SOA Exam MFE
Solutions: May 2007

Solution 1

B Chapter 1, Put-Call Parity

Let each dividend amount be D. The first dividend occurs at the end of 2 months, and the second dividend occurs at the end of 5 months.

We can use put-call parity to find D:

$$C_{Eur}(K,T) + Ke^{-rT} = S_0 - PV_{0,T}(Div) + P_{Eur}(K,T)$$
$$4.50 + 50e^{-0.06(0.5)} = 52 - De^{-0.06(2/12)} - De^{-0.06(5/12)} + 2.45$$
$$D[e^{-0.06(2/12)} + e^{-0.06(5/12)}] = 52 + 2.45 - 4.50 - 50e^{-0.06(0.5)}$$
$$D = 0.7264$$

Solution 2

A Chapter 3, Realistic Probability

We can solve for p, the true probability of the stock price going up, using the following formula:

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.10-0)\times 1} - 0.756}{1.433 - 0.756} = 0.51576$$

Solution 3

C Chapter 7, Black-Scholes Formula

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$
$$= \frac{\ln(100/98) + (0.055 - 0.01 + 0.5\times 0.5^2)\times 0.5}{0.5\sqrt{0.5}} = 0.29756$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.29756 - 0.5\sqrt{0.5} = -0.05600$$

We have:

$$N(-d_1) = N(-0.29756) \approx N(-0.30) = 1 - N(0.30) = 1 - 0.6179 = 0.3821$$

$$N(-d_2) = N(0.05600) \approx N(0.06) = 0.5239$$

The value of the European put option is:

$$\begin{aligned} P_{Eur} &= Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1) \\ &= 98e^{-0.055(0.5)} \times 0.5239 - 100e^{-0.01(0.5)} \times 0.3821 \\ &= 11.93 \end{aligned}$$

Solution 4

E Chapter 1, Put-Call Parity

For each put option, the choice is between having the exercise value now or having a 1-year European put option. Therefore, the decision depends on whether the exercise value is greater than the value of the European put option. The value of each European put option is found using put-call parity:

$$C_{Eur}(K,T) + Ke^{-rT} = S_0e^{-\delta T} + P_{Eur}(K,T)$$

$$P_{Eur}(K,T) = C_{Eur}(K,T) + Ke^{-rT} - S_0e^{-\delta T}$$

The values of each of the 1-year European put options are:

$$P_{Eur}(40,1) = 9.12 + 40e^{-0.04(1)} - 50e^{-0.08(1)} = 1.40$$

$$P_{Eur}(50,1) = 4.91 + 50e^{-0.04(1)} - 50e^{-0.08(1)} = 6.79$$

$$P_{Eur}(60,1) = 0.71 + 60e^{-0.04(1)} - 50e^{-0.08(1)} = 12.20$$

$$P_{Eur}(70,1) = 0.00 + 70e^{-0.04(1)} - 50e^{-0.08(1)} = 21.10$$

In the third and fourth columns of the table below, we compare the exercise value with the value of the European put options. The exercise value is $Max(K - S_0, 0)$.

K	C	Exercise Value	European Put
40.00	9.12	0.00	1.40
50.00	4.91	0.00	6.79
60.00	0.71	10.00	12.20
70.00	0.00	20.00	21.10

In each case, the exercise value is less than the value of the European put option. Therefore, it is not optimal to exercise any of these put options.

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Solution 5

D Chapter 8, Volatility of an Option

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$
$$= \frac{\ln(85/80) + (0.055 - 0.00 + 0.5 \times 0.5^2) \times 1}{0.5\sqrt{1}} = 0.48125$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.48125 - 0.5\sqrt{1} = -0.01875$$

We have:

$$N(d_1) = N(0.48125) \approx N(0.48) = 0.6844$$
$$N(d_2) = N(-0.01875) \approx N(-0.02) = 1 - N(0.02) = 1 - 0.5080 = 0.4920$$

The value of the call option is:

$$C_{Eur}(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$
$$= 85e^{-0(1)} \times 0.6844 - 80e^{-0.055(1)} \times 0.4920$$
$$= 20.9203$$

The value of delta is:

$$\Delta_{Call} = e^{-\delta T} N(d_1) = e^{-0.0(1)} \times 0.6844 = 0.6844$$

The elasticity of the option is:

$$\Omega = \frac{S\Delta}{V} = \frac{85 \times 0.6844}{20.9203} = 2.7807$$

The volatility of the call option is:

$$\sigma_{Option} = \sigma_{Stock} \times \left|\Omega_{Option}\right| = 0.5 \times 2.7807 = 1.3904$$

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Solution 6

C Chapter 1, Exchange Options

The first page of Study Note MFE-27-07 tells us that "for each share of the stock the amount of dividends paid between time t and time $t+dt$ is assumed to be $S(t)\delta dt$." Therefore, the continuously compounded dividend rate for Stock 1 is 5%, and the continuously compounded dividend rate for Stock 2 is 10%.

