, Option Strategies

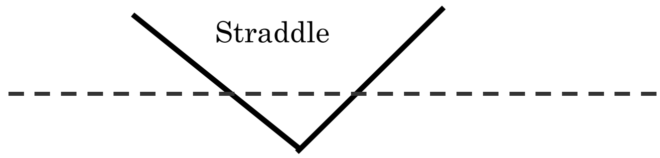
This question is probably more appropriate for Exam FM than for Exam MFE/3F. This material is covered on page 87 of the second edition of the Derivatives Markets textbook, which is assigned for Exam FM but is not assigned for Exam MFE/3F.

A bull spread consists of purchasing a low-strike option and selling a high-strike option. In the profit diagram below, the dotted line represents zero profit. The profit diagram shows the pattern of profits at the end of 1 year:



The diagram above matches Portfolio IV, so the correct answer must be Choice (D) or Choice (E).

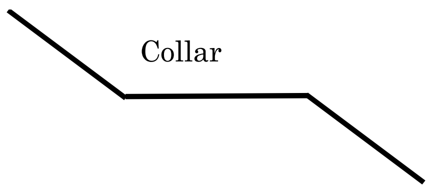
A straddle consists of the purchase of a call and a put with the same strike price. The profit diagram shows the pattern of the profits at the end of 1 year:



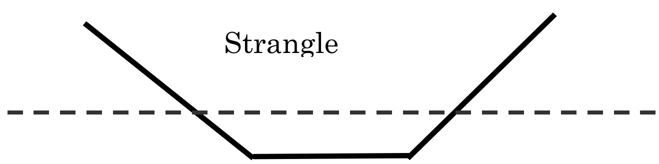
The diagram above matches Portfolio II, so the correct answer must be Choice (D).

For the sake of completeness, let's consider the other two strategies as well.

A collar consists of the purchase of a put and the sale of a call with a higher strike price. We would need more information to know where it would produce zero profit, so the dotted line has been left off of the graph below. Nonetheless, the pattern indicates that Portfolio I is the collar:



A strangle consists of the purchase of a put and the purchase of a call with a higher strike price. It's profit diagram matches Portfolio III.



Solution 41

C Chapter 8, Elasticity

The contingent claim can be replicated with a portfolio consisting of the present value of \$42 and a short position in a European put option with a strike price of \$42.

To find the value of this contingent claim, we must find the value of the put option:

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(45/42) + (0.07 - 0.03 + 0.5 \times 0.25^2) \times 1}{0.25\sqrt{1}}$$
$$= 0.56097$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.56097 - 0.25\sqrt{1} = 0.31097$$

We have:

$$N(-d_1) = N(-0.56097) = 0.28741$$

$$N(-d_2) = N(-0.31097) = 0.37791$$

The value of the European put option is:

$$P(42) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$
$$= 42e^{-0.07} \times 0.37791 - 45e^{-0.03} \times 0.28741 = 2.24795$$

The current value of the contingent claim is the present value of $42 minus the value of the put:

$$V = 42e^{-0.07} - 2.24795 = 36.91259$$

The delta of the put option is:

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{-0.03} \times 0.28741 = -0.27892$$

The delta of the contingent claim is the delta of the present value of $42 (i.e., zero) minus the delta of the put:

$$\Delta = 0 - (-0.27892) = 0.27892$$

The elasticity of the contingent claim is:

$$\Omega = \frac{S\Delta}{V} = \frac{45 \times 0.27892}{36.91259} = 0.34003$$

Solution 42

D Chapter 10, Barrier Options

As the barrier of an up-and-out call approaches infinity, the price of the up-and-out call approaches the price of a regular call.

Therefore, a regular call option with a strike price of $60 can be read from the bottom line of the table:

$$C(60) = 4.0861$$

The value of the corresponding up-and-in calls can be determined with the parity relationship for barrier options:

$$\text{Up-and-in call} + \text{Up-and-out call} = \text{Ordinary call}$$

For the $70 and $80 barriers, we have:

H	Up-and-out Call	Up-and-in Call
70	0.1294	$4.0861 - 0.1294 = 3.9567$
80	0.7583	$4.0861 - 0.7583 = 3.3278$

The special option consists of 2 of the 70-barrier up-and-in calls and a short position in 1 of the 80-barrier up-and-in calls. Therefore, the price of the special option is:

$$2 \times 3.9567 - 3.3278 = 4.5856$$

Solution 43

E Chapter 14, Geometric Brownian Motion

For this question, we make use of the following relationship:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \quad \Leftrightarrow \quad S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)}$$

We can write $x(t)$ as:

$$x(t) = x(0)e^{(r - r_€ - 0.5\sigma^2)t + \sigma Z(t)}$$

Therefore, $y(t)$ is:

$$y(t) = \frac{1}{x(t)} = \frac{1}{x(0)}e^{-(r - r_€ - 0.5\sigma^2)t - \sigma Z(t)} = \frac{1}{x(0)}e^{(r_€ - r + 0.5\sigma^2)t - \sigma Z(t)}$$

This implies that:

$$\frac{dy(t)}{y(t)} = \left[r_€ - r + 0.5\sigma^2 + 0.5(-\sigma)^2\right]dt - \sigma dZ(t) = (r_€ - r + \sigma^2)dt - \sigma dZ(t)$$

Solution 44

D Chapter 4, Three-Period Binomial Tree

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.10 - 0.065)\times 1} - \frac{210}{300}}{\frac{375}{300} - \frac{210}{300}} = 0.6102$$

The tree of prices for the American put option is shown below:

American Put			0.0000
		0.0000	
	14.4603		0.0000
39.7263		41.0002	
	90.0000		116.2500
		153.0000	
			197.1000

Early exercise is optimal if the stock price falls to \$210 at the end of 1 year or falls to \$147 at the end of 2 years. This is indicated by the bolding of those two nodes in the tree above.

