SOA Exam MFE
Solutions: Spring 2009

Solution 1

E Chapter 4, American Call Option in Binomial Model

The values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.05-0.05)(1)+0.30\sqrt{1}} = 1.34986$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.05-0.05)(1)-0.30\sqrt{1}} = 0.74082$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{1}{e^{\sigma\sqrt{h}}+1} = \frac{1}{e^{0.30\sqrt{1}}+1} = 0.42556$$

The stock price tree and its corresponding tree of option prices are:

Stock			American Call		
		182.2119			82.2119
	134.9859			**34.9859**	
100.0000		100.0000	14.1624		0.0000
	74.0818			0.0000	
		54.8812			0.0000

The tree of prices for the call option is found by working from right to left. The rightmost column is found as follows:

$$Max\left[0,\ 182.2119-100\right] = 82.2119$$
$$Max\left[0,\ 100.0000-100\right] = 0.0000$$
$$Max\left[0,\ 54.8812-100\right] = 0.0000$$

At the end of 1 year, the value of the 2-year cash flows can be found using the risk-neutral probabilities:

$$e^{-0.05(1)}\left[(0.42556)(82.2119)+(1-0.42556)(0.0000)\right] = 33.2796$$
$$e^{-0.05(1)}\left[(0.42556)(0.0000)+(1-0.42556)(0.0000)\right] = 0.0000$$

There is only one opportunity to call the option early, and this occurs after 1 year if the stock price has risen. Early exercise is optimal since:

$$134.9859-100 = 34.9859 \quad \text{and} \quad 34.9859 > 33.2796$$

The value of 34.9859 is bolded in the tree above to indicate that early exercise is optimal at that point.

The current price of the option is:

$$e^{-0.05(1)}\left[(0.42556)(34.9859)+(1-0.42556)(0.0000)\right]=14.1624$$

Solution 2

B Chapter 10, Path-Dependent Options

The arithmetic average price is:

$$\bar{S}=\frac{105+120+115+110+115+110+100+90+105+125+110+115}{12}=110$$

The payoff of Option (i) is:

$$\text{Max}[\bar{S}-K,0]=\text{Max}[110-100,0]=10$$

The barrier of 125 for Option (ii) is reached on October 31, 2008, so the option is knocked out. Therefore, the payoff of Option (ii) is 0.

The barrier of 120 for Option (iii) is reached on February 29, 2008, so the option is knocked in. Therefore, the payoff of Option (iii) is:

$$\text{Max}[S-K,0]=\text{Max}[115-110,0]=5$$

The highest payoff is 10 from Option (i). The lowest payoff is 0 from Option (ii). The difference is:

$$10-0=10$$

Solution 3

B Chapter 3, Arbitrage in the Binomial Model

If the stock price moves up, then the option pays Max[0, 55 – 50] = \$5. If the stock price moves down, then the option pays Max[0, 40 – 50] = \$0:

	55		5
50		V	
	40		0

Let's use replication method to find the value of the option:

$$\Delta=e^{-\delta h}\frac{V_u-V_d}{S(u-d)}=e^{-0.10(1)}\frac{5-0}{55-40}=0.30161$$

$$B=e^{-rh}\frac{uV_d-dV_u}{u-d}=e^{-rh}\frac{SuV_d-SdV_u}{Su-Sd}=e^{-0.05(1)}\frac{55(0)-40(5)}{55-40}=-12.68306$$

If no arbitrage is available, then the price of the option is:

$$V=S_0\Delta+B=50(0.30161)-12.68306=2.39756$$

The price of \$1.90 is less than \$2.39756, so the option's price is too low. Arbitrage is obtained by buying the option (buy low) and also replicating the sale of the option. The sale of the option is replicated by selling Δ shares and borrowing B. In this case, B is negative, so borrowing B is equivalent to lending 12.68306.

To summarize, arbitrage is obtained by:

- buying the option for \$1.90
- selling 0.30161 shares of stock
- lending \$12.68306 at the risk-free rate.

These actions are described in Choice B.