The claim has the following payoff at time 3:

$$Max\left[S_1(3), S_2(3)\right]$$

A portfolio consisting of a share of Stock 2 and the option to exchange Stock 2 for Stock 1 effectively gives its owner the stock with this maximum value. If the value of Stock 2 is greater than the value of Stock 1 at time 3, then the owner keeps Stock 2 and allows the exchange option to expire unexercised. If the value of Stock 1 is greater than the value of Stock 2, then the owner exercises the option, giving up Stock 2 for Stock 1.

Since Stock 2 has a continuously compounded dividend rate of 10%, the cost now of a share of Stock 2 at time 3 is:

$$F_{0,T}^{P}(S) = e^{-\delta T} S_0$$

$$F_{0,3}^{P}(S_2) = e^{-0.10(3)} 200 = 148.16$$

The cost of an exchange option allowing its owner to exchange Stock 2 for Stock 1 at time 3 is \$10.

Adding the costs together, we obtain the cost of the claim:

$$148.16 + 10 = 158.16$$

Solution 7

E Chapter 17, Black Formula

The option expires in 1 year, so $T = 1$. The underlying bond matures 1 year after the option expires, so $s = 1$. The bond forward price is:

$$F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0,T)} = \frac{0.8817}{0.9434} = 0.93460$$

The volatility of the forward price is:

$$\sigma = 0.05$$

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We have:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.93460}{0.9259}\right) + 0.5(0.05)^2(1)}{0.05\sqrt{1}} = 0.21201$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.21201 - 0.05\sqrt{1} = 0.16201$$

$$N(d_1) = N(0.21201) \approx N(0.21) = 0.5832$$

$$N(d_2) = N(0.16201) \approx N(0.16) = 0.5636$$

The Black formula for the call price is:

$$\begin{aligned} C &= P(0,T)\left[F \times N(d_1) - K \times N(d_2)\right] \\ &= 0.9434\left[0.9346 \times 0.5832 - 0.9259 \times 0.5636\right] \\ &= 0.0219 \end{aligned}$$

Solution 8

C Chapter 7, Black-Scholes Formula

The volatility of the stock is:

$$\sigma = \sqrt{\frac{Var\left[\ln S(t)\right]}{t}} = \sqrt{\frac{0.4t}{t}} = \sqrt{0.4}$$

We can use the version of the Black-Scholes formula that is based on prepaid forward prices to find the value of the call option:

$$d_1 = \frac{\ln\left(\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100}{100}\right) + 0.5 \times 0.4 \times 10}{\sqrt{0.4} \times \sqrt{10}} = 1$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1 - \sqrt{0.4} \times \sqrt{10} = -1$$

$$N(d_1) = N(1) = 0.8413$$

$$N(d_2) = N(-1) = 1 - N(1) = 0.1587$$

The price of the call option is:

$$\begin{aligned} C_{Eur}\left(F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T\right) &= F_{0,T}^P(S)N(d_1) - F_{0,T}^P(K)N(d_2) \\ &= 100 \times 0.8413 - 100 \times 0.1587 \\ &= 68.26 \end{aligned}$$

Solution 9

A Chapter 18, Interest Rate Cap

The tree of interest rates is:

		9.892%
	7.704%	
6.000%		6.000%
	4.673%	
		3.639%

Although the cap payments are made at the end of each year, we will follow the textbook's convention of valuing them at the beginning of each year using the following formula:

$$T\text{-year caplet payoff at time }(T-1) = \frac{Max\left[0, R_{T-1} - K_R\right]}{1 + R_{T-1}} \times Notional$$

The interest rate cap consists of a 1-year caplet, a 2-year caplet, and a 3-year caplet.

