Course MFE/3F Practice Exam 2 – Solutions

The chapter references below refer to the chapters of the ActuarialBrew.com Study Manual.

Solution 1

A Chapter 16, Black-Scholes Equation

The expressions for the value of the derivative and the partial derivatives are:

$$V = S + S^{-1}$$
$$V_S = 1 - S^{-2}$$
$$V_{SS} = 2S^{-3}$$
$$V_t = 0$$

From Statement (ii), we observe that $\sigma = 0.4$.

From the Black-Scholes equation, we have:

$$0.5\sigma^2 S^2 V_{SS} + (r-\delta)SV_S + V_t = rV$$
$$0.5(0.4)^2 S^2\left(2S^{-3}\right) + (r-0)S\left(1 - S^{-2}\right) + 0 = r\left(S + S^{-1}\right)$$
$$0.16S^{-1} + rS - rS^{-1} = rS + rS^{-1}$$
$$0.16S^{-1} - rS^{-1} = rS^{-1}$$
$$0.16 = 2r$$
$$r = 0.08$$

Solution 2

E Chapter 10, Compound Options

Using the formula for an American call option, we can find the value of the call on a put. Then we can use compound option parity to find the value of the ordinary put option.

We know that the value of an American call option is:

$$C_{Amer}(S_0, K, T) = S_0 - Ke^{-rt_1} + CallOnPut$$

The call on a put expires at time t_1 and has a strike price of:

$$x = D - K\left(1 - e^{-r(T-t_1)}\right)$$

In this case, we have:

$$x = D - K\left(1 - e^{-r(T-t_1)}\right) = 7 - 80.99\left(1 - e^{-0.085(1-0.5)}\right) = 3.63$$

Therefore the value of the compound call with a strike price of \$3.63 can be found as follows:

$$C_{Amer}(S_0, K, T) = S_0 - Ke^{-rt_1} + CallOnPut$$
$$8.75 = 80 - 80.99e^{-0.085(0.5)} + CallOnPut$$
$$CallOnPut = 6.37$$

We can now use the parity relationship to find the value of the European put option:

$$CallOnPut - PutOnPut = Put - xe^{-rt_1}$$
$$6.37 - 0.48 = Put - 3.63e^{-0.085(0.5)}$$
$$Put = 9.37$$

Solution 3

B Chapter 3, Two-Period Binomial Model

Since no dividend is mentioned, we assume that the dividend rate is zero.

The values of u and d are:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.07-0.00)(1)+0.25\sqrt{1}} = 1.37713$$
$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.07-0.00)(1)-0.25\sqrt{1}} = 0.83527$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.07-0.00)(1)} - 0.83527}{1.37713 - 0.83527} = 0.43782$$

The stock price tree and its corresponding tree of option prices are:

Stock			European Put		
		104.3064			0.0000
	75.7420			0.0000	
55.0000		63.2651	3.7443		0.0000
	45.9399			7.1433	
		38.3722			13.6278

The put option can be exercised only if the stock price falls to \$38.3722. At that point, the put option has a payoff of \$13.6278. Although the entire table of put prices is filled in above, we can directly find the value of the put option as follows:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \left[\binom{n}{j} (p^*)^j (1-p^*)^{n-j} V(S_0 u^j d^{n-j}, K, hn) \right]$$
$$= e^{-0.07(1)(2)} \left[0 + 0 + (1 - 0.43782)^2 (13.6278) \right] = 3.7443$$

Solution 4

A Chapter 7, Black-Scholes Call Price

The Black-Scholes Formula uses the risk-free interest rate, not the expected annual return.

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(48.96/50) + (0.07 - 0.06 + 0.5 \times 0.20^2) \times 0.25}{0.20\sqrt{0.25}}$$
$$= -0.13519$$
$$d_2 = d_1 - \sigma\sqrt{T} = -0.13519 - 0.20\sqrt{0.25} = -0.23519$$

We have:

$$N(d_1) = N(-0.13519) = 0.44623$$

$$N(d_2) = N(-0.23519) = 0.40703$$

The value of the European call option is:

$$C_{Eur} = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$
$$= 48.96e^{-0.06(0.25)} \times 0.44623 - 50e^{-0.07(0.25)} \times 0.40703 = 1.5237$$

Solution 5

E Chapter 5, Effect of Parameters on Call Option

Statement B is true because options on nondividend-paying stocks increase in value as their time to maturity increases when the strike price grows at the risk-free interest rate.

