Course MFE/3F Practice Exam 3 – Solutions

Solution 1

C Chapter 20, Valuing a Claim on S^a

The payoff consists of $Q(2)$ shares of stock. The quantity of shares is therefore:

$$Q(2) = [S(2)]^2$$

The price of each share at time 2 is $S(2)$. The value of the payoff at the end of 2 years is equal to the quantity of shares times the price of each share:

$$\text{Payoff} = \text{Quantity} \times \text{Price} = Q(2) \times S(2) = [S(2)]^2 \times S(2) = [S(2)]^3$$

The value of the derivative is therefore the prepaid forward price of $S(T)^a$, where $a = 3$ and $T = 2$. The prepaid forward price is:

$$\begin{aligned} F_{0,T}^{P}\left[S(T)^a\right] &= e^{-(\delta^*)(T-0)}S(0)^a \\ &= e^{\left[-r+a(r-\delta)+0.5a(a-1)\sigma^2\right]T}S(0)^a \\ &= e^{[-0.04+3(0.04-0.07)+0.5(3)(3-1)0.3^2]2}(4)^3 \\ &= e^{0.28}(4)^3 \\ &= 84.6803 \end{aligned}$$

Solution 2

E Chapter 21, Black-Scholes Equation

The dividend yield on the underlying asset is 3%, so we can find α:

$$\alpha - \delta = 0.08 \quad \Rightarrow \quad \alpha - 0.03 = 0.08 \quad \Rightarrow \quad \alpha = 0.11$$

The Sharpe ratio of the claim and the underlying asset must be the same, so:

$$\frac{\alpha - r}{\sigma} = 0.20 \quad \Rightarrow \quad \frac{0.11 - r}{0.20} = 0.20 \quad \Rightarrow \quad r = 0.07$$

The following formula from the Chapter 20 Review Note allows us to find the lease rate of the claim:

$$\delta^* = r - a(r-\delta) - 0.5a(a-1)\sigma^2 = 0.07 - 3(0.07-0.03) - 0.5(3)(2)(0.2)^2 = -0.17$$

Therefore, the dividend rate for the claim is:

$$D(t) = -0.17[S(t)]^3 = -0.17V(t)$$

The Black-Scholes equation is:

$$0.5\sigma^2[S(t)]^2 V_{SS} + (r-\delta)S(t)V_S + V_t + D(t) = rV(t)$$
$$0.5(0.20)^2[S(t)]^2 V_{SS} + (0.07-0.03)S(t)V_S + V_t - 0.17V(t) = 0.07V(t)$$
$$0.02[S(t)]^2 V_{SS} + (0.04)S(t)V_S + V_t - 0.24V(t) = 0$$

Let's find the derivative of V with respect to t:

$$V = S^3$$
$$V_t = 0$$

Substituting 0 in for V_t, we have:

$$0.02[S(t)]^2 V_{SS} + (0.04)S(t)V_S - 0.24V(t) = 0$$

<u>Alternate Solution</u>

The partial derivatives are:

$$V = S^3$$
$$V_S = 3S^2$$
$$V_{SS} = 6S$$
$$V_t = 0$$

Substituting into the equation in Choice (A), we obtain an inequality for all $S(t) > 0$, indicating that Choice (A) is not correct:

$$V_t + 0.02[S(t)]^2 V_{SS} + 0.04S(t)V_S - 0.07V(t) = 0$$
$$0 + 0.02S^2(6S) + 0.04S(3S^2) - 0.07S^3 = 0$$
$$0.17S^3 = 0$$

Similar inequalities result from making the same substitutions into Choices (B), (C), and (D). Substituting into Choice (E) results in the following equality:

$$0.02[S(t)]^2 V_{SS} + 0.04S(t)V_S - 0.24V(t) = 0$$
$$0.02S^2(6S) + 0.04S(3S^2) - 0.24S^3 = 0$$
$$0.12S^3 + 0.12S^3 - 0.24S^3 = 0$$

Therefore, only Choice (E) can be correct.

Solution 3

B Chapter 10, Options on Futures Contracts

The values of u_F and d_F are:

$$u_F = e^{\sigma\sqrt{h}} = e^{0.18\sqrt{0.5}} = 1.13573$$
$$d_F = e^{-\sigma\sqrt{h}} = e^{-0.18\sqrt{0.5}} = 0.88049$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{1-d_F}{u_F - d_F} = \frac{1-0.88049}{1.13573-0.88049} = 0.46822$$

The futures price tree and the put option tree are below:

F_{0,T_F}	$F_{0.5,T_F}$	F_{1,T_F}
		16.7686
	14.7645	
13.0000		13.0000
	11.4463	
		10.0784

American Put Option

		0.0000
	0.2593	
1.1835		0.5000
	2.0537	
		3.4216

If the futures price reaches \$11.4463 at the end of 6 months, then early exercise of the American put option is optimal.

The price of the American put option is:

$$e^{-0.05(0.5)}\left[(0.46822)(0.2593) + (1-0.46822)(2.0537)\right] = 1.1835$$

Solution 4

B Chapter 12, Black-Scholes Put Price

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(10.67/9.75) + (0.10 - 0.00 + 0.5\times 0.40^2)\times 0.25}{0.40\sqrt{0.25}}$$
$$= 0.67584$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.67584 - 0.40\sqrt{0.25} = 0.47584$$

We have:

$$N(d_1) = N(0.67584) = 0.75043$$

$$N(d_2) = N(0.47584) = 0.68291$$

The value of the European put option is:

$$\begin{aligned}P_{Eur} &= Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)\\ &= 9.75e^{-0.10(0.25)} \times (1-0.68291) - 10.67e^{-0.00(0.25)} \times (1-0.75043)\\ &= 0.3524\end{aligned}$$

Solution 5

C Chapter 22, Asset-or-Nothing Options

The investor can replicate the payoff by purchasing a 2-year asset call and selling 100 2-year cash puts. Therefore, the current value of the payoff is:

$$\begin{aligned}&AssetCall(100) - 100 \times CashPut(100) = S_t e^{-\delta(T-t)}N(d_1) - 100 \times e^{-r(T-t)}N(-d_2)\\ &\quad = 92e^{-2\delta}N(d_1) - 100e^{-2r}N(-d_2)\end{aligned}$$

We can use the 2-year forward price to find $92e^{-2\delta}$:

$$\begin{aligned}F_{0,2} &= S_0 e^{(r-\delta)(2-0)}\\ 100 &= 92e^{2(r-\delta)}\\ 100e^{-2r} &= 92e^{-2\delta}\end{aligned}$$