Solution 4

D Chapter 11, Asset-or-Nothing Put

Since $K = 1{,}000(1 - 0.4) = 600$, the value of d_1 is:

$$d_1 = \frac{\ln\left(S_t / K\right) + [r - \delta + 0.5\sigma^2](T-t)}{\sigma\sqrt{T-t}}$$
$$= \frac{\ln\left(1{,}000/600\right) + [0.025 - 0.02 + 0.5(0.2)^2](1-0)}{0.2\sqrt{1-0}}$$
$$= 2.68$$

The value of $N(-d_1)$ is:

$$N(-d_1) = N(-2.68) = 1 - N(2.68) = 1 - 0.9963 = 0.0037$$

The price of one of the asset-or-nothing put options is:

$$AssetPut(K) = S_t e^{-\delta(T-t)} N(-d_1) = 1{,}000 e^{-0.02(1-0)} \times 0.0037 = 3.626735$$

Therefore, the price of one million of the options is:

$$1{,}000{,}000 \times 3.626735 = 3{,}626{,}735$$

Solution 5

E Chapter 18, Binomial Interest Rate Model

A 1-year bond in 2 years is a 3-year bond now. The tree of prices for the 3-year bond is:

		0.8353	1.0000
	0.7323		
		0.8869	1.0000
0.6743			
		0.8869	1.0000
	0.8256		
		0.9418	1.0000

There is no need to calculate the first two columns of the tree to answer this question, but they are shown above for illustrative purposes.

The third column is easily obtained by discounting a 1-year bond at the possible interest rates at time 2:

$$e^{-0.18} = 0.8353$$

$$e^{-0.12} = 0.8869$$

$$e^{-0.06} = 0.9418$$

The put option pays off only if the time-2 price falls below \$0.90. This occurs at 3 nodes, and the payoffs are:

up-up: $V_2 = 0.90 - 0.8353 = 0.0647$

up-down: $V_2 = 0.90 - 0.8869 = 0.0131$

down-up: $V_2 = 0.90 - 0.8869 = 0.0131$

The price of the put option is:

$$V_0 = E^*\left[V_2 \times e^{-\sum_{i=0}^{2-1} r_i}\right] = \frac{(0.7)^2 0.0647}{e^{(0.12+0.15)}} + \frac{(0.7)(0.3)0.0131}{e^{(0.12+0.15)}} + \frac{(0.3)(0.7)0.0131}{e^{(0.12+0.09)}}$$

$$= 0.0285$$

Solution 6

D Chapter 15, Itô's Lemma

We have:

$$Y(t) = [X(t)]^{-1}$$

The partial derivatives are:

$$Y_X = -X^{-2} \qquad Y_{XX} = 2X^{-3} \qquad Y_t = 0$$

From Itô's Lemma and the multiplication rules:

$$\begin{aligned}
dY(t) &= Y_X dX + \frac{1}{2} Y_{XX} (dX)^2 + Y_t dt \\
&= -X^{-2}\left[2(4-X)dt + 8dZ\right] + \frac{1}{2} \times 2X^{-3} \times 64dt + 0 \\
&= -X^{-2}\left[8dt - 2Xdt + 8dZ\right] + 64X^{-3}dt \\
&= -8X^{-2}dt + 2X^{-1}dt - 8X^{-2}dZ + 64X^{-3}dt \\
&= \left[2X^{-1} - 8X^{-2} + 64X^{-3}\right]dt - 8X^{-2}dZ \\
&= \left[2Y - 8Y^2 + 64Y^3\right]dt - 8Y^2 dZ
\end{aligned}$$

From the expression above, we can determine $\alpha(Y)$:

$$\alpha(Y) = 2Y - 8Y^2 + 64Y^3$$
$$\alpha(0.5) = 2(0.5) - 8(0.5)^2 + 64(0.5)^3 = 1 - 2 + 8 = 7$$

Solution 7

D Chapter 4, State Prices

This question is a bit unfair, because it incorrectly states that "all other assumptions were correct." Based on the solution provided, the original realistic expected return on the stock, the original state prices, and the original utility values were, in fact, not correct. The question should have stated that "the other assumptions listed above were correct, and the assumption used for the risk-free rate of return was correct." Students were (perhaps unreasonably) expected to know that the unspecified risk-free rate of return was correct, while other unspecified assumptions were not correct.