The probability that the call option expires in the money is:

$$\text{Prob}\left[S_T > K\right] = N(\hat{d}_2) \qquad \text{where: } K = S_0 e^{rT}$$

The value of $\hat{d}_2$ is:

$$\hat{d}_2 = \frac{\ln\left(\frac{S_0}{K}\right) + (\alpha - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S_0}{S_0 e^{rT}}\right) + (\alpha - 0 - 0.5\sigma^2)T}{\sigma\sqrt{T}}$$
$$= \frac{[(\alpha - r) - 0.5\sigma^2]\sqrt{T}}{\sigma}$$

The probability that the call option expires in the money is:

$$\text{Prob}\left[S_T > S_0 e^{rT}\right] = N(\hat{d}_2) = N\left(\frac{(\alpha - r) - 0.5\sigma^2}{\sigma}\sqrt{T}\right)$$

The question tells us that the risk premium is greater than $0.5\sigma^2$:

$$\alpha - r > 0.5\sigma^2 \qquad \Rightarrow \qquad (\alpha - r) - 0.5\sigma^2 > 0$$

Therefore $\hat{d}_2$ is positive:

$$\frac{[(\alpha - r) - 0.5\sigma^2]\sqrt{T}}{\sigma} > 0$$

We have established that the numerator in the fraction below is positive:

$$\text{Prob}\left[S_T > S_0 e^{rT}\right] = N\left(\frac{(\alpha - r) - 0.5\sigma^2}{\sigma}\sqrt{T}\right)$$

Therefore, the value in the parentheses becomes more positive as T increases, meaning that an increase in T increases the probability that the call option expires in the money. Therefore Statement A is true.

The value in the parenthesis becomes more positive as α increases, so an increase in α increases the probability that the call option expires in the money. Therefore, Statement D is true.

The risk-neutral probability is obtained by substituting r for α:

$$N\left(\frac{(r - r) - 0.5\sigma^2}{\sigma}\sqrt{T}\right) = N\left(\frac{-0.5\sigma^2}{\sigma}\sqrt{T}\right)$$

The value in the parentheses is now negative, meaning that an increase in T decreases the risk-neutral probability that the call option expires in the money. Therefore, Statement C is true.

The value in the parentheses is not affected by changes in r, so Statement E is false.

Solution 6

B Chapter 9, Delta-Hedging

The delta of a put option covering 1 share is:

$$\Delta_{Put} = -e^{-\delta T} N(-d_1) = -e^{-0.05\times 1} N(-0.41000) = e^{-0.05\times 1} \times (1 - 0.65910) = -0.324274$$

The market-maker sold 100 put options, each of which covers 100 shares. Therefore, from the perspective of the market-maker, the delta of the position to be hedged is:

$$-100 \times 100 \times (-0.324274) = 3{,}242.74$$

To delta-hedge the position, the market-maker sells 3,242.74 shares of stock.

When the stock price increases, the market-maker has a loss on the stock and a gain on the put options:

Gain on Stock:	$-3{,}242.74 \times (52-51) = -3{,}242.74$
Gain on Puts:	$100 \times 31.23 = 3{,}123.00$
Net Profit:	-119.74

The market-maker has a net profit of –119.74.

Solution 7

A Chapter 9, Delta-Gamma Approximation

The delta-gamma approximation for the new price is:

$$V(t+h) \approx V(t) + \varepsilon\Delta_t + \frac{\varepsilon^2}{2}\Gamma_t$$

The change in the stock price is:

$$\varepsilon = S_{t+h} - S_t = 51.50 - 50.00 = 1.50$$

The delta-gamma approximation is:

$$\begin{aligned} V(t+h) &\approx V(t) + \varepsilon\Delta_t + \frac{\varepsilon^2}{2}\Gamma_t \\ &= 2.49 + (1.50)(-0.418) + \frac{1.50^2}{2}(0.052) \\ &= 1.9215 \end{aligned}$$

Solution 8

B Chapter 15, Sharpe Ratio

When the price follows geometric Brownian motion, the natural log of the price follows arithmetic Brownian motion:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t) \quad \Leftrightarrow \quad d\left[\ln S(t)\right] = \left(\alpha - 0.5\sigma^2\right)dt + \sigma dZ(t)$$

Therefore:

$$\frac{dY(t)}{Y(t)} = Adt + BdZ(t) \quad \Leftrightarrow \quad d\left[\ln Y(t)\right] = \left(A - 0.5B^2\right)dt + BdZ(t)$$

The arithmetic Brownian motion provided in the question for $d\left[\ln Y(t)\right]$ allows us to find an expression for C and to solve for B.