Substituting $100e^{-2r}$ in for $92e^{-2\delta}$, we have:

$$\begin{aligned}&AssetCall(100) - 100 \times CashPut(100)\\ &\quad = 92e^{-2\delta}N(d_1) - 100e^{-2r}N(-d_2)\\ &\quad = 100e^{-2r}N(d_1) - 100e^{-2r}N(-d_2)\\ &\quad = 100e^{-2r}\left[N(d_1) - N(-d_2)\right]\end{aligned}$$

We can make use of the substitution $92e^{-2\delta} = 100e^{-2r}$ in determining the values of d_1 and d_2:

$$d_1 = \frac{\ln\left(\dfrac{Se^{-\delta(T-t)}}{Ke^{-r(T-t)}}\right) + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\dfrac{92e^{-2\delta}}{100e^{-2r}}\right) + 0.5 \times \sigma^2 \times 2}{\sigma\sqrt{2}}$$

$$= \frac{\ln\left(\dfrac{100e^{-2r}}{100e^{-2r}}\right) + 0.5 \times \sigma^2 \times 2}{\sigma\sqrt{2}} = \frac{0 + 0.5 \times \sigma^2 \times 2}{\sigma\sqrt{2}} = 0.5\sigma\sqrt{2}$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.5\sigma\sqrt{2} - \sigma\sqrt{2} = -0.5\sigma\sqrt{2}$$

Therefore:

$$d_1 = -d_2$$

The current value of the payoff is:

$$100e^{-2r}\left[N(d_1)-N(-d_2)\right]=100e^{-2r}\left[N(d_1)-N(d_1)\right]=0$$

Solution 6

C Chapter 20, Itô's Lemma

The drift is the expected change in the asset price per unit of time.

For the first equation, the partial derivatives are:

$$U_Z=10 \qquad U_{ZZ}=0 \qquad U_t=0$$

This results in:

$$dU(Z,t)=U_Z dZ+\frac{1}{2}U_{ZZ}(dZ)^2+U_t dt=10dZ+0+0=10dZ$$

Since there is no dt term, the drift is zero for dU.

For the second equation, the partial derivatives are:

$$V_Z=10Z \qquad V_{ZZ}=10 \qquad V_t=-5$$

This results in:

$$\begin{aligned}dV(Z,t)&=V_Z dZ+\frac{1}{2}V_{ZZ}(dZ)^2+V_t dt=10Z(t)dZ+\frac{1}{2}\times 10(dZ)^2-5dt\\&=10Z(t)dZ+5dt-5dt=10Z(t)dZ \qquad (\text{since } (dZ)^2=dt)\end{aligned}$$

Since there is no dt term, the drift is zero for dV.

For the third equation, the partial derivatives are:

$$W_Z=t \qquad W_{ZZ}=0 \qquad W_t=Z$$

This results in:

$$\begin{aligned}dW(Z,t)&=W_Z dZ+\frac{1}{2}W_{ZZ}(dZ)^2+W_t dt=tdZ+\frac{1}{2}\times 0\times (dZ)^2+Z(t)dt\\&=Z(t)dt+tdZ(t)\end{aligned}$$

Since the dt term is nonzero, the drift is nonzero for dW.

Solution 7

D Chapter 14, Gap Options

For the gap option, we have:

$$K_1 = \text{Strike Price} = 100$$
$$K_2 = \text{Trigger Price} = 110$$

The delta of the regular put option is:

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -0.8288 \qquad \Rightarrow \qquad e^{-\delta T}N(-d_1) = 0.8288$$

For the regular European put option, we have:

$$P_{Eur}(K,T) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$
$$23.43 = 110e^{-rT}N(-d_2) - 85(0.8288)$$
$$e^{-rT}N(-d_2) = 0.853436$$

Since the regular put option and the gap put option have the same values for d_1 and d_2, we can substitute the final line above into the equation for the value of the gap put option:

$$\begin{aligned}\text{Gap put price} &= K_1e^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)\\ &= 100(0.853436) - 85(0.8288)\\ &= 14.90\end{aligned}$$

Solution 8

E Chapter 13, Market-Maker Profit

The market-maker's profit is zero if the stock price movement is one standard deviation:

$$\text{One-standard-deviation move} = \sigma S_t\sqrt{h}$$

To answer the question, we must determine the value of σ. We first determine the value of $N(d_1)$:

$$\begin{aligned}\Delta_{Put} &= -e^{-\delta T}N(-d_1)\\ -0.3446 &= -e^{-0.0(1.00)}N(-d_1)\\ -0.3446 &= -[1 - N(d_1)]\\ 0.3446 &= 1 - N(d_1)\\ N(d_1) &= 1 - 0.34458\\ N(d_1) &= 0.65542\end{aligned}$$

From the normal distribution table, this implies that:

$$d_1 = 0.40000$$

We can use the formula for d_1 to solve for the value of σ :

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$0.40000 = \frac{\ln(65/65) + (0.08 - 0 + 0.5\sigma^2)1.00}{\sigma\sqrt{1.00}}$$

$$0.4\sigma = 0.08 + 0.5\sigma^2$$

$$0.5\sigma^2 - 0.4\sigma + 0.08 = 0$$

$$\sigma^2 - 0.8\sigma + 0.16 = 0$$

$$(\sigma - 0.4)(\sigma - 0.4) = 0$$

$$\sigma = 0.4$$

The price of the stock moves by:

$$\sigma S_t\sqrt{h} = 0.4 \times 65 \times \sqrt{\frac{1}{365}} = 1.36$$

The stock price moves either up or down by 1.36.

Solution 9

C Chapter 18, Probability that Stock Price is > K

The current time is $t = 0.25$, and we are interested in the price at the end of 9 months, so $T = 0.75$.

The value of $\hat{d}_2$ is:

$$\hat{d}_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{75}{77}\right) + (0.12 - 0.05 - 0.5 \times 0.32^2)(0.75 - 0.25)}{0.32\sqrt{0.75 - 0.25}}$$

$$= -0.07476$$

The probability that $S(0.75)$ is greater than \$77 is:

$$\text{Prob}\left(S_T > K\right) = N(\hat{d}_2) = N(-0.07476) = 0.47020$$

Solution 10

D Chapter 24, Vasicek Model

The process can be written as:

$$dr = 0.15(0.08 - r)dt + 0.04dZ$$

The expected change in the interest rate is:

$$E[dr] = E[0.15(0.08 - r)dt + 0.04dZ] = (0.012 - 0.15r)dt + 0.04 \times 0$$
$$= (0.012 - 0.15r)dt$$

Since $r = 0.07$, we have:

$$E[dr] = (0.012 - 0.15r)dt = \left[0.012 - 0.15(0.07)\right]dt = 0.0015dt$$

To convert the expected change in the interest rate into an annual rate, we must divide by the increment of time:

$$\frac{0.0015dt}{dt} = 0.0015$$

Solution 11

B Chapter 10, Multiple-Period Binomial Tree

If we work through the entire binomial tree, this is a very time-consuming problem. Even using the direct method takes a lot of time. But if we notice that the value of the corresponding put option can be calculated fairly quickly, then we can use put-call parity to find the value of the call option.