The payoff for the 1-year caplet is zero since 6% is less than 7.5%.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 7.704%, since 4.673% is less than 7.5%:

$$R_1 = 0.07704 \Rightarrow \quad \frac{0.07704 - 0.075}{1.07704} \times 100 = 0.1894$$

The payoff for the 3-year caplet is positive only when the short-term rate increases to 9.892%, since the other short-term interest rates are less than 7.5%:

$$R_2 = 0.09892 \Rightarrow \quad \frac{0.09892 - 0.075}{1.09892} \times 100 = 2.1767$$

The possible payments from the cap are illustrated in the tree below:

		2.1767
	0.1894	
0.0000		0.0000
	0.0000	
		0.0000

The value of the 2-year caplet is:

$$\text{Value of 2-year caplet} = 0.5 \times \frac{0.1894}{1.06} = 0.0893$$

The value of the 3-year caplet is:

$$\text{Value of the 3-year caplet} = (0.5)^2 \times \frac{2.1767}{1.06 \times 1.07704} = 0.4766$$

The value of the 3-year cap is:

(1-year caplet) + (2-year caplet) + (3-year caplet)

$$= 0.0000 + 0.0893 + 0.4766 = 0.5660$$

Solution 10

B Chapter 9, Delta-Gamma Hedging

The gamma of the position to be hedged is:

$$-1{,}000 \times 0.0651 = -65.10$$

We can solve for the quantity, Q, of the other call option that must be purchased to bring the hedged portfolio's gamma to zero:

$$-65.10 + 0.0746Q = 0.00$$
$$Q = 872.7$$

Since only choice B specifies the purchase of 872.7 units of Call-II, we already have enough information to see that Choice B must be the correct answer.

The delta of the position becomes:

$$-1{,}000 \times 0.5825 + 872.7 \times 0.7773 = 95.8$$

The quantity of underlying stock that must be purchased, Q_S, is the opposite of the delta of the position being hedged:

$$Q_S = -95.8$$

Therefore, in order to delta-hedge and gamma-hedge the position, we must sell 95.8 units of stock and purchase 872.7 units of Call-II.

Solution 11

D Chapter 4, Two-Period Binomial Tree

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.05(0.5)} - 0.890}{1.181 - 0.890} = 0.46500$$

The stock price tree is:

		97.6333
	82.6700	
70.0000		73.5763
	62.3000	
		55.4470

The tree of prices for the American put option is:

		0.0000
	3.3518	
10.7558		6.4237
	17.7000	
		24.5530

If the stock price reaches $82.67 in 6 months, then the value of option is:

$$e^{-0.05(0.5)}\left[(0.46500)0.0000+(1-0.46500)6.4237\right]=3.3518$$

If the stock price reaches $62.30 in 6 months, then early exercise is optimal because the value of holding the option is only:

$$e^{-0.05(0.5)}\left[(0.46500)6.4237+(1-0.46500)24.5530\right]=15.7248$$

Since the exercise value is $17.70, the option is exercised early if the stock price reaches $62.30.

The value of the option is:

$$e^{-0.05(0.5)}\left[(0.46500)3.3518+(1-0.46500)17.7000\right]=10.7558$$

Solution 12

A Chapter 14, Itô's Lemma

Since we are given the risk-free rate in both the U.S. and Great Britain, we have:

$$r-r^{*}=0.08-0.10=-0.02$$

The expression for $G(t)$ is therefore:

$$G(t)=S(t)e^{-0.02(T-t)}$$

The partial derivatives are:

$$G_S=e^{-0.02(T-t)}$$
$$G_{SS}=0$$
$$G_t=\frac{\partial\left[S(t)e^{-0.02T}e^{0.02t}\right]}{\partial t}=S(t)e^{-0.02T}(0.02)e^{0.02t}=S(t)(0.02)e^{-0.02(T-t)}$$

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From Itô's Lemma, we have:

$$\begin{aligned} dG(t) &= G_S dS(t) + \frac{1}{2} G_{SS} [dS(t)]^2 + G_t dt \\ &= e^{-0.02(T-t)} dS(t) + \frac{1}{2}(0)[dS(t)]^2 + S(t)(0.02)e^{-0.02(T-t)} dt \\ &= e^{-0.02(T-t)} dS(t) + G(t)(0.02)dt \\ &= e^{-0.02(T-t)} S(t)\left[0.1dt + 0.4dZ(t)\right] + G(t)(0.02)dt \\ &= G(t)\left[0.1dt + 0.4dZ(t)\right] + G(t)(0.02)dt \\ &= G(t)\left[0.12dt + 0.4dZ(t)\right] \end{aligned}$$