The price of the option is:

$$e^{-0.10\times 1}\left[0.6102\times 14.4603+(1-0.6102)\times 90\right]=39.7263$$

Solution 45

C Chapter 4, Greeks in the Binomial Model

We need to calculate the two possible values of delta at the end of 1 year:

$$\Delta(Su,h)=e^{-\delta h}\frac{V_{uu}-V_{ud}}{Su^2-Sud}=e^{-0.065\times 1}\frac{0.00-41.00}{468.75-262.50}=-0.1863$$

$$\Delta(Sd,h)=e^{-\delta h}\frac{V_{ud}-V_{dd}}{Sud-Sd^2}=e^{-0.065\times 1}\frac{41.00-153.00}{262.50-147.00}=-0.9087$$

We can now calculate gamma:

$$\Gamma(S,0)\approx\Gamma(S_h,h)=\frac{\Delta(Su,h)-\Delta(Sd,h)}{Su-Sd}=\frac{-0.1863-(-0.9087)}{375-210}=0.004378$$

Solution 46

E Chapter 4, Options on Futures Contracts

We are given that the ratio of the factors applicable to the futures price is:

$$\frac{u_F}{d_F}=\frac{4}{3}$$

The formula for the risk-neutral probability of an up move can be used to find u_F and d_F:

$$p^*=\frac{1-d_F}{u_F-d_F}$$

$$p^*=\frac{\frac{1}{d_F}-\frac{d_F}{d_F}}{\frac{u_F}{d_F}-\frac{d_F}{d_F}}$$

$$\frac{1}{3}=\frac{\frac{1}{d_F}-1}{\frac{4}{3}-1}$$

$$d_F=0.9$$

$$u_F=\frac{4}{3}\times d_F=1.2$$

The tree of futures prices is therefore:

Futures Prices		
		115.2000
	96.0000	
80.0000		86.4000
	72.0000	
		64.8000

The tree of prices for the European call option is:

European Call		
		30.2000
	10.7284	
3.7838		1.4000
	0.4551	
		0.0000

The price of the European call is:

$$e^{-0.05\times 0.5}\left[(1/3)10.7284+(2/3)(0.4551)\right]=3.7838$$

The tree of prices for the American call option is:

American Call		
		30.2000
	11.0000	
3.8721		1.4000
	0.4551	
		0.0000

The price of the American call is:

$$e^{-0.05\times 0.5}\left[(1/3)11.0000+(2/3)(0.4551)\right]=3.8721$$

If the futures price moves up at the end of 6 months, then early exercise of the American option is optimal, and this is indicated by the bolding of that node above.

The price of the American call option exceeds the price of the European call option by:

$$3.8721-3.7838=0.08830$$

Solution 47

B Chapter 9, Market-Maker Profit

Let's assume that "several months ago" was time 0. Further, let's assume that the options expire at time T and that the current time is time t. We can use put-call parity to obtain a system of 2 equations.

$$8.88+Ke^{-rT}=40+1.63$$
$$14.42+Ke^{-r(T-t)}=50+0.26$$

This can be solved to find e^{rt}:

$$\left.\begin{aligned} Ke^{-rT} &= 32.75 \\ Ke^{-r(T-t)} &= 35.84 \end{aligned}\right\} \Rightarrow \qquad e^{rt} = \frac{35.84}{32.75}$$

The market-maker sold 100 of the call options. From the market-maker's perspective, the value of this short position was:

$$-100 \times 8.88 = -888.00$$

The delta of the position, from the perspective of the market-maker, was:

$$\text{Delta of a short position in 100 calls} = -100 \times (0.794) = -79.4$$

To delta-hedge the position, the market-maker purchased 79.4 shares of stock. The value of this position was:

$$79.4 \times 40 = 3{,}176.00$$

Since the market-maker received only $888.00 from the sale of the calls, the differential was borrowed. The value of the loan, from the perspective of the market-maker, was:

$$888.00 - 3{,}176.00 = -2{,}288.00$$

The initial position, from the perspective of the market-maker, was:

Component	Value
Options	–888.00
Shares	3,176.00
Risk-Free Asset	–2,288.00
Net	0.00

After t years elapsed, the value of the options changed by:

$$-100 \times (14.42 - 8.88) = -554.00$$

After t years elapsed, the value of the shares of stock changed by:

$$79.4 \times (50.00 - 40.00) = 794.00$$

After t years elapsed, the value of the funds that were borrowed at the risk-free rate changed by:

$$-2{,}288.00\left(e^{rt} - 1\right) = -2{,}288.00\left(\frac{35.84}{32.75} - 1\right) = -215.88$$

The sum of these changes is the profit.

Component	Change
Gain on Options	–554.00
Gain on Stock	794.00
Interest	–215.88
Overnight Profit	24.12

The change in the value of the position is $24.12, and this is the profit.

Solution 48

E Chapter 15, Sharpe Ratio

The Sharpe ratio of Asset 1 must be equal to that of Asset 2:

$$\frac{0.06-0.04}{0.02}=\frac{0.03-0.04}{k}$$
$$k=-0.01$$

Over the next instant, the returns on Stock 1 and Stock 2 are:

$$\text{Return on } S_1 = 0.06S_1dt+0.02S_1dZ = 0.06(100)dt+0.02(100)dZ = 6.0dt+2dZ$$
$$\text{Return on } S_2 = 0.03S_2dt-0.01S_2dZ = 0.03(50)dt-0.01(50)dZ = 1.5dt-0.5dZ$$

Since the investor purchases 1 share of Stock 1, the amount of Stock 2 that must be purchased to remove the random term (i.e., the dZ term) is:

$$\frac{2}{0.50}=4$$

When the return on Stock 1 is added to the return on 4 shares of Stock 2, there is no random term:

$$\begin{aligned}\text{Return on } S_1 &= 6.0dt+2dZ\\ \text{Return on 4 shares of } S_2 &= \underline{4\times(1.5dt-0.5dZ)}\\ &\quad 12dt\end{aligned}$$

Solution 49

B Chapter 4, American Put Option

The values of u and d are:

$$u=e^{(r-\delta)h+\sigma\sqrt{h}}=e^{(0.04-0.00)0.25+0.3\sqrt{0.25}}=1.1735$$
$$d=e^{(r-\delta)h-\sigma\sqrt{h}}=e^{(0.04-0.00)0.25-0.3\sqrt{0.25}}=0.8694$$

The risk-free probability of an upward movement is:

$$p^*=\frac{e^{(r-\delta)h}-d}{u-d}=\frac{e^{(0.04-0.00)0.25}-0.8694}{1.1736-0.8694}=0.4626$$

The stock price tree is:

	117.35
100	
	86.94

The strike price K, for which an investor will exercise the put option at the beginning of the period must be at least \$100, since otherwise the payoff to immediate exercise would be zero. Since we are seeking the lowest strike price that results in immediate exercise, let's begin by determining whether there is a strike price that is greater than \$100 but less than \$117.35 that results in immediate exercise. If there is such a strike price, the value of exercising now must exceed the value of holding the option:

$$K - 100 > (K - 86.94)(1 - p^*)e^{-0.04(0.25)}$$
$$K - 100 > (K - 86.94)(1 - 0.4626)e^{-0.04(0.25)}$$
$$K - 100 > 0.5321K - 46.26$$
$$0.4679K > 53.74$$
$$K > 114.85$$

The smallest integer that satisfies this inequality is \$115.