This implies that B is 0.20:

$$d\left[\ln Y(t)\right] = Cdt + 0.20dZ(t) \quad \text{and} \qquad d\left[\ln Y(t)\right] = \left(A - 0.5B^2\right)dt + BdZ(t)$$

$$A - 0.5B^2 = C$$

$$B = 0.20$$

Since X and Y are perfectly positively correlated, they must have the same Sharpe ratio. This allows us to solve for A:

$$\frac{0.10 - 0.07}{0.15} = \frac{A - 0.07}{0.20}$$

$$A = 0.11$$

We can now determine C:

$$C = A - 0.5B^2 = 0.11 - 0.5 \times 0.2^2 = 0.09$$

Solution 9

C Chapter 4, Binomial Model & Currency Options

The values of u and d are:

$$u = e^{(r - r_f)h + \sigma\sqrt{h}} = e^{(0.10 - 0.06)1 + 0.30\sqrt{1}} = 1.40495$$

$$d = e^{(r - r_f)h - \sigma\sqrt{h}} = e^{(0.10 - 0.06)1 - 0.30\sqrt{1}} = 0.77105$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.30\sqrt{1}}} = 0.42556$$

The tree of prices for one British pound is below:

		3.9478
	2.8099	
2.0000		2.1666
	1.5421	
		1.1890

The tree of prices for an American put option on one British pound is below:

		0.0000
	0.1733	
0.5646		0.3334
	0.9579	
		1.3110

The bold entry indicates that early exercise is optimal at that node.

If the pound goes down in value in the first month, then the value of holding the put option is:

$$e^{-0.10}\left[0.42556\times 0.3334+(1-0.42556)\times 1.3110\right]=0.8098$$

The value of exercising the option then is:

$$2.50-1.5421=0.9579$$

The value of the option is the maximum of the value of holding it and the value of exercising it, so its value after 1 downward movement is $0.9579.

Continuing with the recursive calculations, the value of the put option at time 0 is:

$$e^{-0.10}\left[0.42556\times 0.1733+(1-0.42556)\times 0.9579\right]=0.5646$$

The price of a put option on 100 British pounds is:

$$100\times 0.5646=56.46$$

Solution 10

A Chapters 3 and 4, Greeks in the Jarrow-Rudd Binomial Model

How do we know that gamma in the question refers to Γ and not γ? Because we would need to know the realistic probability of an upward movement in order to determine the expected return on the option, γ, but there is no way of knowing the realistic probability of an upward movement.

The values of u and d are:

$$u=e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}}=e^{(0.11-0.04-0.5\times 0.32^2)(1)+0.32\sqrt{1}}=1.40326$$

$$d=e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}}=e^{(0.11-0.04-0.5\times 0.32^2)(1)-0.32\sqrt{1}}=0.73993$$

The stock price tree and its corresponding tree of option prices are:

Stock			Call		
		78.7658			40.7658
	56.1305			19.8879	
40.0000		41.5326	9.6414		3.5326
	29.5972			1.5867	
		21.8998			0.0000

There is no need to calculate the current value or the time 1 values of the call to answer this question, but they are provided in the tree above for completeness.

We need to calculate the two possible values of delta at time 1:

$$\Delta(Su,h)=e^{-\delta h}\frac{V_{uu}-V_{ud}}{Su^2-Sud}=e^{-0.04\times 1}\frac{40.7658-3.5326}{78.7658-41.5326}=0.9608$$

$$\Delta(Sd,h)=e^{-\delta h}\frac{V_{ud}-V_{dd}}{Sud-Sd^2}=e^{-0.04\times 1}\frac{3.5326-0.0000}{41.5326-21.8998}=0.1729$$

We can now calculate gamma:

$$\Gamma(S,0) \approx \Gamma(S_h,h) = \frac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd} = \frac{0.9608 - 0.1729}{56.1305 - 29.5972} = 0.0297$$

Solution 11

D Chapter 19, Vasicek Model

The Vasicek model of short-term interest rates is:

$$dr = a(b - r)dt + \sigma dZ$$

Therefore, we can determine the value of a:

$$dr = 0.4(b - r)dt + \sigma dZ \qquad \Rightarrow \qquad a = 0.4$$

In the Vasicek model, the Sharpe ratio is constant:

$$\phi(r,t) = \phi$$

Therefore, for any r, t, and T, we have:

$$\phi = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

Since the Sharpe ratio is constant:

$$\frac{\alpha(0.08,0,3) - 0.08}{q(0.08,0,3)} = \frac{\alpha(0.09,2,6) - 0.09}{q(0.09,2,6)}$$