Solution 13

E Chapter 19, Vasicek Model

In the Vasicek Model, we have:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$

We use the following two facts about the Vasicek model:

- $A(t,T)$ and $B(t,T)$ do not depend on *r*.
- $A(t,T) = A(0,T-t)$ and $B(t,T) = B(0,T-t)$. This implies:

$$A(0,2) = A(1,3) = A(2,4)$$
$$B(0,2) = B(1,3) = B(2,4)$$

We have two equations and two unknowns:

$$A(0,2)e^{-B(0,2)(0.04)} = 0.9445$$
$$A(0,2)e^{-B(0,2)(0.05)} = 0.9321$$

Dividing the second equation into the first equation allows us to find $B(0,2)$:

$$e^{-B(0,2)(0.04)+B(0,2)(0.05)} = \frac{0.9445}{0.9321}$$
$$B(0,2)(0.01) = \ln\left(\frac{0.9445}{0.9321}\right)$$
$$B(0,2) = 1.32156$$

We can now solve for the value of $A(0,2)$:

$$A(0,2)e^{-B(0,2)(0.04)} = 0.9445$$
$$A(0,2) = 0.9445e^{B(0,2)(0.04)}$$
$$A(0,2) = 0.9445e^{(1.32156)(0.04)}$$
$$A(0,2) = 0.99577$$

We can now solve for r^*:

$$A(0,2)e^{-B(0,2)(r^*)} = 0.8960$$
$$0.99577e^{-1.32156r^*} = 0.8960$$
$$-1.32156r^* = \ln\left(\frac{0.8960}{0.99577}\right)$$
$$r^* = 0.07989$$

Solution 14

E Chapter 3, One-Period Binomial Tree

If the stock price moves up, then the straddle pays \$20. If the stock price moves down, then the straddle pays \$5:

	70		$20 = \lvert 50-70 \rvert$
60		V	
	45		$5 = \lvert 50-45 \rvert$

In this case, $u = 70/60$ and $d = 45/60$.

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.00)1} - \frac{45}{60}}{\frac{70}{60} - \frac{45}{60}} = 0.79989$$

The value of the straddle is:

$$V = e^{-rh}\left[(p^*)V_u + (1-p^*)V_d\right] = e^{-0.08(1)}\left[0.79989(20) + (1-0.79989)(5)\right]$$
$$= 15.69$$

Solution 15

C Chapter 7, Black-Scholes Formula

The volatility parameter σ is not defined in the question, but let's assume that it is the volatility of the prepaid forward.

The prepaid forward prices of the stock and the strike price are:

$$F_{0,T}^P(S) = 50 - 1.50e^{-0.05(4/12)} = 48.5248$$
$$F_{0,T}^P(K) = 50e^{-0.05(0.5)} = 48.7655$$

The volatility of the prepaid forward price is:

$$\sigma_{PF} = 0.30$$

We use the prepaid forward volatility in the Black-Scholes Formula:

$$d_1 = \frac{\ln\left(\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right) + 0.5\sigma_{PF}^2 T}{\sigma_{PF}\sqrt{T}} = \frac{\ln\left(\frac{48.5248}{48.7655}\right) + 0.5 \times 0.30^2 \times 0.5}{0.30\sqrt{0.5}}$$

$$= 0.0827$$

$$d_2 = d_1 - \sigma_{PF}\sqrt{T} = 0.0827 - 0.30\sqrt{0.5} = -0.1294$$

We have:

$$N(-d_1) = N(-0.0827) \approx N(-0.08) = 1 - N(0.08) = 1 - 0.5319 = 0.4681$$

$$N(-d_2) = N(0.1294) \approx N(0.13) = 0.5517$$

The value of the European put option is:

$$P_{Eur}\left(F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T\right) = Ke^{-rT}N(-d_2) - \left[S_0 - PV_{0,T}(Div)\right]N(-d_1)$$

$$= 48.7655 \times 0.5517 - 48.5248 \times 0.4681$$

$$= 4.1895$$

Solution 16

D Perpetual Options

This question is based on Section 12.6 of the textbook, which is no longer assigned for Exam MFE/3F.