Solution 50

A Chapter 5, Lognormal Confidence Intervals

We are given:

$$S_0 = 0.25$$
$$\sigma = 0.35$$
$$\alpha - \delta = 0.15$$

The z-value associated with the upper limit of the 90% confidence interval is found below:

$$P\left(z > z^U\right) = \frac{p}{2}$$
$$P\left(z > z^U\right) = \frac{0.10}{2}$$
$$P\left(z > z^U\right) = 0.05$$
$$P\left(z < z^U\right) = 0.95$$
$$z^U = 1.64485$$

The upper limit of the confidence interval is calculated using the associated z-value:

$$S_T^U = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T-t) + \sigma z^U \sqrt{T-t}}$$
$$= 0.25e^{(0.15 - 0.5\times 0.35^2)(0.5) + 0.35\times 1.64485\sqrt{0.5}} = 0.39265$$

Solution 51

E Chapter 13, Estimating Parameters of a Lognormal Distribution

First, we determine the monthly mean and the monthly standard deviation using a calculator, and then we determine the annualized standard deviation and the annualized expected return.

Using the TI-30X IIS calculator, press [2nd] [STAT] and choose 1-VAR and press [ENTER] [DATA]. Perform the following sequence:

X1 = LN(56/54) [ENTER] ↓↓ (Hit the down arrow twice)

X2 = LN(48/56) [ENTER] ↓↓

X3 = LN(55/48) [ENTER] ↓↓

X4 = LN(60/55) [ENTER] ↓↓

X5 = LN(58/60) [ENTER] ↓↓

X6 = LN(62/58) [ENTER]

Press [STATVAR] and → and the monthly mean is shown as 0.02302506. Press → again and the monthly standard deviation is shown as 0.103541414. Multiply this number by the square root of 12 to get the annualized standard deviation of 0.35867798. To exit the statistics mode, press [2nd] [EXITSTAT] [ENTER].

Since the dividend rate is zero, we determine the annualized expected return by multiplying the monthly mean return by 12 and adding to that amount one half of the annualized variance:

$$\hat{\alpha} = \frac{\overline{r}}{h} + \delta + 0.5\hat{\sigma}^2 = 0.02302506 \times 12 + 0 + 0.5 \times 0.35867798^2 = 0.340625623$$

Solution 52

C Chapter 12, Normal Random Variables as Quantiles

We convert the draws from the uniform distribution into draws from the standard normal distribution:

$$F(0.98300) = N(\hat{z}_1) \quad \Rightarrow \quad \hat{z}_1 = 2.12007$$
$$F(0.03836) = N(\hat{z}_2) \quad \Rightarrow \quad \hat{z}_2 = -1.77004$$
$$F(0.77935) = N(\hat{z}_3) \quad \Rightarrow \quad \hat{z}_3 = 0.77000$$

We use these draws to find the new stock prices:

$$S_T = S_t e^{(\alpha-\delta-0.5\sigma^2)(T-t)+\sigma z\sqrt{T-t}}$$

1. $S_2 = 50e^{(0.15-0.00-0.5\times0.3^2)2+0.30\times(2.12007)\sqrt{2}} = 151.63746$
2. $S_2 = 50e^{(0.15-0.00-0.5\times0.3^2)2+0.30\times(-1.77004)\sqrt{2}} = 29.10933$
3. $S_2 = 50e^{(0.15-0.00-0.5\times0.3^2)2+0.30\times(0.77000)\sqrt{2}} = 85.51624$

The average of the stock prices is:

$$\frac{151.63746+29.10933+85.51624}{3} = 88.75434$$

Solution 53

B Chapter 11, Cash-Or-Nothing Call Option

A European call option has the same gamma as an otherwise equivalent European put option. Therefore, the gamma of a call option with the payoff described below is 0.07:

$$S_T - 40 \qquad \text{if } S_T > 40$$

The gap call option described in the question has a gamma of 0.08, and its payoff is:

$$S_T - 45 \qquad \text{if } S_T > 40$$

The cash-or-nothing call option has the following payoff:

$$1{,}000 \qquad \text{if } S_T > 40$$

The cash-or-nothing call option can be replicated by purchasing 200 of the call options and selling 200 of the gap call options:

$$200[Call - GapCall] = 200[(S_T - 40) - (S_T - 45)] = 200 \times 5 = 1{,}000 \qquad \text{if } S_T > 40$$

The gamma of the position is:

$$200[0.07 - 0.08] = 200 \times (-0.01) = -2.00$$

Solution 54

A Chapter 1, Exchange Options

Consider an asset, which we will call Asset X, which has a payoff of:

$$X_1 = Min[2S_1(1), S_2(1)]$$

From statement (vi), we know that the price of a European call option on Asset X, with a strike price of 17, is 1.632. The payoff of this call option is:

$$Max\{Min[2S_1(1), S_2(1)] - 17, 0\} = Max\{X_1 - 17, 0\}$$

We are asked to find the price of an option that has the payoff described below:

$$Max\{17 - Min[2S_1(1), S_2(1)], 0\} = Max\{17 - X_1, 0\}$$

We recognize this as a European put option on Asset X. From put-call parity, we have:

$$C_{Eur} + Ke^{-r(T-t)} = S_t + P_{Eur}$$
$$1.632 + 17e^{-0.05\times 1} = X_0 + P_{Eur}$$

If we can find the current value of Asset X, then we can solve for the value of the put option. Asset X can be replicated by purchasing 2 shares of Stock 1, and selling an exchange call option that allows its owner to exchange 1 share of Stock 2 for 2 shares of Stock 1:

$$X_1 = Min[2S_1(1), S_2(1)] = 2S_1(1) - Max[2S_1(1) - S_2(1), 0]$$

The current value of Asset X is:

$$X_0 = 2S_1(0) - \text{ExchangeCallPrice}$$

The exchange call option is an exchange call option with 2 shares of Stock 1 as the underlying asset and 1 share of Stock 2 as the strike asset. To find the value of this exchange call option, we find the appropriate volatility parameter:

$$\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K} = \sqrt{0.18^2 + 0.25^2 - 2(-0.40)(0.18)(0.25)} = 0.36180$$

The values of d_1 and d_2 are:

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta_S T}}{Ke^{-\delta_K T}}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{(2\times 10)e^{-0\times 1}}{20e^{-0\times 1}}\right) + \frac{0.36180^2 \times 1}{2}}{0.36180\sqrt{1}} = 0.18090$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.18090 - 0.36180\sqrt{1} = -0.18090$$

From the normal table, we have:

$$N(d_1) = N(0.18090) = 0.57178$$
$$N(d_2) = N(-0.18090) = 0.42822$$

The value of the exchange call option is:

$$\begin{aligned}\text{ExchangeCallPrice} &= Se^{-\delta_S T}N(d_1) - Ke^{-\delta_K T}N(d_2)\\ &= 2\times 10e^{0\times 1}(0.57178) - 20e^{0\times 1}(0.42822)\\ &= 2.8712\end{aligned}$$