We now make use of the following formula for $q(r,t,T)$:

$$q(r,t,T) = B(t,T)\sigma(r) = B(t,T)\sigma$$

Substituting this expression for $q(r,t,T)$ into the preceding equation allows us to solve for $\alpha(0.09,2,6)$:

$$\frac{\alpha(0.08,0,3)-0.08}{q(0.08,0,3)}=\frac{\alpha(0.09,2,6)-0.09}{q(0.09,2,6)}$$

$$\frac{\alpha(0.08,0,3)-0.08}{B(0,3)\sigma}=\frac{\alpha(0.09,2,6)-0.09}{B(2,6)\sigma}$$

$$\frac{0.0826205-0.08}{\frac{1-e^{-0.4(3-0)}}{0.4}\sigma}=\frac{\alpha(0.09,2,6)-0.09}{\frac{1-e^{-0.4(6-2)}}{0.4}\sigma}$$

$$\frac{0.0026205}{1-e^{-0.4(3)}}=\frac{\alpha(0.09,2,6)-0.09}{1-e^{-0.4(4)}}$$

$$\frac{0.0026205}{0.6988}=\frac{\alpha(0.09,2,6)-0.09}{0.7981}$$

$$\alpha(0.09,2,6)=0.0929929$$

Solution 12

A Chapter 10, Collect-on-Delivery Call

The COD call is a gap-call that has a strike price of $K_1 = 75 + P$ and a trigger price of $K_2 = 75$.

The first step to pricing the gap call is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K_2)+(r-\delta+0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(80/75)+(0.04-0.09+0.5\times 0.24^2)\times 1}{0.24\sqrt{1}}$$

$$= 0.18058$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.18058 - 0.24\sqrt{1} = -0.05942$$

We have:

$$N(d_1) = N(0.18058) = 0.57165$$

$$N(d_2) = N(-0.05942) = 0.47631$$

The initial price of the gap call is:

$$\text{GapCall} = Se^{-\delta T}N(d_1) - K_1 e^{-rT}N(d_2)$$

$$= 80e^{-0.09\times 1}\times 0.57165 - (75+P)e^{-0.04\times 1}\times 0.47631$$

Since this gap call is a COD call, its initial price is zero:

$$0 = 80e^{-0.09\times 1} \times 0.57165 - (75 + P)e^{-0.04\times 1} \times 0.47631$$

$$-\frac{80e^{-0.09} \times 0.57165}{e^{-0.04} \times 0.47631} = -75 - P$$

$$P = 16.3305$$

If the final stock price is \$111, then the payoff is:

$$S(T) - K - P = 111 - 75 - 16.3305 = 19.6695$$

Solution 13

B Chapter 1, Exchange Options

Let Stock A be the strike asset and let 8 shares of Stock B be the underlying asset.

The value of 8 shares of Stock B in dollars is:

$$8 \times 25 \times 1.20 = 240 \text{ dollars}$$

We can use put-call parity to find the value of an option that allows its owner to exchange 8 shares of Stock B for 1 share of Stock A:

$$C_{Eur}\left(240, 230, \tfrac{10}{12}\right) - P_{Eur}\left(240, 230, \tfrac{10}{12}\right) = F^P_{0,\frac{10}{12}}(8B) - F^P_{0,\frac{10}{12}}(A)$$

$$35.95 - P_{Eur}\left(240, 230, \tfrac{10}{12}\right) = 240 - 230e^{-0.05\times 10/12}$$

$$P_{Eur}\left(240, 230, \tfrac{10}{12}\right) = 16.5636$$

The put option gives its owner the right to give up 8 shares of Stock B and receive 1 share of Stock A. The right to give up 24 shares of Stock B and receive 3 shares of Stock A is the same as owning 3 of these put options:

$$3 \times 16.5636 = 49.69$$

The value of a an exchange option that gives its owner the right to exchange 24 shares of Stock B for 3 shares of Stock A at the end of 10 months is \$49.69.

Solution 14

E Chapter 18, Lognormal Prediction Intervals

For a 90% lognormal prediction interval, we set $p = 10\%$ in the expression below:

$$S_T^U = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T-t) + \sigma z^U \sqrt{T-t}} \qquad \text{where: } P\left(z > z^U\right) = \frac{p}{2}$$

First, we use the standard normal calculator to determine z^U:

$$P\left(z > z^U\right) = \frac{0.10}{2}$$
$$P\left(z > z^U\right) = 0.05$$
$$1 - P\left(z < z^U\right) = 0.05$$
$$P\left(z < z^U\right) = 0.95$$
$$z^U = 1.64485$$

The upper bound is:

$$S_T^U = 50e^{(0.12-0-0.5(0.30)^2)(1-0)+0.3(1.64485)\sqrt{1-0}} = 88.2769$$

Solution 15

E Chapter 4, Utility Values and State Prices

The payoffs of the put option are:

$$V_u = Max(0, 60-80) = 0$$
$$V_d = Max(0, 60-35) = 25$$

The price of the put option is:

$$V = Q_u V_u + Q_d V_d = pU_u \times 0 + (1-p)U_d \times 25 = 0.40(0.86)\times 0 + 0.60(0.97)\times 25$$
$$= 14.55$$