Since the strike price is 10, the call option pays Max[0, 12 – 10] = \$2 if the up state occurs. If the down state occurs, the call option pays Max[0, 8 – 10] = \$0. We can use the stock price and the option price to solve for the original state prices:

$$\left.\begin{array}{r} 10 = 12Q_u + 8Q_d \\ 1.13 = 2Q_u \end{array}\right\} \Rightarrow \begin{array}{l} Q_u = 0.5650 \\ Q_d = 0.4025 \end{array}$$

The discount factor for one year is:

$$e^{-r} = Q_u + Q_d = 0.5650 + 0.4025 = 0.9675$$

After the correction is made, the new state prices still result in the same risk-free rate of return:

$$e^{-r} = \hat{Q}_u + \hat{Q}_d = 0.9675$$

We can use the stock price with the corrected value after a down-move and the expression for the risk-free discount factor to obtain another system of 2 equations and 2 unknowns:

$$\left.\begin{array}{r} 10 = 12\hat{Q}_u + 6\hat{Q}_d \\ 0.9675 = \hat{Q}_u + \hat{Q}_d \end{array}\right\} \Rightarrow \hat{Q}_u = 0.69917$$

The new state price can now be used to find the value of the option:

$$2\hat{Q}_u = 2 \times 0.69917 = 1.39833$$

Solution 8

B Chapter 16, Black-Scholes Equation

The expressions for the value of the derivative and the partial derivatives are:

$$V = e^{rt} \ln S$$
$$V_S = e^{rt} S^{-1}$$
$$V_{SS} = -e^{rt} S^{-2}$$
$$V_t = re^{rt} \ln S$$

From the Black-Scholes equation, we have:

$$0.5\sigma^2 S^2 V_{SS} + (r-\delta)SV_S + V_t = rV$$
$$0.5(0.3)^2 S^2(-e^{rt}S^{-2}) + (0.055-\delta)S(e^{rt}S^{-1}) + re^{rt}\ln S = r\left(e^{rt}\ln S\right)$$
$$0.5(0.3)^2 S^2(-e^{rt}S^{-2}) + (0.055-\delta)S(e^{rt}S^{-1}) = 0$$
$$-0.5(0.3)^2 e^{rt} + (0.055-\delta)e^{rt} = 0$$
$$-0.5(0.3)^2 + (0.055-\delta) = 0$$
$$0.055-\delta = 0.5(0.09)$$
$$\delta = 0.01$$

Solution 9

E Chapter 1, Currency Options

In Statement (ii), we given the price of a dollar-denominated put option that allows its owner the right to:

Give up 1 yen

Get \$0.008

We can use put-call parity to find the value of a dollar-denominated call option that allows its owner to:

Get 1 yen

Give up \$0.008

For both of the options described above, the strike price is \$0.008 and the strike asset is 1 yen:

$$C + Ke^{-rT} = x_0 e^{-r_f T} + P$$
$$C + 0.008e^{-0.03\times 4} = 0.011e^{-0.015\times 4} + 0.0005$$
$$C = 0.003764$$

The yen-denominated put option allows its owner the right to:

Get 125 yen

Give up \$1

Observe that the yen-denominated put option has the same payoff as 125 of the dollar-denominated call options:

Get 1 yen $\times 125$ = Get 125 yen

Give up \$0.008 $\times 125$ = Give up \$1

Since we have the price of the dollar-denominated call option, we can easily find the price of 125 of them by multiplying by 125. Then, we convert the value to yen by dividing by the exchange rate:

$$\frac{0.003764 \times 125}{0.011} = 42.7733$$

Solution 10

C Chapter 14, Synthetic Risk-free Asset from 2 Risky Assets

Let X be the percentage invested in Asset 1. Over the next instant, the percentage return on the investment is:

$$X\left(0.08dt + 0.2dZ\right) + (1 - X)\left(0.0925dt - 0.25dZ\right)$$

We can determine X so that the dZ terms cancel:

$$0.2XdZ - 0.25(1 - X)dZ = 0$$
$$0.2X - 0.25 + 0.25X = 0$$
$$0.45X = 0.25$$
$$X = 0.55556$$