The current value of Asset X is:

$$X_0 = 2S_1(0) - \text{ExchangeCallPrice} = 2\times 10 - 2.8717 = 17.1288$$

We can now use put-call parity to find the value of the put option:

$$1.632 + 17e^{-0.05\times 1} = X_0 + P_{Eur}$$
$$1.632 + 17e^{-0.05} = 17.1288 + P_{Eur}$$
$$P_{Eur} = 0.6741$$

Solution 55

D Chapter 7, Options on Futures Contracts

At time 0, the values of d_1 and d_2 are:

$$d_1 = \frac{\ln\left(F_{0,T_F} / K\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln(20/20) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = 0.5\sigma\sqrt{T}$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.5\sigma\sqrt{T} - \sigma\sqrt{T} = -0.5\sigma\sqrt{T}$$

We can find $N(-d_1)$ in terms of $N(-d_2)$:

$$N(-d_1) = N\left(-0.5\sigma\sqrt{T}\right)$$
$$N(-d_2) = N\left(0.5\sigma\sqrt{T}\right)$$
$$N(-d_1) = 1 - N(-d_2)$$

We can substitute this value of $N(-d_1)$ into the time 0 formula for the price in order to find d_2:

$$P_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) = Ke^{-rT}N(-d_2) - F_{0,T_F}e^{-rT}N(-d_1)$$
$$1.625 = 20e^{-0.10\times 0.75}N(-d_2) - 20e^{-0.10\times 0.75}N(-d_1)$$
$$1.625 = 20e^{-0.075}\left[N(-d_2) - \left(1 - N(-d_2)\right)\right]$$
$$\frac{1.625}{20}e^{0.075} = 2N(-d_2) - 1$$
$$N(-d_2) = 0.54379$$
$$-d_2 = 0.10999$$
$$d_2 = -0.10999$$

We can now find σ:

$$d_2 = -0.5\sigma\sqrt{T}$$
$$-0.10999 = -0.5\sigma\sqrt{0.75}$$
$$\sigma = 0.25401$$

After 3 months, the new values of d_1 and d_2 are:

$$d_1 = \frac{\ln\left(F_{0,T_F} / K\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(17.7/20\right) + 0.5 \times 0.25401^2 \times 0.5}{0.25401\sqrt{0.5}} = -0.59037$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.59037 - 0.25401\sqrt{0.5} = -0.76998$$

We can look up the values of $N(-d_1)$ and $N(-d_2)$ from the normal distribution table:

$$N(-d_1) = N(0.59037) = 0.72253$$
$$N(-d_2) = N(0.76998) = 0.77934$$

After 3 months, the value of the put option is:

$$\begin{aligned} P_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) &= Ke^{-rT} N(-d_2) - F_{0,T_F} e^{-rT} N(-d_1) \\ &= 20e^{-0.10\times 0.5} \times 0.77934 - 17.70e^{-0.10\times 0.5} \times 0.72253 \\ &= 2.66156 \end{aligned}$$

Solution 56

A Chapter 5, Covariance of S_t and S_T

To answer this question, we use the following formulas:

$$E\left[\frac{S_T}{S_t}\right] = e^{(\alpha-\delta)(T-t)}$$

$$Var\left[S_T \,|S_t\right] = S_t^2 e^{2(\alpha-\delta)(T-t)}\left(e^{\sigma^2(T-t)} - 1\right)$$

$$Cov[S_t, S_T] = E\left[\frac{S_T}{S_t}\right] Var\left[S_t \,|S_0\right]$$

The variance of $A(2)$ is:

$$Var\left[A(2)\right] = \frac{1}{4}\left\{Var[S(1) + Var[S(2)] + 2Cov[S(1), S(2)]\right\}$$

The variances of $S(1)$ and $S(2)$ are:

$$Var\left[S_1 \,|S_0\right] = 5^2 e^{2(0.05)(1-0)}\left(e^{0.2^2(1-0)} - 1\right) = 1.12757$$

$$Var\left[S_2 \,|S_0\right] = 5^2 e^{2(0.05)(2-0)}\left(e^{0.2^2(2-0)} - 1\right) = 2.54318$$

The covariance is:

$$Cov[S_1, S_2] = E\left[\frac{S_2}{S_1}\right] Var\left[S_1 \,|S_0\right] = e^{0.05(2-1)} \times 1.12757 = 1.18538$$

We can now find the variance of $A(2)$:

$$Var\left[A(2)\right] = \frac{1}{4}\left\{Var[S(1) + Var[S(2)] + 2Cov[S(1), S(2)]\right\}$$
$$= \frac{1}{4}\left\{1.12757 + 2.54318 + 2 \times 1.18538\right\} = 1.51038$$

Solution 57

E Chapter 12, Stratified Sampling Method

The stratified sampling method assigns the first and fifth uniform (0, 1) random numbers to the segment (0.00, 0.25), the second and sixth uniform (0, 1) random variables to the segment (0.25, 0.50), the third and seventh uniform (0, 1) random variables to the segment (0.50, 0.75), and the fourth and the eighth uniform (0, 1) random variables to the segment (0.75, 1.00).

The lowest simulated normal random variables will come from the segment (0.00, 0.25). The lower of the two values in this segment is the fifth one: 0.3172, so we use it to find the corresponding standard normal random variable:

$$u_5 = 0.3172$$
$$\hat{u}_5 = \frac{0.3172 + (1-1)}{4} = 0.0793$$
$$Z_5 = N^{-1}(0.0793) = -1.40980$$

The highest simulated normal random variable will come from the segment (0.75, 1.00). The higher of the two values in this segment is the fourth one: 0.4482, so we use it to find the corresponding standard normal random variable:

$$u_4 = 0.4482$$
$$\hat{u}_4 = \frac{0.4482 + (4-1)}{4} = 0.86205$$
$$Z_4 = N^{-1}(0.86205) = 1.08958$$

The difference between the largest and smallest simulated normal random variates is:

$$1.08958 - (-1.40980) = 2.49938$$

Solution 58

B Chapter 12, Control Variable Method with Beta

Let the variable Y be associated with the \$42-strike option and the variable X be associated with the \$40-strike option. In the regression analysis, Y will be the dependent variable and X will be the independent variable.