The values of u and d are:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.08-0.00)(1/12)+0.30\sqrt{1/12}} = 1.09776$$
$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.08-0.00)(1/12)-0.30\sqrt{1/12}} = 0.92318$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.08-0.00)(1/12)} - 0.92318}{1.09776 - 0.92318} = 0.47836$$

If the stock price decreases each month, then at the end of 7 months, the price is:

$$130d^7 = 130(0.92318)^7 = 74.2905$$

If the stock price decreases for 6 of the months and increases for 1 of the months, then the price is:

$$130ud^6 = 130(1.09776)(0.92318)^6 = 88.3396$$

Consider the corresponding European put option with a strike price of \$88. Since the price of \$88.3396 is out-of-the-money, all higher prices are also out-of-the-money. This means that the only price at which the put option expires in-the-money is the lowest possible price, which is \$74.2905.

Calculating the price of the European call option is daunting, but the corresponding European put option price can be found fairly easily. The value of the put option is:

$$\begin{aligned} V(S_0, K, 0) &= e^{-r(hn)} \sum_{j=0}^{n} \left[\binom{n}{j} (p^*)^j (1-p^*)^{n-j} V(S_0 u^j d^{n-j}, K, hn) \right] \\ &= e^{-0.08(7/12)} [(1-0.47836)^7 (88-74.2905) + 0 + 0 + 0 + 0 + 0 + 0 + 0] \\ &= 0.1375 \end{aligned}$$

Now we can use put-call parity to find the value of the corresponding call option:

$$\begin{aligned} C_{Eur}(K,T) + Ke^{-rT} &= S_0 e^{-\delta T} + P_{Eur}(K,T) \\ C_{Eur}(K,T) + 88e^{-0.08(7/12)} &= 130e^{-0.00(7/12)} + 0.1375 \\ C_{Eur}(K,T) &= 46.1498 \end{aligned}$$

The value of the European call option is \$46.15. Since the stock does not pay dividends, the value of the American call option is equal to the value of the European call option.

Solution 12

C Chapter 9, Different Strike Prices

The prices of the options violate Proposition 3:

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} \le \frac{P(K_3) - P(K_2)}{K_3 - K_2}$$

because:

$$\begin{aligned} \frac{P(75) - P(70)}{75 - 70} &> \frac{P(95) - P(75)}{95 - 75} \\ \frac{14 - 10}{75 - 70} &> \frac{25 - 14}{95 - 75} \\ \frac{4}{5} &> \frac{11}{20} \\ 0.8 &> 0.55 \end{aligned}$$

Arbitrage is available using an asymmetric butterfly spread:

Buy λ of the 70-strike options

Sell 1 of the 75-strike options

Buy $(1-\lambda)$ of the 95-strike options

where:

$$\lambda = \frac{95-75}{95-70} = \frac{20}{25} = 0.8$$

The payoffs for buying and selling put options are:

Option Position	Payoff
Buy put	$Max(K-S_t,0)$
Sell put	$-Max(K-S_t,0)$

In the payoff table below, we have scaled the strategy up by multiplying by 5:

		Time 1			
Transaction	**Time 0**	$S_1 < 70$	$70 \le S_1 \le 75$	$75 \le S_1 \le 95$	$95 < S_1$
Buy 4 of $P(70)$	$-4(10.00)$	$4(70-S_1)$	0.00	0.00	0.00
Sell 5 of $P(75)$	$5(14.00)$	$-5(75-S_1)$	$-5(75-S_1)$	0.00	0.00
Buy 1 of $P(95)$	$-1(25.00)$	$95-S_1$	$95-S_1$	$95-S_1$	0.00
Total	5.00	0.00	$4S_1-280$	$95-S_1$	0.00

The cash flows at time 1 for each of the possible stock prices are in the table below:

Answer Choice	Stock Price	Time 1 Cash Flow
A	69	0.00
B	74	$4S_1-280 = 4(74)-280 = 16.00$
C	77	$95-S_1 = 100-77 = 18.00$
D	93	$95-S_1 = 95-93 = 2.00$
E	96	0.00

The highest cash flow at time 1, \$18.00, occurs if the final stock price is \$77. This results in the highest arbitrage profits since the time 0 cash flow is the same for each future stock price.

Solution 13

C Chapter 14, Forward Start Option

In one year, the value of the call option will be:

$$
\begin{aligned}
C_{Eur}(S_1) &= S_1 N(d_1) - Ke^{-rT} N(d_2) \\
&= S_1 N(d_1) - S_1 e^{-0.10} N(d_2)
\end{aligned}
$$

In one year, the values of d_1 and d_2 will be:

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\dfrac{F^P_{t_1,T}(S)}{F^P_{t_1,T}(K)}\right) + 0.5\sigma^2 (T - t_1)}{\sigma\sqrt{T - t_1}} \\
&= \frac{\ln\left(\dfrac{S_1}{S_1 e^{-0.10\times 1}}\right) + 0.5(0.3)^2(1)}{0.3\sqrt{1}} \\
&= 0.48333
\end{aligned}
$$

$$
d_2 = d_1 - \sigma\sqrt{T - t_1} = 0.48333 - 0.30\sqrt{1} = 0.18333
$$

From the normal distribution table:

$$
\begin{aligned}
N(d_1) &= N(0.48333) = 0.68557 \\
N(d_2) &= N(0.18333) = 0.57273
\end{aligned}
$$

In one year, the value of the call option can be expressed in terms of the stock price at that time:

$$
\begin{aligned}
C_{Eur}(S_1) &= S_1 N(d_1) - S_1 e^{-0.10} N(d_2) \\
&= S_1 \times 0.68557 - S_1 e^{-0.10} \times 0.57273 \\
&= 0.16735 \times S_1
\end{aligned}
$$

In one year, the call option will be worth 0.1674 shares of stock.

The prepaid forward price of one share of stock is:

$$
\begin{aligned}
F^P_{0,t_1}(S) &= S_0 - PV_{0,t_1}(Div) \\
F^P_{0,1}(S) &= 50 - 5e^{-0.10(0.5)} = 45.2439
\end{aligned}
$$

The value today of 0.1674 shares of stock in one year is:

$$
0.16735 \times 45.2439 = 7.5716
$$

Solution 14

E Chapter 11, Estimating Volatility

Since we have 7 months of data, we can calculate 6 monthly returns. Each monthly return is a calculated as a continuously compounded rate:

$$r_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

The next step is to calculate the average of the returns:

$$\bar{r} = \frac{\sum_{i=1}^{k} r_i}{k}$$

The returns and their average are shown in the third column below:

Date	Price	$r_i = \ln\left(\frac{S_{i+h}}{S_i}\right)$	$(r_i - \bar{r})^2$
1	85		
2	81	–0.048202	0.005668
3	87	0.071459	0.001969
4	93	0.066691	0.001569
5	102	0.092373	0.004262
6	104	0.019418	0.000059
7	100	–0.039221	0.004397
		$\bar{r} = 0.027086$	$\sum_{i=1}^{6}(r_i - \bar{r})^2 = 0.017924$

The fourth column shows the squared deviations and the sum of squares.