The portion of the \$1,000 to be invested in Asset 1 is:

$$1{,}000 \times X = 1{,}000 \times 0.55556 = 555.56$$

Solution 11

C Chapter 15, Valuing a Claim on S^a

The expected value can be used to solve for a:

$$E\left[S(T)^a\right] = S(t)^a e^{[a(\alpha-\delta)+0.5a(a-1)\sigma^2](T-t)}$$

$$1.4 = 0.5^a e^{[a(0.05)+0.5a(a-1)0.2^2](1-0)}$$

$$\ln(1.4) = a\ln(0.5) + a(0.05) + 0.02a(a-1)$$

$$\ln(1.4) = a\ln(0.5) + a(0.05) + 0.02a^2 - 0.02a$$

$$0 = 0.02a^2 + \left(\ln(0.5) + 0.03\right)a - \ln(1.4)$$

$$a = \frac{-\left(\ln(0.5)+0.03\right) \pm \sqrt{\left(\ln(0.5)+0.03\right)^2 - 4(0.02)\left(-\ln 1.4\right)}}{2(0.02)}$$

$$a = -0.49985 \quad \text{or} \quad a = 33.65721$$

Since a is negative, its value must be –0.49985.

We can now find the time-0 price of the contingent claim:

$$F_{t,T}^P\left[S(T)^a\right] = e^{-r(T-t)}\left[S(t)\right]^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right](T-t)}$$

$$F_{0,1}^P\left[S(1)^a\right] = e^{-0.03}\left[0.5\right]^{-0.49985} e^{\left[-0.49985(0.03-0)+0.5(-0.49985)(-0.49985-1)0.2^2\right]}$$

$$= 1.37227$$

Alternatively, we can rearrange the formula for the prepaid forward price to allow us to make use of the expected value provided in the question:

$$F_{0,T}^P\left[\left(S(T)\right)^a\right] = e^{-rT}\left[S(0)\right]^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right]T}$$

$$= e^{-rT}\left[S(0)\right]^a e^{\left[a(\alpha-\delta)+0.5a(a-1)\sigma^2\right]T} e^{\left[a(r-\alpha)\right]T}$$

$$= e^{-rT} E\left[\left(S(T)\right)^a\right] e^{\left[a(r-\alpha)\right]T} = e^{-0.03} \times 1.4 e^{\left[-0.49985(0.03-0.05)\right]}$$

$$= 1.37227$$

Solution 12

E Chapter 2, Bounds on Option Prices

From Propositions 1 and 2, we see that Statement I is true:

Proposition 1: $C(K_1) \geq C(K_2)$ for $K_1 < K_2$

$$\Rightarrow \quad 0 \leq C(50) - C(55)$$

Proposition 2: $C_{Eur}(K_1) - C_{Eur}(K_2) \leq (K_2 - K_1)e^{-rT}$ for $K_1 < K_2$

$$\Rightarrow \quad C_{Eur}(50) - C_{Eur}(55) \leq (55-50)e^{-rT}$$

We make use of put-call parity for Statements II and III:

$$C(K) + K^{-rT} = S + P(K)$$

When the strike price for a put is decreased, its price goes down, so the first inequality in Statement II is false:

$$P(50) - C(50) + S = 50e^{-rT} \quad \Rightarrow \quad P(45) - C(50) + S \leq 50e^{-rT}$$

Furthermore, from the expression above, we can see that the second inquality in Statement II is true (but Statement II is still false, because the first inequality is false).