The payoffs depend on the simulated stock prices:

Stock Price	Payoff With Strike = 40 $\left(X_i e^{0.02}\right)$	Payoff With Strike = 42 $\left(Y_i e^{0.02}\right)$
33.29	0.00	0.00
37.30	0.00	0.00
40.35	0.35	0.00
43.65	3.65	1.65
48.90	8.90	6.90

The values in the two rightmost columns are the payoffs, which means that they are the discounted Monte Carlo prices, X_i and Y_i, times e^{rT}.

The estimate for β is:

$$\beta = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{\left(e^{rT}\right)^2 \sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\left(e^{rT}\right)^2 \sum_{i=1}^{n}(X_i - \bar{X})^2}$$

We get the same estimate for β regardless of whether we use the time 0 prices or the time 0.25 payoffs (as shown in the rightmost expression above). To save time, we use the time 0.25 payoffs from the table above.

We perform a regression using the 2nd column in the table above as the *x*-values and the third column as the *y*-values. The resulting slope coefficient is:

$$\beta = 0.764211$$

During the exam, it is more efficient to let the calculator perform the regression and determine the slope coefficient.

Using the TI-30XS Multiview calculator, we first clear the data by pressing [data] [data] ↓↓↓ until Clear ALL is shown, and then press [enter].

Fill out the table as shown below:

L1	L2	L3
0.00	0.00	-----------
0.00	0.00	
0.35	0.00	
3.65	1.65	
8.90	6.90	

Next, press:

[2nd] [stat] 2 (i.e., select 2-Var Stats)

Select L1 for x-data and press [enter].

Select L2 for y-data and press [enter].

Select CALC and then [enter]

Scroll down to: $a = 0.764211005$.

Exit the statistics mode by pressing [2nd] [quit].

Solution 59

B Chapter 12, Control Variable Method with Beta

Since the 40-strike call is our control, we have:

$$\bar{X} = \frac{0+0+0.35+3.65+8.90}{5} \times e^{-0.02} = 2.5289$$

$$\bar{Y} = \frac{0+0+0+1.65+6.90}{5} \times e^{-0.02} = 1.6761$$

The estimate for the price of the \$42-strike option is:

$$C^*(42) = Y^* = \bar{Y} + \beta(X - \bar{X}) = 1.6761 + 0.764211(2.7847 - 2.5289) = 1.8716$$

Solution 60

E Chapter 19, Cox-Ingersoll-Ross Model

The CIR model is:

$$dr = a(b-r)dt + \sigma\sqrt{r}dZ$$

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

$$a = 0.1 \qquad b = 0.11 \qquad \sigma = 0.08$$

We can define c in terms of $\bar{\phi}$:

$$\phi(r,t) = \bar{\phi}\frac{\sqrt{r}}{\sigma} \qquad \& \qquad \phi(r,t) = c\sqrt{r} \qquad \Rightarrow \qquad c = \frac{\bar{\phi}}{\sigma}$$

In the CIR model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

$$\bar{r} = \frac{2ab}{(a - \bar{\phi} + \gamma)} = \lim_{T\to\infty}\left[\frac{-\ln[P(r,t,T)]}{T-t}\right]$$

From the information provided in part (ii) of the question, we know that the rightmost portion is equal to 0.10. Therefore:

$$\frac{2ab}{(a-\bar{\phi}+\gamma)} = 0.10$$

$$\frac{2\times 0.1\times 0.11}{(0.1-\bar{\phi}+\gamma)} = 0.10$$

$$0.22 = 0.1-\bar{\phi}+\gamma$$

$$\gamma = \bar{\phi}+0.12$$

We can substitute this value into the formula for γ:

$$\gamma = \sqrt{(a-\bar{\phi})^2+2\sigma^2}$$

$$\bar{\phi}+0.12 = \sqrt{(0.1-\bar{\phi})^2+2(0.08)^2}$$

$$\bar{\phi}^2+0.24\bar{\phi}+0.0144 = (0.1-\bar{\phi})^2+2(0.08)^2$$

$$\bar{\phi}^2+0.24\bar{\phi}+0.0144 = (0.01-0.2\bar{\phi}+\bar{\phi}^2)+0.0128$$

$$0.44\bar{\phi} = 0.0084$$

$$\bar{\phi} = 0.01909$$

The value of c is:

$$c = \frac{\bar{\phi}}{\sigma} = \frac{0.01909}{0.08} = 0.2386$$

Solution 61

D Chapter 15, Risk-Neutral Process

In the Black-Scholes framework:

$$\frac{dS(t)}{S(t)} = (\alpha-\delta)dt+\sigma dZ(t)$$

We can use statements (ii) and (iii) in the question to find α and σ:

$$\alpha-\delta = 0.05 \quad \Rightarrow \quad \alpha-0.01 = 0.05 \quad \Rightarrow \quad \alpha = 0.06$$

$$\sigma = 0.25$$

When the stock price process is a geometric Brownian motion, the relationship between $Z(T)$ and $\tilde{Z}(T)$ is described by:

$$\tilde{Z}(T) = Z(T)+\frac{\alpha-r}{\sigma}T$$

Under the risk-neutral probability measure $\tilde{Z}(T)$ is a pure Brownian motion, so:

$$E^*\left[\tilde{Z}(T)\right] = E^*\left[Z(T) + \frac{\alpha - r}{\sigma}T\right]$$

$$0 = E^*\left[Z(T)\right] + \frac{\alpha - r}{\sigma}T$$

$$E^*\left[Z(T)\right] = \frac{r - \alpha}{\sigma}T$$

Since the expectation of $Z(0.5)$ is -0.03, we have:

$$E^*\left[Z(0.5)\right] = \frac{r - \alpha}{\sigma} \times 0.5$$

$$-0.03 = \frac{r - 0.06}{0.25} \times 0.5$$

$$r = 0.045$$

Solution 62

A Chapter 15, Risk-Neutral Process

The risk-neutral price process is:

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t)$$

We observe that:

$$\mu = r - \delta \quad \text{and} \quad \sigma = 0.4$$

The formula for the forward price of S^2 is:

$$F_{t,T}\left[S(T)^{a}\right] = \left[S(t)\right]^{a} e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right](T-t)} = \left[S(t)\right]^2 e^{\left[2\mu+0.5\times2(2-1)0.40^2\right](T-t)}$$

$$= \left[S(t)\right]^2 e^{\left[2\mu+0.16\right](T-t)}$$

Setting this expression equal to the forward price provided in statement (ii), we have:

$$\left[S(t)\right]^2 e^{\left[2\mu+0.16\right](T-t)} = \left[S(t)\right]^2 e^{0.18(T-t)}$$

$$2\mu + 0.16 = 0.18$$

$$\mu = 0.01$$

Solution 63

A Chapter 15, Quadratic Variation

The quadratic variation is the sum of the squared increments of the process.