The estimate for the standard deviation of the monthly returns is:

$$\hat{\sigma}_h = \sqrt{\frac{\sum_{i=1}^{k}(r_i - \bar{r})^2}{k-1}}$$

$$\hat{\sigma}_{\frac{1}{12}} = \sqrt{\frac{0.017924}{5}} = 0.059873$$

We adjust the monthly volatility to obtain the annual volatility:

$$\hat{\sigma} = \hat{\sigma}_h\sqrt{\frac{1}{h}} = 0.059873\sqrt{12} = 0.2074$$

This problem isn't very difficult if you are familiar with the statistical function of your calculator.

On the TI-30X IIS, the steps are:

[2nd][STAT] (Select 1-VAR) [ENTER]

[DATA]

X1= $ln\left(\frac{81}{85}\right)$ [ENTER] ↓↓ (Hit the down arrow twice)

X2= $ln\left(\frac{87}{81}\right)$ [ENTER] ↓↓

X3= $ln\left(\frac{93}{87}\right)$ [ENTER] ↓↓

X4= $ln\left(\frac{102}{93}\right)$ [ENTER] ↓↓

X5= $ln\left(\frac{104}{102}\right)$ [ENTER] ↓↓

X6= $ln\left(\frac{100}{104}\right)$ [ENTER]

[STATVAR] → → (Arrow over to *Sx*)

$\times \sqrt{(12)}$ [ENTER]

The result is: 0.207404852

To exit the statistics mode:

[2nd] [EXITSTAT] [ENTER]

On the BA II Plus calculator, the steps are:

[2nd][DATA] [2nd][CLR WORK]

X01 81/85 = LN [ENTER] ↓↓ (Hit the down arrow twice)

X02 87/81 = LN [ENTER] ↓↓

X03 93/87 = LN [ENTER] ↓↓

X04 102/93 = LN [ENTER] ↓↓

X05 104/102 = LN [ENTER] ↓↓

X06 100/104 = LN [ENTER]

[2nd][STAT] ↓↓↓ $\times \sqrt{(12)}$ =

The result is: 0.20740485

To exit the statistics mode: [2nd][QUIT]

On the TI-30XS MultiView, the steps are:

[data] [data] 4 (to clear the data table)

(enter the data below)

L1	L2	L3
85	81	-----------
81	87	
87	93	
93	102	
102	104	
104	100	

(place cursor in the L3 column)

[data] → (to highlight FORMULA)

1 [ln] [data] 2 / [data] 1) [enter]

[2nd] [quit] [2nd] [stat] 1

DATA: (highlight L3) FRQ: (highlight one) (select CALC) [enter]

3 (to obtain Sx)

$Sx \times \sqrt{12}$ [enter] The result is 0.207404852

Solution 15

D Chapter 12, Currency Options and Black-Scholes

The currency option is a put option with a euro as its underlying asset. The domestic currency is dollars, and the current value of the underlying asset is:

$$x_0 = \frac{1}{0.80} = 1.25 \text{ dollars}$$

Since the option is at-the-money, the strike price is equal to the value of one euro:

$$K = \frac{1}{0.80} = 1.25 \text{ dollars}$$

The domestic interest rate is 3%, and the foreign interest rate is 7%:

$$r = 3\%$$
$$r_f = 7\%$$

The volatility of the euro per dollar exchange rate is:

$$\sigma = 0.08$$

The values of d_1 and d_2 are:

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1) + (0.03 - 0.07 + 0.5 \times 0.08^2)1}{0.08\sqrt{1}} = -0.46000$$
$$d_2 = d_1 - \sigma\sqrt{T} = -0.46 - 0.08\sqrt{1} = -0.54000$$

In this case, we don't need to round d_1 and d_2 when using the normal distribution table:

$$N(-d_1) = N(0.46000) = 0.67724$$
$$N(-d_2) = N(0.54000) = 0.70540$$

The value of the put option is:

$$\begin{aligned} P_{Eur}(S,K,\sigma,r,T,\delta) &= Ke^{-rT}N(-d_2) - Se^{-r_f T}N(-d_1) \\ &= 1.25e^{-0.03(1)}(0.70540) - 1.25e^{-0.07(1)}(0.67724) \\ &= 0.066372361 \end{aligned}$$

Since the option is for €100,000,000, Company A purchases 100,000,000 put options, and the value of the put options in dollars is:

$$0.066372361 \times 100,000,000 = 6,637,236$$

The solution is $6,639,236.

Solution 16

C Chapter 9, Put-Call Parity

The options expire in 6 months, so we do not need to consider dividends paid after 6 months. Therefore, only the first dividend is included in the calculations below.

We can use put-call parity to understand this problem:

$$C_{Eur}(K,T) + Ke^{-rT} = S_0 - PV_{0,T}(Div) + P_{Eur}(K,T)$$

Put-call parity tells us that the following strategies produce the same cash flow at the end of 6 months:

- Purchase 1 call option and lend the present value of the $90 strike price. The cost of this strategy is:

$$C_{Eur}(K,T) + Ke^{-rT} = 7.22 + 90e^{-0.06(0.5)} = 94.5601$$

- Purchase 1 share of stock, borrow the present value of the first dividend, and purchase 1 put option. The cost of this strategy is:

$$S_0 - PV_{0,T}(div) + P_{Eur}(K,T) = 92 - 5e^{-0.06(2/12)} + 5.50 = 92.5498$$

Since the first strategy costs more than the second strategy, arbitrage profits can be earned by shorting the first strategy and going long the second strategy. The net payoff in 6 months is zero, and the arbitrage profit now is:

$$94.5601 - 92.5498 = 2.0103$$

Solution 17

A Chapter 13, Delta-Hedging

The table in the question is similar to Table 13.2 on page 420 of the textbook.

We can treat each day's re-hedging as a liquidation of the position, followed by the recreation of the position. The liquidation produces each day's profits or losses. The recreation of the position creates no cash flows at the outset since any funds needed are borrowed.