The expected return on the put option can now be calculated:

$$(1+\gamma_{Put})^{0.5} = \frac{pV_u + (1-p)V_d}{V}$$
$$(1+\gamma_{Put})^{0.5} = \frac{0.40\times 0 + 0.60\times 25}{14.55}$$
$$\gamma_{Put} = 0.0628$$

Solution 16

A Chapter 8, Volatility of an Option

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$
$$= \frac{\ln(50/55) + (0.03 - 0.00 + 0.5\times 0.35^2)\times 1}{0.35\sqrt{1}} = -0.01160$$
$$d_2 = d_1 - \sigma\sqrt{T} = -0.01160 - 0.35\sqrt{1} = -0.36160$$

We have:

$$N(-d_1) = N(0.01160) = 0.50463$$

$$N(-d_2) = N(0.36160) = 0.64117$$

The value of the put option is:

$$\begin{aligned} P_{Eur}(S,K,\sigma,r,T,\delta) &= Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1) \\ &= 55e^{-0.03(1)} \times 0.64117 - 50e^{-0.0(1)} \times 0.50463 \\ &= 8.9906 \end{aligned}$$

The value of delta is:

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{0.0(1)}(0.50463) = -0.50463$$

The elasticity of the option is:

$$\Omega = \frac{S\Delta}{V} = \frac{50 \times (-0.50463)}{8.9906} = -2.8064$$

The volatility of the put option is:

$$\sigma_{Option} = \sigma_{Stock} \times \left|\Omega_{Option}\right| = 0.35 \times 2.8064 = 0.9822$$

Solution 17

E Chapter 19, Cox-Ingersoll-Ross Model

In the Cox-Ingersoll-Ross Model, we have:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$

We use the following two facts about the CIR model:

- $A(t,T)$ and $B(t,T)$ do not depend on r.
- $A(t,T) = A(0,T-t)$ and $B(t,T) = B(0,T-t)$. This implies:

$$A(0,3) = A(2,5) = A(3,6)$$
$$B(0,3) = B(2,5) = B(3,6)$$

We have two equations and two unknowns:

$$A(0,3)e^{-B(0,3)(0.08)} = 0.7798$$

$$A(0,3)e^{-B(0,3)(0.10)} = 0.7465$$

Dividing the second equation into the first equation allows us to find $B(0,3)$:

$$e^{-B(0,3)(0.08)+B(0,3)(0.10)} = \frac{0.7798}{0.7465}$$

$$B(0,3)(0.02) = \ln\left(\frac{0.7798}{0.7465}\right)$$

$$B(0,3) = 2.18209$$

We can now solve for the value of $A(0,3)$:

$$A(0,3)e^{-B(0,3)(0.08)} = 0.7798$$

$$A(0,3) = 0.7798e^{B(0,3)(0.08)}$$

$$A(0,3) = 0.7798e^{2.18209(0.08)}$$

$$A(0,3) = 0.92853$$

We can now solve for r^*:

$$A(0,3)e^{-B(0,3)(r^*)} = 0.7146$$

$$0.92853e^{-2.18209r^*} = 0.7146$$

$$-2.18209r^* = \ln\left(\frac{0.7146}{0.92853}\right)$$

$$r^* = 0.1200$$

Solution 18

B Chapter 1, Synthetic T-bills

The T-bill is replicated by purchasing stock, selling a call option, and buying a put:

$$Ke^{-rT} = S_0e^{-\delta T} - C_{Eur}(K,T) + P_{Eur}(K,T)$$

$$40e^{-r(0.75)} = 36e^{-0.04(0.75)} - 2.87 + 4.68$$

$$e^{-r(0.75)} = 0.918651$$

$$-0.75r = \ln(0.918651)$$

$$r = 0.1131$$

Solution 19

B Chapter 7, Black-Scholes Formula

The volatility parameter is:

$$\sigma = \sqrt{\frac{Var[\ln S_t]}{t}} = \sqrt{\frac{0.3t}{t}} = \sqrt{0.3}$$

We can use the version of the Black-Scholes formula that is based on prepaid forward prices to find the value of the put option:

$$d_1 = \frac{\ln\left(\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{75}{75}\right) + 0.5 \times 0.3 \times 5}{\sqrt{0.3} \times \sqrt{5}} = 0.61237$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.61237 - \sqrt{0.3} \times \sqrt{5} = -0.61237$$

$$N(-d_1) = N(-0.61237) = 0.27015$$

$$N(-d_2) = N(0.61237) = 0.72985$$

The price of the put option is:

$$\begin{aligned} P_{Eur}\left(F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T\right) &= F_{0,T}^P(K)N(-d_2) - F_{0,T}^P(S)N(-d_1) \\ &= 75 \times 0.72985 - 75 \times 0.27015 \\ &= 34.4775 \end{aligned}$$