(i) W is not stochastic, and its differential is found below:

$$W(t) = t^2$$
$$dW = 2tdt$$

Making use of the multiplication rule, $dt \times dt = 0$, we obtain the quadratic variation:

$$V_{2.4}^2(W) = \int_0^{2.4} [dW]^2 = \int_0^{2.4} [2tdt]^2 = \int_0^{2.4} 4t^2(dt \times dt)] = 0$$

(ii) The increments to X are 0 across the interval [0,1). But when $t = 1$, X jumps from 0 to 1. The subsequent increments are again 0 until $t = 2$, at which point X jumps from 1 to 2. The subsequent increments, up to $t = 2.4$, are 0. An infinite number of zeros is summed, and their sum is zero. The two non-zero increments (of 1 each) sum to 2:

$$V_{2.4}^2(X) = \int_0^{2.4} [dX]^2 = 0^2 + \cdots + 0^2 + 1^2 + 0^2 + \cdots + 0^2 + 1^2 + 0^2 + \cdots + 0^2 = 2$$

(iii) We can determine dY by finding the partial differential of each of the terms of Y:

$$Y = 2t + 0.9Z$$
$$dY = 2dt + 0.9dZ$$

Making use of the multiplication rules, we have:

$$(dY)^2 = (2dt + 0.9dZ)^2 = 0.81(dZ)^2 = 0.81dt$$

The quadratic variation is:

$$V_{2.4}^2(Y) = \int_0^{2.4} [dY]^2 = \int_0^{2.4} 0.81dt = 0.81 \times (2.4 - 0) = 1.944$$

Since $0 < 1.944 < 2$, we have:

$$V_{2.4}^2(W) < V_{2.4}^2(Y) < V_{2.4}^2(X)$$

Solution 64

C Chapters 15, Prediction Intervals

$Y(t)$ follows geometric Brownian motion with the following parameters:

$$\alpha_Y - \delta_Y = 1.2 \qquad \bar{\sigma}_Y = -0.50 \qquad \sigma_Y = |\bar{\sigma}_Y| = |-0.50| = 0.50$$

The upper *z*-value for the 90% prediction interval is found below:

$$P\left(z > z^U\right) = \frac{p}{2} \;\Rightarrow\; P\left(z > z^U\right) = \frac{1-0.90}{2} \;\Rightarrow\; P\left(z < z^U\right) = 0.95 \;\Rightarrow\; z^U = 1.64485$$

The upper limit of the prediction interval for $Y(2)$ is:

$$Y(0)e^{(\alpha_Y - \delta_Y - 0.5\sigma_Y^2)T + \sigma_Y z^U \sqrt{T}} = 64e^{(1.2-0.5\times 0.50^2)(2)+0.50\times 1.64485\times\sqrt{2}} = 1{,}758.0626$$

The corresponding upper limit of the prediction interval for $S(2)$ is the square root of the upper limit for $Y(2)$:

$$U = \sqrt{1{,}758.0626} = 41.9293$$

Solution 65

C Chapter 8, Elasticity and Risk Premium

Since the Black-Scholes framework applies, we have:

$$\ln\left(\frac{S_T}{S_t}\right) \sim N\left[(\alpha - \delta - 0.5\sigma^2)(T-t),\ \sigma^2(T-t)\right]$$

Therefore, from (iii), we have:

$$\alpha - \delta - 0.5\sigma^2 = 0.10$$

Under the risk-neutral probability measure, the expected return is *r*, and so from (iv), we have:

$$(r - \delta - 0.5\sigma^2)(5-3) = 0.06$$
$$r - \delta - 0.5\sigma^2 = 0.03$$

Subtracting the expression above from the one obtained from (iii), we have the risk premium for the stock:

$$\alpha - r = 0.10 - 0.03 = 0.07$$

In (vii), we are not told if the put option being hedged has been purchased or sold. Therefore, we know that one of the following is true:

$$S\Delta = 20 \quad \text{or} \quad -S\Delta = 20$$

Since the Δ of a put is negative, the second expression above must be true. Therefore, we have:

$$S\Delta = -20$$

We can now find the elasticity of the put option:

$$\Omega = \frac{S\Delta}{V} = \frac{-20}{10} = -2$$

The formula for the risk premium of the option allows us to solve for the absolute value of its expected return:

$$\gamma - r = \Omega \times (\alpha - r)$$
$$\gamma - 0.04 = -2 \times (0.07)$$
$$\gamma = -0.10$$
$$|\gamma| = 0.10$$

Solution 66

A Chapter 15, Sharpe Ratio

From (i), we have:

$$\delta_X = 0.02 \qquad \delta_Y = 0.01$$

From (ii), we have:

$$\alpha_X - \delta_X = 0.06 \qquad \Rightarrow \qquad \alpha_X = 0.06 + \delta_X = 0.06 + 0.02 = 0.08$$
$$\sigma_X = 0.20$$

From (iii), we have:

$$\mu = \alpha_Y - \delta_Y - 0.5\sigma_Y^2$$
$$\sigma_Y = -0.10$$

Stocks X and Y must have the same Sharpe ratio:

$$\frac{\alpha_X - r}{\sigma_X} = \frac{\alpha_Y - r}{\sigma_Y}$$
$$\frac{0.08 - 0.04}{0.20} = \frac{\alpha_Y - 0.04}{-0.10}$$
$$\alpha_Y = 0.02$$

We can now find μ:

$$\mu = \alpha_Y - \delta_Y - 0.5\sigma_Y^2 = 0.02 - 0.01 - 0.5 \times (-0.10)^2 = 0.005$$

Solution 67

A Chapter 15, Sharpe Ratio

Stock 1 does not pay dividends, so its parameters are:

$$\alpha_1 = \mu \quad \text{and} \quad \sigma_1 = 20\mu$$

Stock 2 has a dividend yield of 0.01, so its parameters are:

$$\alpha_2 - \delta_2 - 0.5\sigma_2^2 = 0.03 \qquad \sigma_2 = 0.20$$
$$\alpha_2 - 0.01 - 0.5 \times (0.20)^2 = 0.03$$
$$\alpha_2 = 0.06$$

Since the two stocks have the same source of uncertainty, they must have the same Sharpe ratio:

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$$
$$\frac{\mu - 0.04}{20\mu} = \frac{0.06 - 0.04}{0.20}$$
$$\mu - 0.04 = 2\mu$$
$$\mu = -0.04$$

Solution 68

D Chapter 14, Ornstein-Uhlenbeck Process

The key to this question is recognizing that the process is an Ornstein-Uhlenbeck process:

$$dX(t) = \lambda \times [\alpha - X(t)]dt + \sigma dZ(t) \quad \Leftrightarrow \quad X(t) = X(0)e^{-\lambda t} + \alpha\left(1 - e^{-\lambda t}\right) + \sigma\int_0^t e^{-\lambda(t-s)}dZ(s)$$