When the strike price for a call is increased, its price goes down, so the first inequality in Statement III is true:

$$P(45) - C(45) + S = 45e^{-rT} \quad \Rightarrow \quad P(45) - C(50) + S \geq 45e^{-rT}$$

The second inequality in Statement III is also true, because when we considered Statement II above, we found that:

$$P(45) - C(50) + S \leq 50e^{-rT}$$

Solution 13

A Chapter 7, Black-Scholes Formula and Holding Period Profit

We begin by finding the value of the call option today. The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(85/75) + (0.05 - 0 + 0.5 \times 0.26^2) \times 4/12}{0.26\sqrt{4/12}}$$

$$= 1.02$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1.02 - 0.26\sqrt{4/12} = 0.87$$

We have:

$$N(d_1) = N(1.02) = 0.8461$$

$$N(d_2) = N(0.87) = 0.8078$$

The value of the European call option today is:

$$C_{Eur} = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

$$= 85e^{-0.00(4/12)} \times 0.8461 - 75e^{-0.05(4/12)} \times 0.8078 = 12.3349$$

The holding period profit is the current value of the position minus the cost of the position, including interest:

$$12.3349 - 8e^{0.05(8/12)} = 4.0637$$

Solution 14

E Chapter 18, Black-Derman-Toy Model

The ratio of the short rate to the rate below it is constant within a column, so:

$$\frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}} \quad \Rightarrow \quad \frac{0.80}{r_{ud}} = \frac{r_{ud}}{0.20} \quad \Rightarrow \quad r_{ud} = 0.40$$

The value of a 2-year bond is:

$$P(0,2) = 0.5\left[\left(\frac{1}{1+r_0}\right)\left(\frac{1}{1+r_u}\right) + \left(\frac{1}{1+r_0}\right)\left(\frac{1}{1+r_d}\right)\right]$$

$$= 0.5\frac{1}{1+r_0}\left[\frac{1}{1.60} + \frac{1}{1.30}\right] = \frac{0.69712}{1+r_0}$$

The value of a 3-year bond is:

$$P(0,3) = 0.25\left[\left(\frac{1}{1+r_0}\right)\left(\frac{1}{1+r_u}\right)\left(\frac{1}{1+r_{uu}}\right) + \left(\frac{1}{1+r_0}\right)\left(\frac{1}{1+r_u}\right)\left(\frac{1}{1+r_{ud}}\right)\right.$$
$$\left. + \left(\frac{1}{1+r_0}\right)\left(\frac{1}{1+r_d}\right)\left(\frac{1}{1+r_{ud}}\right) + \left(\frac{1}{1+r_0}\right)\left(\frac{1}{1+r_d}\right)\left(\frac{1}{1+r_{dd}}\right)\right]$$
$$= 0.25\frac{1}{1+r_0}\left[\left(\frac{1}{1.60}\right)\left(\frac{1}{1.80}\right) + \left(\frac{1}{1.60}\right)\left(\frac{1}{1.40}\right) + \left(\frac{1}{1.30}\right)\left(\frac{1}{1.40}\right) + \left(\frac{1}{1.30}\right)\left(\frac{1}{1.20}\right)\right]$$
$$= \frac{0.49603}{1+r_0}$$

The forward price is:

$$F_{0,2}\left[P(2,3)\right] = P_0(2,3) = \frac{P(0,3)}{P(0,2)} = \frac{\frac{0.49603}{1+r_0}}{\frac{0.69712}{1+r_0}} = 0.71155$$

Multiplying by 1,000, we have:

$$1{,}000 \times F_{0,2}\left[P(2,3)\right] = 1{,}000 \times 0.71155 = 711.55$$

Solution 15

C Chapter 19, Risk-Neutral Version of the Vasicek Model

The process for the short rate follows the Vasicek Model. The true process can be rewritten in the familiar form:

$$dr = 0.1(0.08 - r)dt + 0.05dZ \quad \Rightarrow \quad a = 0.1,\ b = 0.08,\ \sigma = 0.05$$

The risk-neutral process is:

$$dr = \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z} = \left[a(b-r) + \sigma\phi\right]dt + \sigma d\tilde{Z}$$

The portion in front of the *dt* term in the rightmost expression above indicates that that if we subtract the drift term of the realistic process from the drift term of the risk-neutral process, we are left with the volatility parameter times the Sharpe ratio:

$$\left[a(b-r)+\sigma\phi\right]-\left[a(b-r)\right]=\sigma\phi$$
$$\left[0.013-0.1r\right]-\left[0.008-0.1r\right]=\sigma\phi$$
$$0.005=0.05\phi$$
$$\phi=0.1$$