For writing the 100 options at the end of Day 4, the market-maker receives:

$$100 \times 3.65 = 365.00$$

The market-maker wrote 100 of the call options, so the delta at the end of Day 4 is:

$$-100 \times 0.4830 = -48.30$$

The market-maker hedges this position by purchasing 48.30 shares of the stock for:

$$48.30 \times 72.00 = 3,477.60$$

Since the cost of the stock exceeds the amount received for writing the options, the market-maker borrows:

$$3,477.60 - 365.00 = 3,112.60$$

At the end of Day 5, the change in the value of the options is:

$$-100 \times (3.16 - 3.65) = 49.00$$

At the end of Day 5, the change in the value of the stock is:

$$48.30 \times (71.00 - 72.00) = -48.30$$

At the end of Day 5, the change in the value of the borrowed funds is:

$$-3,112.60 \times \left(e^{0.10/365} - 1\right) = -0.8529$$

The fifth day's profit is the sum of theses changes:

Component	Day 5 Change
Gain on Options	49.0000
Gain on Stock	–48.3000
Interest	–0.8529
Daily Profit	–0.1529

The fifth day's profit is –$0.1529.

Solution 18

C Chapter 24, Cox-Ingersoll-Ross Model

We begin with the Sharpe ratio and parameterize it for the CIR model:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\bar{\phi}\frac{\sqrt{r}}{\sigma} = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma(r)}$$

$$\bar{\phi} r = \frac{\alpha(r,t,T) - r}{B(t,T)}$$

We use the value of $\alpha(0.08,1,5)$ provided in the question:

$$\bar{\phi} \times 0.08 = \frac{0.0818538 - 0.08}{B(1,5)}$$

$$0.0818538 = 0.08 + B(1,5)\bar{\phi}(0.08)$$

$$B(1,5)\bar{\phi} = 0.0231725$$

Making use of the fact that $B(1,5) = B(2,6)$, we have:

$$\bar{\phi} \times 0.07 = \frac{\alpha(0.07,2,6) - 0.07}{B(2,6)}$$

$$\begin{aligned}\alpha(0.07,2,6) &= 0.07 + B(2,6)\bar{\phi}(0.07) \\ &= 0.07 + 0.0231725(0.07) \\ &= 0.0716221\end{aligned}$$

Solution 19

D Chapter 12, Black-Scholes Formula w/ Discrete Dividends

The prepaid forward price can be written in terms of the forward price:

$$F_{t,T}^{P}(S) = e^{-r(T-t)}F_{t,T} \qquad \Rightarrow \qquad F_{t,T} = e^{r(T-t)}F_{t,T}^{P}(S)$$

The variance of the prepaid forward price is equal to the variance of the forward price:

$$\begin{aligned}Var\left[\ln\left(F_{t,1}^{P}(S)\right)\right] &= Var\left[\ln\left(e^{r(1-t)}F_{t,1}(S)\right)\right] \\ &= Var\left[\ln\left(e^{r(1-t)}\right) + \ln\left(F_{t,1}(S)\right)\right] \\ &= 0 + Var\left[\ln\left(F_{t,1}(S)\right)\right] = 0.04 \times t\end{aligned}$$

The prepaid forward volatility is:

$$\sigma_{PF} = \sqrt{\frac{Var\left[\ln\left(F_{t,1}^{P}(S)\right)\right]}{t}} = \sqrt{\frac{0.04t}{t}} = 0.20$$

The prepaid forward prices of the stock and the strike price are:

$$F_{0,T}^{P}(S) = 80 - 5.00e^{-0.10(3/12)} = 75.1235$$
$$F_{0,T}^{P}(K) = 85e^{-0.10\times 1} = 76.9112$$

We use the prepaid forward volatility in the Black-Scholes Formula:

$$d_1 = \frac{\ln\left(\dfrac{F_{0,T}^{P}(S)}{F_{0,T}^{P}(K)}\right) + 0.5\sigma_{PF}^2 T}{\sigma_{PF}\sqrt{T}} = \frac{\ln\left(\dfrac{75.1235}{76.9112}\right) + 0.5\times 0.20^2 \times 1}{0.20\sqrt{1}} = -0.01759$$

$$d_2 = \frac{\ln\left(\dfrac{F_{0,T}^{P}(S)}{F_{0,T}^{P}(K)}\right) - 0.5\sigma_{PF}^2 T}{\sigma_{PF}\sqrt{T}} = \frac{\ln\left(\dfrac{75.1235}{76.9112}\right) - 0.5\times 0.20^2 \times 1}{0.20\sqrt{1}} = -0.21759$$

We have:

$$N(-d_1) = N(0.01759) = 0.50702$$
$$N(-d_2) = N(0.21759) = 0.58613$$

The value of the European put option is:

$$\begin{aligned} P_{Eur}\left(F_{0,T}^{P}(S), F_{0,T}^{P}(K), \sigma, T\right) &= Ke^{-rT}N(-d_2) - \left[S_0 - PV_{0,T}(Div)\right]N(-d_1) \\ &= 76.9112\times 0.58613 - 75.1235\times 0.50702 \\ &= 6.9909 \end{aligned}$$

Solution 20

E Chapter 24, Risk-Neutral CIR Model

Ray uses the Cox-Ingersoll-Ross model for the short rate. The risk-neutral process for the short rate in the CIR model is:

$$\begin{aligned} dr &= \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z} \\ &= \left[a(b-r) + \bar{\phi} r\right]dt + \sigma\sqrt{r}d\tilde{Z} \end{aligned}$$

The risk-neutral version of Ray's model is:

$$
\begin{aligned}
dr &= \left[a(b-r)+\bar{\phi}r\right]dt+\sigma\sqrt{r}d\tilde{Z} \\
&= \left[0.3(0.12-r)+0.05r\right]dt+0.04\sqrt{r}d\tilde{Z} \\
&= 0.25(0.144-r)dt+0.04\sqrt{r}d\tilde{Z}
\end{aligned}
$$

Shelagh's model produces the same price as Ray's model if the risk-neutral version of her model is the same as the risk-neutral version of Ray's model.

Choices A and B use the Vasicek model instead of the CIR model, so we can rule them out.

The risk-neutral version of Choice C is not the same as the risk-neutral version of Ray's model:

$$
\begin{aligned}
dr &= \left[a(b-r)+\bar{\phi}r\right]dt+\sigma\sqrt{r}d\tilde{Z} \\
&\left[0.30(0.144-r)+0\times r\right]dt+0.04\sqrt{r}d\tilde{Z} \\
&0.30(0.144-r)dt+0.04\sqrt{r}d\tilde{Z}
\end{aligned}
$$

The risk-neutral version of Choice D is not the same as the risk-neutral version of Ray's model:

$$
\begin{aligned}
dr &= \left[a(b-r)+\bar{\phi}r\right]dt+\sigma\sqrt{r}d\tilde{Z} \\
&= \left[0.36(0.10-r)+0.05r\right]dt+0.04\sqrt{r}d\tilde{Z} \\
&= 0.31(0.1161-r)dt+0.04\sqrt{r}d\tilde{Z}
\end{aligned}
$$

The risk-neutral version of Choice E is the same as the risk-neutral version of Ray's model:

$$
\begin{aligned}
dr &= \left[a(b-r)+\bar{\phi}r\right]dt+\sigma\sqrt{r}d\tilde{Z} \\
&= \left[0.36(0.10-r)+0.11r\right]dt+0.04\sqrt{r}d\tilde{Z} \\
&= 0.25(0.144-r)dt+0.04\sqrt{r}d\tilde{Z}
\end{aligned}
$$

Therefore, Choice E is the solution.