The parameters of the process are:

$$\lambda = 3 \qquad \alpha = 0 \qquad \sigma = 2$$

Therefore the solution is:

$$X(t) = X(0)e^{-3t} + 0 \times \left(1 - e^{-3t}\right) + 2\int_0^t e^{-3(t-s)}dZ(s) = e^{-3t}\left[X(0) + 2\int_0^t e^{3s}dZ(s)\right]$$

In this form, we can observe that:

$$A = 3 \qquad B = X(0) \qquad C = 2 \qquad D = 3$$

The sum of A, C, and D is:

$$A + C + D = 3 + 2 + 3 = 8$$

Solution 69

C Chapter 4, Theta in the Binomial Model

The formula for theta is:

$$\theta(S,0)=\frac{V_{ud}-V-(Sud-S)\Delta(S,0)-\frac{(Sud-S)^2}{2}\Gamma(S,0)}{2h}$$

Since we have $S=Sud=120$, this simplifies to:

$$\theta(S,0)=\frac{V_{ud}-V}{2h}$$

The risk-neutral probability of an upward move is:

$$p^*=\frac{120e^{0.10}-96}{150-96}=0.67816$$

Since the put option is an American option, we solve for its value from right to left. The completed tree is shown below.

			0.0000
		0.0000	
	2.0353		0.0000
8.2381		6.9892	
	24.0000		24.0000
		43.2000	
			58.5600

The bolded entries in the table above indicate where early exercise is optimal.

The value of theta can now be found:

$$\theta(S,0)=\frac{V_{ud}-V}{2h}=\frac{6.9892-8.2381}{2\times 1}=-0.6245$$

Solution 70

C Chapter 14, Portfolio Returns

The instantaneous percentage increase of the fund is the weighted average of the return on the stock (including its dividend yield) and the return on the risk-free asset:

$$\begin{aligned}\frac{dW(t)}{W(t)}&=0.80\left[\frac{dS(t)}{S(t)}+\delta dt\right]+0.20rdt\\&=0.80\left[0.10dt+0.20dZ(t)+0.02dt\right]+0.20(0.05)dt\\&=0.106dt+0.16dZ(t)\end{aligned}$$

Since the fund's value follows geometric Brownian motion, we can convert the partial differential equation into an expression for the value of the fund:

$$\frac{dW(t)}{W(t)} = (\alpha_W - \delta_W)dt + \sigma_W dZ(t) \quad \Leftrightarrow \quad W(t) = W(0)e^{(\alpha_W - \delta_W - 0.5\sigma_W^2)t + \sigma_W Z(t)}$$

$$\frac{dW(t)}{W(t)} = 0.106dt + 0.16dZ(t) \quad \Leftrightarrow \quad W(t) = W(0)e^{(0.106 - 0.5\times 0.16^2)t + 0.16Z(t)}$$

Simplifying the lower right-hand expression above, we have:

$$W(t) = W(0)e^{0.0932t + 0.16Z(t)}$$

Solution 71

C Chapter 15, Risk-Neutral Pricing

The risk-neutral price process is:

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t)$$

From statement (iii), we can obtain the risk-free rate and the volatility parameter:

$$r - \delta = 0.08 \quad \Rightarrow \quad r = 0.08 + 0.04 = 0.12$$

$$\sigma = 0.40$$

We can use the following equivalency:

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t) \quad \Leftrightarrow \quad S(t) = S(0)e^{(r - \delta - 0.5\sigma^2)t + \sigma\tilde{Z}(t)}$$

$$\frac{dS(t)}{S(t)} = 0.08dt + 0.40d\tilde{Z}(t) \quad \Leftrightarrow \quad S(t) = S(0)e^{(0.08 - 0.5\times 0.4^2)t + 0.4\tilde{Z}(t)} = e^{0.4\tilde{Z}(t)}$$

Let's find the expected value of the derivative security under the risk-neutral probability measure:

$$E^*\left[1 + S(1)\times\left\{\ln\left[S(1)\right]\right\}^2\right] = E^*\left[1 + e^{0.4\tilde{Z}(1)}\times\left\{\ln\left[e^{0.4\tilde{Z}(1)}\right]\right\}^2\right]$$

$$= E^*\left[1 + e^{0.4\tilde{Z}(1)}\times\left\{0.4\tilde{Z}(1)\right\}^2\right] = 1 + E^*\left[0.16e^{0.4\tilde{Z}(1)}\times\left\{\tilde{Z}(1)\right\}^2\right]$$

$$= 1 + 0.16E^*\left[\left\{\tilde{Z}(1)\right\}^2\times e^{0.4\tilde{Z}(1)}\right]$$

Since $\tilde{Z}(1)$ is a standard normal random variable, we can use statement (v) to obtain:

$$1 + 0.16E^*\left[\left\{\tilde{Z}(1)\right\}^2\times e^{0.4\tilde{Z}(t)}\right] = 1 + 0.16\left[\left(1 + 0.4^2\right)e^{0.4^2/2}\right] = 1.20106$$

The time-0 price is obtained by discounting the expected value at the risk-free rate:

$$1.20106 \times e^{-r} = 1.20106 \times e^{-0.12} = 1.06524$$

Solution 72

A Chapter 15, Gap Put-Call Parity and S^a

The usual put-call parity expression for gap calls and gap puts is:

$$GapCall + K_1 e^{-rT} = Se^{-\delta T} + GapPut$$

The first term on the right side of the equation is the prepaid forward price of the underlying asset, so we can also write the put-call parity expression as:

$$GapCall + K_1 e^{-rT} = F_{0,T}^P(S) + GapPut$$

In this case, the underlying asset is S^2, so we replace S with S^2:

$$GapCall + K_1 e^{-rT} = F_{0,T}^P(S(T)^2) + GapPut$$

$$GapCall + K_1 e^{-rT} = e^{-rT}\left[S(0)\right]^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right]T} + GapPut$$

$$5.542 + 95e^{-0.07(0.5)} = e^{-0.07(0.5)} \times 10^2 e^{[2(0.07-\delta)+0.5\times 2(2-1)0.10^2]0.5} - 4.745$$

$$97.2745 = e^{-0.035} \times 100e^{[0.15-2\delta]0.5} - 4.745$$

$$105.6534 = 100e^{(0.075-\delta)}$$

$$\delta = 0.02$$

Solution 73

B Chapter 15, Claim on S^a

The usual relationship is:

$$\frac{dS(t)}{S(t)} = (\alpha-\delta)dt + \bar{\sigma}dZ(t) \quad \Rightarrow \quad \frac{d(S^a)}{S^a} = \left[a(\alpha-\delta) + 0.5a(a-1)\sigma^2\right]dt + a\bar{\sigma}dZ(t)$$