The values of h_1 and h_2 are:

$$h_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

Therefore:

$$h_1 + h_2 = 1 - \frac{2(r-\delta)}{\sigma^2}$$

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We can solve for δ:

$$h_1 + h_2 = 1 - \frac{2(r-\delta)}{\sigma^2}$$
$$\frac{7}{9} = 1 - \frac{2(0.05-\delta)}{0.30^2}$$
$$\frac{2}{9} = \frac{2(0.05-\delta)}{0.30^2}$$
$$0.01 = 0.05 - \delta$$
$$\delta = 0.04$$

We now have the information necessary to solve for h_1:

$$h_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$
$$= \frac{1}{2} - \frac{0.05-0.04}{0.30^2} + \sqrt{\left(\frac{0.05-0.04}{0.30^2} - \frac{1}{2}\right)^2 + \frac{2(0.05)}{0.30^2}} = 1.5124$$

Solution 17

B Chapter 10, Gap Options

For the gap option, we have:

Strike Price: $K_1 = 90$
Trigger Price: $K_2 = 100$

The delta of the regular call option is:

$$\Delta_{Call} = e^{-\delta T} N(d_1) = 0.2$$

For the regular European call option, we have:

$$4 = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$
$$4 = 80(0.20) - 100e^{-rT} N(d_2)$$
$$e^{-rT} N(d_2) = 0.12$$

Since the regular call option and the gap call option have the same values for d_1 and d_2, we can substitute the final line above into the equation for the value of the gap call option:

$$\text{Gap call price} = Se^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)$$
$$= 80(0.2) - 90(0.12)$$
$$= 5.20$$

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Solution 18

A Chapter 15, Sharpe Ratio

When the price follows geometric Brownian motion, the natural log of the price follows arithmetic Brownian motion:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t) \quad \Leftrightarrow \quad d\left[\ln S(t)\right] = \left(\alpha - 0.5\sigma^2\right)dt + \sigma dZ$$

Therefore:

$$\frac{dY(t)}{Y(t)} = Gdt + HdZ(t) \quad \Leftrightarrow \quad d\left[\ln Y(t)\right] = \left(G - 0.5H^2\right)dt + HdZ$$

The arithmetic Brownian motion provided in the question for $d\left[\ln Y(t)\right]$ allows us to find an expression for H and G:

$$d\left[\ln Y(t)\right] = 0.06dt + \sigma dZ(t) \quad \text{and} \quad d\left[\ln Y(t)\right] = \left(G - 0.5H^2\right)dt + HdZ$$

$$H = \sigma$$

$$G - 0.5H^2 = 0.06$$

$$G = 0.06 + 0.5H^2$$

Since X and Y have the same source of randomness, $dZ(t)$, they must have the same Sharpe ratio.

$$\frac{0.07 - 0.04}{0.12} = \frac{G - 0.04}{H}$$

$$0.25 = \frac{G - 0.04}{H}$$

$$0.25 = \frac{0.06 + 0.5H^2 - 0.04}{H}$$

$$0.25H = 0.02 + 0.5H^2$$

$$0.5H^2 - 0.25H + 0.02 = 0$$

We can use the quadratic formula to solve for H:

$$H = \frac{0.25 \pm \sqrt{0.25^2 - 4(0.5)(0.02)}}{2(0.5)} = 0.1 \text{ or } 0.4$$

Since we are given that $\sigma < 0.25$ and we know that $H = \sigma$, it must be the case that:

$$H = 0.1$$

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We can now find the value of G:

$$0.25 = \frac{G - 0.04}{H}$$
$$0.25 = \frac{G - 0.04}{0.1}$$
$$G = 0.065$$

Solution 19

D Chapter 9, Delta-Gamma Approximation

The delta-gamma approximation for the new price is:

$$V_{t+h} \approx V_t + \varepsilon\Delta_t + \frac{\varepsilon^2}{2}\Gamma_t$$

The change in the stock price is:

$$\varepsilon = S_{t+h} - S_t = 31.50 - 30.00 = 1.50$$

The delta-gamma approximation is:

$$\begin{aligned} V_{t+h} &\approx V_t + \varepsilon\Delta_t + \frac{\varepsilon^2}{2}\Gamma_t \\ &= 4.00 + (1.50)(-0.28) + \frac{1.50^2}{2}(0.10) \\ &= 3.6925 \end{aligned}$$

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