Solution 20

B Chapter 15, Forward Price of S^a

The drift of the risk-neutral Itô process for the stock is equal to the difference between the risk-free rate and the dividend yield:

$$0.06dt = (r - \delta)dt$$

$$0.06 = r - \delta$$

The expected value of the claim under the risk-neutral valuation measure is equal to the forward price of the claim:

$$\begin{aligned} E^*\left[\left(S(T)\right)^a\right] = F_{0,T}\left[\left(S(T)\right)^a\right] &= \left[S(0)\right]^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right]T} \\ &= \left[5\right]^{-1} e^{\left[-1(0.06)+0.5(-1)(-1-1)0.25^2\right]4} \\ &= 0.20e^{0.01} \\ &= 0.2020 \end{aligned}$$

Solution 21

B Chapter 5, Expected Value and Median of Stock Price

From the distribution of the natural log of the stock price, we observe that:

$$\sigma^2 = 0.16 \quad \Rightarrow \quad \sigma = 0.4$$

The Sharpe ratio of the call option is equal to the Sharpe ratio of the stock:

$$\frac{\alpha - r}{\sigma} = 0.125 \quad \Rightarrow \quad \frac{\alpha - 0.05}{0.4} = 0.125 \quad \Rightarrow \quad \alpha = 0.10$$

The expected value of $S(3)$ is:

$$E[S(t)] = S(0)e^{(\alpha - \delta)t}$$
$$E[S(3)] = 65e^{(0.10-0.02)3} = 82.6312$$

The median value of $S(3)$ is:

$$S(0)e^{(\alpha - \delta - 0.5\sigma^2)t} = 65e^{(0.10-0.02-0.5\times 0.4^2)3} = 65$$

The expected value exceeds the median by:

$$82.6312 - 65 = 17.6312$$

Solution 22

B Chapter 19, Theta in the Cox-Ingersoll-Ross Model

The process describes the Cox-Ingersoll-Ross-Model with:

$$a(r) = a \times (b - r) = 0.13(0.07 - r)$$
$$\sigma(r) = \sigma\sqrt{r} = 0.18\sqrt{r}$$

The price of a 15-year bond is:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r} = 0.6184e^{-4.7193\times 0.08} = 0.4239$$

The formula for the price of a bond must satisfy the following partial differential equation:

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + \left[a(r) + \sigma(r)\phi(r,t)\right]P_r + P_t$$

Delta and gamma for the bond are:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$

Delta: $P_r = -B(t,T)P(r,t,T) = -4.7193 \times 0.4239 = -2.0007$

Gamma: $P_{rr} = [B(t,T)]^2 P(r,t,T) = (4.7193)^2 \times 0.4239 = 9.4419$

We can now solve for theta:

$$rP = \frac{1}{2}[\sigma(r)]^2 P_{rr} + \left[a(r) + \sigma(r)\phi(r,t)\right]P_r + P_t$$

$$P_t = rP - \frac{1}{2}[\sigma(r)]^2 P_{rr} - \left[a(r) + \sigma(r)\phi(r,t)\right]P_r$$

$$P_t = 0.08 \times 0.4239 - \frac{1}{2}\left(0.18\sqrt{0.08}\right)^2 \times 9.4419$$
$$-\left[0.13(0.07 - 0.08) + 0.18\sqrt{0.08} \times (0)\right](-2.0007)$$
$$= 0.0191$$

Solution 23

C Chapter 12, Variance of Control Variate Estimate

The formula from the ActuarialBrew.com Study Manual that has X as the control variate and Y^* as the control variate estimate is:

$$Var\left[Y^*\right] = Var\left[\bar{Y}\right]\left(1-\rho^2_{\bar{X},\bar{Y}}\right)$$

In this question, however, the control variate is denoted by A and the control variate estimate is denoted by B^*, so we have:

$$Var\left[B^*\right] = Var\left[\bar{B}\right]\left(1-\rho^2_{\bar{A},\bar{B}}\right) = 4^2\left(1-0.6^2\right) = 10.24$$

Solution 24

D Chapter 14, Geometric Brownian Motion and Mutual Funds

Since b is greater than 1, the mutual fund takes a leveraged position in the stock by borrowing at the risk-free rate.