In this case, $\alpha - \delta = 0.30$ and $\bar{\sigma} = -\sigma$, so we have:

$$\frac{dS(t)}{S(t)} = 0.30dt - \sigma dZ(t) \quad \Rightarrow \quad \frac{d(S^a)}{S^a} = \left[0.30a + 0.5a(a-1)\sigma^2\right]dt - a\sigma dZ(t)$$

From statement (ii), we have the following 2 equations and 2 unknowns:

$$\begin{cases} 0.30a + 0.5a(a-1)\sigma^2 = -0.66 \\ -a\sigma = 0.6 \end{cases}$$

The second equation can be written as $a = \frac{-0.6}{\sigma}$, and we can substitute this value of a into the first equation:

$$0.30a + 0.5a(a-1)\sigma^2 = -0.66$$
$$0.30\left(\frac{-0.6}{\sigma}\right) + 0.5\left(\frac{-0.6}{\sigma}\right)\left[\frac{-0.6}{\sigma} - 1\right]\sigma^2 = -0.66$$
$$\frac{-0.18}{\sigma} + 0.18 + 0.3\sigma = -0.66$$
$$\frac{-0.18}{\sigma} + 0.84 + 0.3\sigma = 0$$
$$0.3\sigma^2 + 0.84\sigma - 0.18 = 0$$
$$\sigma^2 + 2.8\sigma - 0.6 = 0$$
$$10\sigma^2 + 28\sigma - 6 = 0$$
$$(10\sigma - 2)(\sigma + 3) = 0$$
$$10\sigma - 2 = 0 \text{ or } \sigma + 3 = 0$$
$$\sigma = 0.2 \text{ or } \sigma = -3$$

Since we are told that σ is a positive constant, we conclude that $\sigma = 0.2$.

Solution 74

E Chapter 5, Probability that Stock Price is Greater Than *K*

Let *F* be the fund amount. The time-4 cash flow to the company that sold the put option is:

$$F(1.02)^4 - Max\left[0, 40 - S_4\right]$$

For this cash flow to be negative, the put option must be in-the-money, meaning that S_4 must be less than 40. Furthermore, the put option's nonzero payoff must be greater than $F(1.02)^4$. That is, the following must be negative:

$$F(1.02)^4 - (40 - S_4)$$

The company wants the probability of a nonnegative cash flow to be 99%, so:

$$P\left[F(1.02)^4 - (40 - S_4) > 0\right] = 0.99$$
$$P\left[F(1.02)^4 - 40 + S_4 > 0\right] = 0.99$$
$$P\left[S_4 > 40 - F(1.02)^4\right] = 0.99$$

We make use of the following formula for the probability of the price of a stock exceeding a threshold:

$$\text{Prob}\left(S_T > K\right) = N(\hat{d}_2) \quad \text{with} \quad K = 40 - F(1.02)^4$$

We can use the normal distribution table to find $\hat{d}_2$:

$$N(\hat{d}_2) = 0.99 \quad \Rightarrow \quad \hat{d}_2 = 2.32635$$

We have:

$$\hat{d}_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$2.32635 = \frac{\ln\left(\frac{40}{40 - F(1.02)^4}\right) + (0.10 - 0 - 0.5 \times 0.30^2)(4-0)}{0.30\sqrt{4-0}}$$

$$1.17581 = \ln\left(\frac{40}{40 - F(1.02)^4}\right)$$

$$F = 25.55102$$

Solution 75

D Chapter 12, Variance of Control Variate Estimate

The formula from the ActuarialBrew.com Study Manual that has X as the control variate and Y^* as the control variate estimate is:

$$Var\left[Y^*\right] = Var\left[\bar{Y}\right]\left(1 - \rho^2_{\bar{X},\bar{Y}}\right)$$

In this question, however, the control variate is denoted by Y and the control variate estimate is denoted by X^*, so we switch the roles of X and Y:

$$Var\left[X^*\right] = Var\left[\bar{X}\right]\left(1 - \rho^2_{\bar{X},\bar{Y}}\right) = 5^2\left(1 - 0.8^2\right) = 9$$

Solution 76

D Chapter 18, BDT Interest Rate Cap

The ratio of each interest rate to the rate above it in the tree is constant for any column within the tree, so:

$$\frac{r_{ud}}{r_{dd}} = \frac{r_{uu}}{r_{ud}} \quad \Rightarrow \quad r_{ud} = \sqrt{r_{uu} \times r_{dd}} = \sqrt{0.172 \times 0.106} = 0.1350$$

The completed tree of interest rates is:

		0.1720
	0.1260	
0.0900		0.1350
	0.0930	
		0.1060

Although the cap payments are made at the end of each year, we will value them at the beginning of each year using the following formula:

$$T\text{-year caplet payoff at time } (T-1) = \frac{Max\left[0, R_{T-1} - K_R\right]}{1 + R_{T-1}} \times Notional$$

A 3-year cap consists of a 1-year caplet, a 2-year caplet, and a 3-year caplet.

The payoff for the 1-year caplet is zero since 9% is less than 11.5%. Therefore, the value of the 1-year caplet is zero.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 12.6%:

$$R_1 = 0.126 \quad \Rightarrow \quad \frac{0.126 - 0.115}{1.126} \times 10{,}000 = 97.6909$$

The payoff for the 3-year caplet is positive when the short-term rate is 17.2% or 13.5%:

$$R_2 = 0.172 \quad \Rightarrow \quad \frac{0.172 - 0.115}{1.172} \times 10{,}000 = 486.3481$$

$$R_2 = 0.135 \quad \Rightarrow \quad \frac{0.135 - 0.115}{1.135} \times 10{,}000 = 176.4358$$

The possible payoffs from the cap are illustrated in the tree below:

		486.3481
	97.6909	
0.0000		176.4358
	0.0000	
		0.0000

The value of the 2-year caplet is:

$$\text{Value of 2-year caplet} = 0.5 \times \frac{97.6909}{1.09} = 44.8124$$

The value of the 3-year caplet is:

$$\text{Value of the 3-year caplet}$$
$$= (0.5)^2 \times \frac{486.3481}{1.09 \times 1.126} + (0.5)^2 \times \frac{176.4358}{1.09 \times 1.126} + (0.5)^2 \times \frac{176.4358}{1.09 \times 1.093} = 172.0279$$

The value of the 3-year cap is:

$$(\text{1-year caplet}) + (\text{2-year caplet}) + (\text{3-year caplet}) = 0.000 + 44.8214 + 172.0279$$
$$= 216.8402$$