Solution 21

A Chapter 18, Comparing Stock with a Risk-free Bond

Elizabeth's \$1,000 investment in the stock purchases the following quantity of stock at time 0:

$$
\frac{1{,}000}{S_0}
$$

Since Elizabeth reinvests the dividends, each share purchased will result in owning $e^{\delta T}$ shares at time T. Therefore, the time T payoff (including dividends) for Elizabeth is:

$$\frac{1{,}000}{S_0}e^{\delta T}S_T$$

Sadie's \$1,000 investment at the risk-free rate produces a time T payoff of:

$$1{,}000e^{rT}$$

The probability that Sadie's investment outperforms Elizabeth's investment is equal to the probability that the risk-free investment outperforms the stock:

$$\text{Prob}\left[\frac{1{,}000}{S_0}e^{\delta T}S_T < 1{,}000e^{rT}\right] = \text{Prob}\left[S_T < S_0 e^{(r-\delta)T}\right] = N(-\hat{d}_2)$$

$$\text{where: } K = S_0 e^{(r-\delta)T}$$

The value of $\hat{d}_2$ is:

$$\hat{d}_2 = \frac{\ln\left(\frac{S_0}{K}\right) + (\alpha - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S_0}{S_0 e^{(r-\delta)T}}\right) + (\alpha - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$= \frac{[(\alpha - r) - 0.5\sigma^2]\sqrt{T}}{\sigma}$$

The probability that Sadie's investment outperforms Elizabeth's investment is:

$$\text{Prob}\left[e^{\delta T}S_T < S_0 e^{rT}\right] = N(-\hat{d}_2) = N\left(\frac{0.5\sigma^2 - (\alpha - r)}{\sigma}\sqrt{T}\right)$$

The question tells us that this probability is equal to 50%:

$$N\left(\frac{0.5\sigma^2 - (\alpha - r)}{\sigma}\sqrt{T}\right) = 50\%$$

This implies that the value in parentheses above is 0:

$$\frac{0.5\sigma^2 - (\alpha - r)}{\sigma}\sqrt{T} = 0.00000$$

$$0.5\sigma^2 - (\alpha - r) = 0$$

$$0.5\sigma^2 - (0.12 - 0.08) = 0$$

$$\sigma^2 = 0.08$$

The Sharpe ratio is:

$$\frac{\alpha - r}{\sigma} = \frac{0.12 - 0.08}{\sqrt{0.08}} = 0.1414$$

Solution 22

A Chapter 19, Simulating Lognormal Stock Prices

Dolores uses the following steps to obtain a standard normal random variable based on the inverse cumulative normal distribution:

1. Obtain an observation $\hat{x}$ of a random variable x. This observation is 11.3.
2. Find the probability that x is less than $\hat{x}$:

$$F(\hat{x}) = \text{Prob}(x < \hat{x})$$

For a uniform distribution, we have:

$$F(x) = \frac{x-a}{b-a} \quad \text{for } U(a,b)$$

Therefore, the cumulative probability of 11.3 is:

$$F(11.3) = \frac{11.3-10}{15-10} = 0.26$$

3. Find the value of $\hat{z}$ that is the $F(\hat{x})$ quantile of the normal distribution:

$$N(\hat{z}) = F(\hat{x})$$
$$N(\hat{z}) = 0.26000$$
$$\hat{z} = N^{-1}\left[0.26000\right] = -0.64335$$

Thus –0.64335 is the observation from the standard normal distribution.

The Black-Scholes framework implies that the stock price follows a lognormal distribution, so Dolores' estimate is:

$$\begin{aligned} S_T &= S_t e^{(\alpha-\delta-0.5\sigma^2)(T-t)+\sigma z\sqrt{T-t}} \\ &= 75e^{(0.12-0.04-0.5\times 0.3^2)\times(3-0)+0.3\times(-0.64335)\sqrt{3-0}} \\ &= 59.6321 \end{aligned}$$

Roger sums the 12 uniform random variables:

$$\begin{aligned} &0.382 + 0.101 + 0.596 + 0.899 + 0.885 + 0.958 + 0.014 \\ &\quad + 0.407 + 0.863 + 0.139 + 0.245 + 0.045 = 5.534 \end{aligned}$$

The estimate for the standard normal draw is therefore:

$$\tilde{Z} = \sum_{i=1}^{12} u_i - 6 = 5.534 - 6 = -0.466$$

For the risk-neutral distribution of the stock, we replace α with r. The Black-Scholes framework implies that the stock price follows a lognormal distribution, so Roger's estimate is:

$$\begin{aligned} S_T &= S_t e^{(r-\delta-0.5\sigma^2)(T-t)+\sigma z\sqrt{T-t}} \\ &= 75e^{(0.07-0.04-0.5\times 0.3^2)\times(3-0)+0.3\times(-0.466)\sqrt{3-0}} \\ &= 56.2805 \end{aligned}$$

Dolores' estimate exceeds Roger's estimate by:

$$59.6321 - 56.2805 = 3.35$$

Solution 23

C Chapter 20, Geometric Brownian Motion Equivalencies

The usual formula for the forward price is:

$$F_{t,T} = S(t)e^{(r-\delta)(T-t)}$$

In this case, the underlying asset is one South African rand, and the South African interest rate is analogous to the dividend yield:

$$\begin{aligned} G(t) = F_{t,T} &= S(t)e^{(r-\delta)(T-t)} = S(t)e^{(r-r^*)(T-t)} = S(t)e^{(0.06-0.10)(T-t)} \\ &= S(t)e^{-0.04(T-t)} \end{aligned}$$

The forward price follows geometric Brownian motion:

$$\begin{aligned} G(t) = S(t)e^{-0.04(T-t)} &= S(0)e^{[0.07-0.5(0.35)^2]t+0.35Z(t)}e^{-0.04(T-t)} \\ &= S(0)e^{-0.04T}e^{[0.07-0.5(0.35)^2]t+0.35Z(t)}e^{0.04t} \\ &= G(0)e^{[0.11-0.5(0.35)^2]t+0.35Z(t)} \end{aligned}$$

Since the forward price follows geometric Brownian motion, we can use the following equivalency:

$$dG(t) = (\hat{\alpha}-\hat{\delta})G(t)dt + \hat{\sigma}G(t)dZ(t) \quad \Leftrightarrow \quad \mathrm{G}(t) = G(0)e^{(\hat{\alpha}-\hat{\delta}-0.5\hat{\sigma}^2)t+\hat{\sigma}[Z(t)]}$$

We use the ^ symbol above to avoid confusion with the parameters for S(t).