The instantaneous percentage increase of the mutual fund is the weighted average of the return on the stock (including its dividend yield) and the return on the risk-free asset:

$$\begin{aligned}\frac{dW(t)}{W(t)} &= b\left[\frac{dS(t)}{S(t)}+\delta dt\right]+(1-b)rdt\\ &= 1.3\left[0.07dt+0.25dZ(t)+0.05dt\right]-0.3(0.06)dt\\ &= 1.3\left[0.12dt+0.25dZ(t)\right]-0.3(0.06)dt\\ &= 0.156dt+0.325dZ(t)-0.018dt\\ &= 0.138dt+0.325dZ(t)\end{aligned}$$

As written above, we see that $W(t)$ is a geometric Brownian motion. Therefore, the expected value can be expressed as:

$$E\left[W(t)\right] = W(0)e^{0.138t}$$
$$E\left[W(5)\right] = 50e^{0.138\times 5}$$
$$E\left[W(5)\right] = 99.6858$$

Solution 25

D Chapter 17, Caplets and the Black Model

This question's method of presenting forward price volatilities is the same as that used in Problem 25.2 at the end of the Derivatives Markets textbook Chapter 25.

The price of the caplet is:

$$\text{Caplet price} = (1.11)\times(\text{Put Option Price})$$

The put option:

- has a zero-coupon bond that expires at time 4 as its underlying asset, so $T+s=4$
- expires at time 3, so $T=3$
- has a strike price of:

$$K=\frac{1}{1+K_R}=\frac{1}{1.11}=0.90090$$

The forward price volatilities are:

$$0.10=\sqrt{\frac{Var\left\{\ln\left[P_t(1,2)\right]\right\}}{t}} \quad 0.12=\sqrt{\frac{Var\left\{\ln\left[P_t(2,3)\right]\right\}}{t}} \quad 0.15=\sqrt{\frac{Var\left\{\ln\left[P_t(3,4)\right]\right\}}{t}}$$

The appropriate volatility for an option that expires in 3 years on a bond that matures in 4 years is:

$$\sigma=0.15$$

The bond forward price is:

$$F=P_0(T,T+s)=\frac{P(0,T+s)}{P(0,T)}=\frac{P(0,4)}{P(0,3)}=\frac{0.68}{0.77}=0.88312$$

The values of d_1 and d_2 are:

$$d_1=\frac{\ln\left(\frac{F}{K}\right)+0.5\sigma^2T}{\sigma\sqrt{T}}=\frac{\ln\left(\frac{0.88312}{0.90090}\right)+0.5(0.15)^2(3)}{0.15\sqrt{3}}=0.05316$$
$$d_2=d_1-\sigma\sqrt{T}=0.05316-0.15\sqrt{3}=-0.20665$$

We have:

$$N(-d_1)=N(-0.05316)=0.47880$$

$$N(-d_2)=N(0.20665)=0.58186$$

The Black formula for the put price is:

$$\begin{aligned}P&=P(0,T)\left[K\times N(-d_2)-F\times N(-d_1)\right]\\&=0.77[0.90090\times 0.58186-0.88312\times 0.47880]\\&=0.078049\end{aligned}$$

The price of the caplet for \$1 of borrowing is:

$$\text{Caplet price}=(1.11)\times(\text{Put Option Price})=1.11\times 0.078049=0.086634$$

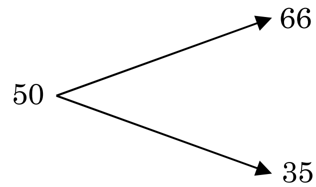
The price of the caplet for \$1,000 of borrowing is:

$$1{,}000\times 0.086634=86.63$$

Solution 26

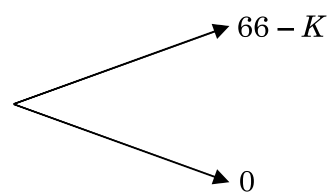
C Chapter 4, American Call Option & State Prices

The stock price tree is:



The strike price K, for which an investor will exercise the call option at the beginning of the period must be less than $50, since otherwise the payoff to immediate exercise would be zero. Since we are seeking the largest strike price that results in immediate exercise, let's begin by determining whether there is a strike price that is less than $50 but more than $35 that results in immediate exercise.

If $35 < K < 50$, then the rightmost portion of the tree for the call option is:



If there is strike price that is greater than $35 that results in immediate exercise, then the value of exercising now must exceed the value of holding the option:

$$50 - K > (66 - K) \times Q_H$$
$$50 - K > (66 - K) \times 0.42$$
$$50 - K > 27.72 - 0.42K$$
$$22.28 > 0.58K$$
$$38.4138 > K$$

The inequality above suggests that $38.4138 is the largest strike price for which the investor will exercise the call option immediately.