We can use the Sharpe ratio to obtain an expression for $\alpha(r,t,T)$:

$$\phi(r,t)=\frac{\alpha(r,t,T)-r}{q(r,t,T)}$$
$$\phi=\frac{\alpha(r,t,T)-r}{B(t,T)\sigma}$$
$$\alpha(r,t,T)=r+B(t,T)\sigma\phi$$
$$\alpha(0.04,2,5)=0.04+B(2,5)0.05(0.10)$$
$$\alpha(0.04,2,5)=0.04+0.005B(2,5)$$

The parameter $B(2,5)$ is:

$$B(t,T)=\frac{1-e^{-a(T-t)}}{a}$$
$$B(2,5)=\frac{1-e^{-0.1(5-2)}}{0.1}=2.59182$$

The expected return on the bond over the next instant is:

$$\alpha(0.04,2,5)=0.04+0.005B(2,5)$$
$$\alpha(0.04,2,5)=0.04+0.005\times 2.59182=0.05296$$

Solution 16

A Chapter 5, Stock Price Probabilities

The call will be exercised if the future stock price is greater than the strike price, and this probability can be found with the following formula:

$$\text{Prob}\left(S_T>K\right)=N(\hat{d}_2) \qquad \text{where: } \hat{d}_2=\frac{\ln\left(\frac{S_t}{K}\right)+(\alpha-\delta-0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

We begin by finding $\hat{d}_2$:

$$\hat{d}_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{100}{125}\right) + (0.10 - 0 - 0.5(0.30)^2)(0.75 - 0)}{0.30\sqrt{0.75-0}}$$

$$= -0.70$$

We can now find the probability that S_t is greater than \$125:

$$\text{Prob}\left(S_T > 125\right) = N(-0.70) = 1 - N(0.70) = 1 - 0.7580 = 0.2420$$

Solution 17

A Chapter 8, Put Option Delta

The value of delta can be used to determine d_1:

$$\Delta_{Put} = -e^{-\delta T} N(-d_1)$$

$$-0.4364 = -e^{-0\times 1} N(-d_1)$$

$$0.4364 = N(-d_1)$$

$$N(d_1) = 1 - 0.4364 = 0.5636$$

$$d_1 = 0.16$$

The formula for d_1 is used to find a quadratic equation in terms of σ:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$0.16 = \frac{\ln(S/S) + (0.012 - 0 + 0.5\sigma^2)\times 1}{\sigma\sqrt{1}}$$

$$0.16 = \frac{(0.012 + 0.5\sigma^2)}{\sigma}$$

$$0.16\sigma = 0.012 + 0.5\sigma^2$$

$$0.5\sigma^2 - 0.16\sigma + 0.012 = 0$$

$$\sigma^2 - 0.32\sigma + 0.024 = 0$$

The two solutions to the quadratic equation are found below:

$$\sigma = \frac{0.32 \pm \sqrt{(-0.32)^2 - 4(1)(0.024)}}{2}$$

$$\sigma = 0.12 \qquad \text{or} \qquad \sigma = 0.20$$

The value of $N(d_2)$ depends on the value of σ:

$$\sigma = 0.12 \quad \Rightarrow \quad d_2 = d_1 - \sigma\sqrt{T} = 0.16 - 0.12 = 0.04 \quad \Rightarrow \quad N(-d_2) = 1 - 0.5160 = 0.4840$$
$$\sigma = 0.20 \quad \Rightarrow \quad d_2 = d_1 - \sigma\sqrt{T} = 0.16 - 0.20 = -0.04 \quad \Rightarrow \quad N(-d_2) = 0.5160$$

For an at-the-money option, $K = S$. We are given that the put price is less than 5% of the stock price:

$$P = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$
$$\frac{P}{S} = e^{-0.012}N(-d_2) - N(-d_1)$$
$$e^{-0.012}N(-d_2) - N(-d_1) < 0.05$$
$$e^{-0.012}N(-d_2) - 0.4364 < 0.05$$
$$N(-d_2) < 0.4923$$

For the inequality above to be true, it must be the case that:

$$\sigma = 0.12$$

Solution 18

A Chapter 15, Sharpe Ratio

The price follows geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \quad \Leftrightarrow \quad S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)}$$