Since we have the rightmost expression above, we can express the left side as:

$$dG(t) = 0.11G(t)dt + 0.35G(t)dZ(t) = G(t)\left[0.11dt + 0.35dZ(t)\right]$$

Solution 24

B Chapter 18, Conditional and Partial Expectations

The risk-neutral partial expectation of the time 1 stock price, conditional on the time 1 stock price being above 98 is:

$$PE^*\left[S_T \middle| S_T > K\right] = \int_K^\infty S_T g^*(S_T; S_t) dS_T = \int_{98}^\infty S_1 g^*(S_1; S_{0.5}) dS_1 = 64.10$$

The risk-neutral probability that the final stock price is above 98 is found using the relationship below:

$$E^*\left[S_T \middle| S_T > K\right] = \frac{PE^*\left[S_T \middle| S_T > K\right]}{\text{Prob}^*\left(S_T > K\right)}$$

$$\text{Prob}^*\left(S_1 > 98\right) = \frac{PE^*\left[S_1 \middle| S_1 > 98\right]}{E^*\left[S_1 \middle| S_1 > 98\right]} = \frac{64.10}{118.14} = 0.5426$$

The risk-neutral expected value of the call at time 1 is therefore:

$$E^*\left[\text{Call Payoff}\right] = \left\{E^*\left[S_T \middle| S_T > K\right] - K\right\} \times \text{Prob}^*\left(S_T > K\right)$$
$$= \left\{118.14 - 98\right\} \times 0.5426 = 10.9275$$

Since we used the risk-neutral probability to obtain this expected value, we can use the risk-free rate of return to discount it back to time 0.5:

$$10.9275 \times e^{-0.09 \times 0.5} = 10.45$$

Solution 25

E Chapter 13, Re-hedging Frequency

The variance of Doug's daily profit on one call option is found by setting h equal to 1/365:

$$Var\left[R_h\right] = \frac{1}{2}\left(S^2\sigma^2\Gamma h\right)^2 = \frac{1}{2}\left(100^2 \times 0.24^2 \times 0.0328 \times \frac{1}{365}\right)^2 = 0.0013396$$

The variance of Bruce's daily profit on one call option is found by dividing Doug's daily variance by 24:

$$Var\left[R_h \middle| n \text{ re-hedgings}\right] = \frac{Var\left[R_h\right]}{n} = \frac{Var\left[R_{\frac{1}{365}}\right]}{24} = \frac{0.0013396}{24} = 0.000055817$$

Since Doug purchased 100 of the call options, the variance and standard deviation of his daily profit are:

$$\text{Variance: } Var\left[100R_h\right] = 100^2 Var\left[R_h\right] = 100^2 \times 0.0013396$$

$$\text{Standard Deviation: } X = 100 \times \sqrt{0.0013396} = 3.66006$$

Since Bruce also purchased 100 of the call options, the variance and standard deviation of his daily profit are:

$$\text{Variance: } 100^2 Var\left[R_h \middle| n \text{ re-hedgings}\right] = 100^2 \times 0.000055817$$

$$\text{Standard Deviation: } Y = 100 \times \sqrt{0.000055817} = 0.74711$$

The difference is:

$$X - Y = 3.66006 - 0.74711 = 2.91296$$

Solution 26

D Chapter 20, Volatility of Prepaid Forward

The prepaid forward price at time t is equal to the stock price minus the present value of the dividend:

$$F_{t,1}^P(S) = S(t) - 12e^{-0.10(0.5-t)}$$

We can take the differential and divide by $F_{t,1}^P(S)$. For $0 \le t < 0.5$, we have:

$$dF_{t,1}^P(S) = dS(t) - 1.2e^{-0.10(0.5-t)}dt$$

$$\frac{dF_{t,1}^P(S)}{F_{t,1}^P(S)} = \frac{dS(t)}{F_{t,1}^P(S)} + \frac{-1.2e^{-0.10(0.5-t)}}{F_{t,1}^P(S)}dt$$

$$\frac{dF_{t,1}^P(S)}{F_{t,1}^P(S)} = \frac{-1.2e^{-0.10(0.5-t)}}{F_{t,1}^P(S)}dt + \frac{dS(t)}{F_{t,1}^P(S)}$$

$$\frac{dF_{t,1}^P(S)}{F_{t,1}^P(S)} = \frac{-1.2e^{-0.10(0.5-t)}}{F_{t,1}^P(S)}dt + \frac{S(t)}{F_{t,1}^P(S)}\frac{dS(t)}{S(t)}$$

$$\frac{dF_{t,1}^P(S)}{F_{t,1}(S)} = \frac{-1.2e^{-0.10(0.5-t)}}{F_{t,1}^P(S)}dt + \frac{S(t)}{F_{t,1}^P(S)}\left[\alpha(t)dt + \sigma(t)dZ(t)\right]$$

$$\frac{dF_{t,1}^P(S)}{F_{t,1}(S)} = \left[\frac{\alpha(t)S(t) - 1.2e^{-0.10(0.5-t)}}{F_{t,1}^P(S)}\right]dt + \frac{S(t)}{F_{t,1}^P(S)}\sigma(t)dZ(t)$$

From the second term in the final expression above, we can see that the volatility parameter for the prepaid forward is:

$$\sigma_{PF} = \beta = \frac{S(t)}{F_{t,1}^P(S)}\sigma(t) \qquad \text{for } 0 \le t < 0.5$$

Since we know the time-0 values on the right side of the equation, we can find β, i.e., σ_{PF}:

$$\sigma_{PF} = \frac{S(0)}{F_{0,1}^{P}(S)}\sigma(0) = \frac{90}{90 - 12e^{-0.10(0.5-0)}}\sigma(0) = 1.1453 \times 0.30 = 0.3436$$

Solution 27

E Chapter 22, Cash-Or-Nothing Call Option

The cash-or-nothing put option pays:

$1{,}000 \qquad \text{if } S_T < 80$

The question provides information about a gap call option, but consider a gap put option with a strike price of 70 and a trigger price of 80. This gap put pays:

$70 - S_T \qquad \text{if } S_T < 80$

A regular put option with a strike price of 80 pays:

$80 - S_T \qquad \text{if } S_T < 80$

The cash-or-nothing put option can be replicated by purchasing 100 of the put options and selling 100 of the gap put options:

$$100\left[Put - GapPut\right] = 100\left[(80 - S_T) - (70 - S_T)\right] = 100 \times 10 = 1{,}000 \qquad \text{if } S_T < 80$$