The largest integer that satisfies this inequality is:

$$\text{INT}(38.4138) = 38$$

The difference is:

$$K - \text{INT}(K) = 38.4138 - 38 = 0.4138$$

Solution 27

B Chapter 15, Sharpe Ratio and Option Volatility

The stock and the option must have the same market price of risk at all times, including time 2:

$$\frac{\alpha - 0.08}{\sigma} = \frac{-0.33 - 0.08}{\sigma_V}$$

When an option is purchased, it is delta-hedged by purchasing $-\Delta$ shares of stock. The cost of the shares required to delta-hedge the option is the number of shares required, $-\Delta$, times the cost of each share, S. Therefore, based on statement (vi) in the question, we have:

$$-\Delta \times S = 27$$
$$\Delta \times S = -27$$

The fact that Δ is negative is not surprising in light of the fact that the option's expected return is negative.

We can find the volatility parameter of the option in terms of the volatility of the underlying stock:

$$\sigma_V = \sigma \times \Omega_V = \sigma \times \frac{S \times \Delta}{V} = \sigma \times \frac{-27}{5.30} = -5.0943\sigma$$

We can now solve for α :

$$\frac{\alpha - 0.08}{\sigma} = \frac{-0.33 - 0.08}{-5.0943\sigma} \qquad \Rightarrow \qquad \alpha = 0.1605$$

The general form for a dividend-paying stock that follows geometric Brownian motion is:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)$$

We can use the differential equation provided in the question to find the dividend yield:

$$\frac{dS(t)}{S(t)} = 0.125dt + \sigma dZ(t) \qquad \Rightarrow \qquad 0.125 = \alpha - \delta$$

$$0.125 = 0.1605 - \delta$$
$$\delta = 0.03548$$

Solution 28

B Chapter 11, Cash-or-Nothing Call Options

The option can be replicated by purchasing 200 cash calls with a strike price of \$50 and selling 100 cash calls with a strike price of \$100. Therefore, the price of the option is:

$$200CashCall(50) - 100CashCall(100)$$

First, we find the value of the cash call with a strike price of \$50:

$$d_2 = \frac{\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100e^{-0.04\times 1}}{50e^{-0.11\times 1}}\right) - \frac{0.42^2}{2}\times 1}{0.42\sqrt{1}} = 1.60702$$

$$N(d_2) = N(1.60702) = 0.94598$$

$$CashCall(50) = e^{-rT}N(d_2) = e^{-0.11\times 1}(0.94598)$$

Next, we find the value of the cash call with a strike price of \$100:

$$d_2 = \frac{\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100e^{-0.04\times 1}}{100e^{-0.11\times 1}}\right) - \frac{0.42^2}{2}\times 1}{0.42\sqrt{1}} = -0.04333$$

$$N(d_2) = N(-0.04333) = 0.48272$$

$$CashCall(100) = e^{-rT}N(d_2) = e^{-0.11\times 1}(0.48272)$$

The value of the option described in the question is:

$$\begin{aligned} &200CashCall(50) - 100CashCall(100) \\ &= 200e^{-0.11}(0.94598) - 100e^{-0.11}(0.48272) \\ &= 126.2445 \end{aligned}$$

Solution 29

E Chapter 1, Currency Options

Greg's call option gives him the right to give up \$1.25 to receive £1.00.

Marcia's put option gives her the right to give up \$1.00 to receive £0.80.

Note that the payoff of Greg's options is 1.25 times the payoff of Marcia's option. Therefore, Greg's option is worth 1.25 times Marcia's option.

The value of Greg's option in dollars is Z, and the value of Greg's option in pounds is:

$$\frac{Z}{1.41}$$

The value of Greg's option in pounds is 1.25 times as much as the value of Marcia's option:

$$\frac{Z}{1.41} = 1.25Y$$

We can now solve for the ratio of Z to Y:

$$\frac{Z}{1.41} = 1.25Y \quad \Rightarrow \quad \frac{Z}{Y} = 1.41 \times 1.25 = 1.7625$$

Solution 30

A Chapter 7, Options on Futures

We use put-call parity to find the current futures price:

$$C_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) + Ke^{-rT} = F_{0,T_F}e^{-rT} + P_{Eur}(F_{0,T_F}, K, \sigma, r, T, r)$$

$$15.61 + 133e^{-0.08\times 1} = F_{0,3}e^{-0.08\times 1} + 13.90$$

$$F_{0,3} = 134.8524$$

We can use the futures price to solve for the stock price:

$$F_{0,T_F} = S_0 e^{(r-\delta)T_F}$$

$$134.8524 = S_0 e^{(0.08-0.05)\times 3}$$

$$S_0 = 123.2458$$