We have:

$$\alpha_1 - 0 - 0.5\sigma_1^2 = 0.10 \quad \& \quad \sigma_1 = 0.2 \Rightarrow \quad \alpha_1 = 0.10 + 0.5(0.2)^2 = 0.12$$
$$\alpha_2 - 0 - 0.5\sigma_2^2 = 0.125 \quad \& \quad \sigma_2 = 0.3 \Rightarrow \quad \alpha_2 = 0.125 + 0.5(0.3)^2 = 0.17$$

Since the two assets are perfectly correlated, they must have the same Sharpe ratio:

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$$
$$\frac{0.12 - r}{0.2} = \frac{0.17 - r}{0.3}$$
$$0.036 - 0.3r = 0.034 - 0.2r$$
$$0.002 = 0.1r$$
$$r = 0.02$$

Solution 19

D Chapter 7, Black-Scholes Formula with Discrete Dividends

The prepaid forward volatility is:

$$\sigma_{PF} = \sqrt{\frac{Var\left[\ln\left(F^P_{t,0.5}(S)\right)\right]}{t}} = \sqrt{\frac{0.01t}{t}} = 0.1$$

The prepaid forward prices of the stock and the strike price are:

$$F^P_{0,T}(S) = 50 - 5.00e^{-0.12(0.75)} = 45.4303$$
$$F^P_{0,T}(K) = 45e^{-0.12(1)} = 39.9114$$

We use the prepaid forward volatility in the Black-Scholes Formula:

$$d_1 = \frac{\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(K)}\right) + 0.5\sigma^2_{PF}T}{\sigma_{PF}\sqrt{T}} = \frac{\ln\left(\frac{45.4303}{39.9114}\right) + 0.5 \times 0.1^2 \times 1}{0.1\sqrt{1}}$$
$$= 1.35$$
$$d_2 = d_1 - \sigma_{PF}\sqrt{T} = 1.35 - 0.1\sqrt{1} = 1.25$$

We have:

$$N(-d_1) = N(-1.35) = 1 - N(1.35) = 1 - 0.9115 = 0.0885$$
$$N(-d_2) = N(-1.25) = 1 - N(1.25) = 1 - 0.8944 = 0.1056$$

The value of the European put option is:

$$P_{Eur} = Ke^{-rT}N(-d_2) - \left[S_0 - PV_{0,T}(Div)\right]N(-d_1)$$
$$= 39.9114 \times 0.1056 - 45.4303 \times 0.0885$$
$$= 0.1941$$

The price of 100 of the options is:

$$100 \times 0.1941 = 19.41$$

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C Chapter 9, Delta-Gamma Approximation

The delta-gamma approximation is:

$$V(t+h) \approx V(t) + \varepsilon\Delta_t + \frac{\varepsilon^2}{2}\Gamma_t$$

The approximation above can be used to create a quadratic equation:

$$2.21 = 2.34 - 0.181\varepsilon + \frac{\varepsilon^2}{2}(0.035)$$

$$0 = 0.0175\varepsilon^2 - 0.181\varepsilon + 0.13$$

$$\varepsilon = \frac{0.181 \pm \sqrt{(-0.181)^2 - 4(0.0175)(0.13)}}{2(0.0175)}$$

$$\varepsilon = 0.7765 \qquad \text{or} \qquad \varepsilon = 9.5663$$

The definition of ε can be rearranged to obtain an expression for S_0:

$$\varepsilon = S_h - S_0 \quad \Rightarrow \quad S_0 = S_h - \varepsilon$$

Since we have 2 possible values for ε, we have 2 possible values for S_0:

$$\varepsilon = 0.7765 \quad \Rightarrow \quad S_0 = 86 - 0.7765 = 85.2235$$

$$\varepsilon = 9.5663 \quad \Rightarrow \quad S_0 = 86 - 9.5663 = 76.4337$$

The value of 76.4337 is less than 80, and Statement (i) in the question tells us that the original stock price is greater than 80. Therefore, the other possibility must be correct, and the stock price is 85.2235.