We use put-call parity for gap calls and gap puts to find the theta of the gap put:

$$GapCall + K_1 e^{-r(T-t)} = Se^{-\delta(T-t)} + GapPut$$

$$GapCall + 70e^{-0.15(1-t)} = 70e^{-0.04(1-t)} + GapPut$$

$$\theta_{GapCall} + (0.15)70e^{-0.15(1-t)} = (0.04)70e^{-0.04(1-t)} + \theta_{GapPut}$$

$$-7.0 + (0.15)70e^{-0.15(1-t)} = (0.04)70e^{-0.04(1-t)} + \theta_{GapPut}$$

Evaluating at time $t = 0$, we have:

$$-7.0 + (0.15)70e^{-0.15(1-0)} = (0.04)70e^{-0.04(1-0)} + \theta_{GapPut}$$

$$\theta_{GapPut} = -0.6528$$

The theta of the cash-or-nothing put is the theta of a position consisting of 100 long puts and 100 short gap puts:

$$100\left[\theta_{Put} - \theta_{GapPut}\right] = 100\left[2.0 - (-0.6528)\right] = 100 \times 2.6528 = 265.28$$

Solution 28

B Chapter 24, Interest Rate Derivative

The realistic process for the short rate follows:

$$dr = a(r)dt + \sigma(r)dZ \qquad \text{where:} \qquad a(r) = 0.25(0.16 - r) \quad \& \quad \sigma(r) = 0.2\sqrt{r}$$

The risk-neutral process follows:

$$dr = \left[a(r) + \sigma(r)\phi(r,t)\right]dt + \sigma(r)d\tilde{Z}$$

We can use the coefficient of the first term of the risk-neutral process to solve for the Sharpe ratio, $\phi(r,t)$:

$$0.04 - 0.1r = a(r) + \sigma(r)\phi(r,t)$$
$$0.04 - 0.1r = 0.25(0.16 - r) + 0.2\sqrt{r}\phi(r,t)$$
$$0.04 - 0.1r = 0.04 - 0.25r + 0.2\sqrt{r}\phi(r,t)$$
$$0.15r = 0.2\sqrt{r}\phi(r,t)$$
$$\phi(r,t) = 0.75\sqrt{r}$$

The derivative, like all interest-rate dependent assets, must have a Sharpe ratio of $0.75\sqrt{r}$. Let's rearrange the differential equation for *g*, so that we can more easily observe its Sharpe ratio:

$$\frac{dg}{g} = \left(r + 0.12\sqrt{r}\right)dt - \beta dZ \quad \Rightarrow \quad \phi(r,t) = \frac{r + 0.12\sqrt{r} - r}{\beta} = \frac{0.12\sqrt{r}}{\beta}$$

Since the Sharpe ratio is $0.75\sqrt{r}$:

$$0.75\sqrt{r} = \frac{0.12\sqrt{r}}{\beta}$$
$$\beta = 0.16$$

Solution 29

D Chapter 12, Arbitrage with Options on Futures

The current value of the futures price is:

$$F_{0,T_F} = S_0 e^{(r-\delta)T_F} = 100e^{(0.03-0.08)4} = 81.8731$$

Gail uses put-call parity to see if arbitrage is available:

$$C_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) + Ke^{-rT} = F_{0,T_F}e^{-rT} + P_{Eur}(F_{0,T_F}, K, \sigma, r, T, r)$$
$$5 + 95e^{-0.03\times1} < 81.8731e^{-0.03\times1} + 20$$
$$97.1923 < 99.4533$$

As shown above, the left side is less than the right side. Therefore, Gail sells the right side and buys the left side (buy low, sell high).

Let's consider the put-call parity expression at time 1:

$$Max\left[0, F_{1,4} - 95\right] + 95 = F_{1,4} + Max\left[0, 95 - F_{1,4}\right]$$

In order to buy the left side, at time 0 Gail does the following:

- Buy the call
- Lend $95e^{-0.03}$

In order to sell the right side at time 0, Gail does the following:

- Sell the put
- Sell something that will have a price of $F_{1,4}$ at time 1.

At time 1, the futures price will be:

$$F_{T,T_F} = S_T e^{(r-\delta)(T_F - T)}$$
$$F_{1,4} = S_1 e^{(0.03-0.08)(4-1)} = S_1 e^{-0.15}$$

This suggests that having $e^{-0.15}$ shares of stock at time 1 is equivalent to having $F_{1,4}$ dollars at time 1. The prepaid forward price of $e^{-0.15}$ shares of stock is:

$$F_{0,1}^P(S)e^{-0.15} = e^{-\delta \times 1} S_0 e^{-0.15} = e^{-0.08} S_0 e^{-0.15} = 0.7945 S_0$$

Selling something now that will have a value of $F_{1,4}$ at time 1 is equivalent to selling 0.7945 shares of stock now.

Summing up, Gail takes the following steps to obtain arbitrage:

- Buy the call
- Lend $95e^{-0.03}$
- Sell the put
- Sell 0.7945 shares of stock

These steps are described in Choice (D).

Solution 30

D Chapter 9, Early Exercise

For each put option, the choice is between having the exercise value now or having a 1-year European put option. Therefore, the decision depends on whether the exercise value is greater than the value of the European put option. The value of each European put option is found using put-call parity:

$$C_{Eur}(K,T) + Ke^{-rT} = S_0 e^{-\delta T} + P_{Eur}(K,T)$$
$$P_{Eur}(K,T) = C_{Eur}(K,T) + Ke^{-rT} - S_0 e^{-\delta T}$$

The values of each of the 1-year European put options are:

$$P_{Eur}(25,1) = 23.32 + 25e^{-0.05(1)} - 50e^{-0.06(1)} = 0.01$$
$$P_{Eur}(50,1) = 4.47 + 50e^{-0.05(1)} - 50e^{-0.06(1)} = 4.94$$
$$P_{Eur}(70,1) = 0.58 + 70e^{-0.05(1)} - 50e^{-0.06(1)} = 20.08$$
$$P_{Eur}(100,1) = 0.01 + 100e^{-0.05(1)} - 50e^{-0.06(1)} = 48.04$$

In the third and fourth columns of the table below, we compare the exercise value with the value of the European put options. The exercise value is $Max(K - S_0, 0)$.

K	C	Exercise Value	European Put
\$25.00	\$23.32	0	0.01
\$50.00	\$4.47	0	4.94
\$70.00	\$0.58	20	20.08
\$100.00	\$0.01	50	48.04

The exercise value is less than the value of the European put option when the strike price is \$70 or less. When the strike price is \$100, the exercise value is greater than the value of the European put option. Therefore, it is optimal to exercise the special put option with an exercise price of \$100.

