MFE/3F Study Manual Sample from Chapter 10

Exotic Options

Online Excerpt of Section 10.4

Introduction

This document provides an excerpt of Section 10.4 of the ActuarialBrew.com Study Manual.

Our Study Manual for the MFE/3F Exam is a detailed explanation of the required material that helps you quickly understand the concepts as you prepare for the exam. Focusing on the more difficult areas, our Study Manual covers all of the MFE/3F learning objectives.

Please note that the Study Manual contains some worked examples but the approximately 650 Questions are available as a separate product. Student may choose whether or not to bundle the Study Manual and Questions together.

Free email support is provided to students who purchase our MFE/3F Study Manual.

Our goal is to help you pass the actuarial exams on your first attempt by brewing better actuarial exam preparation products.

10.4 Gap Options

A **gap option** has a strike price, K_1, and a **trigger price**, K_2. The trigger price determines whether or not the gap option will have a nonzero payoff. The strike price determines the amount of the nonzero payoff. The strike price may be greater than or less than the trigger price.

If the strike price is equal to the trigger price, then the gap option is an ordinary option.

A gap call option has a nonzero payoff (which may be positive or negative) if the final stock price exceeds the trigger price.

Gap Call Option

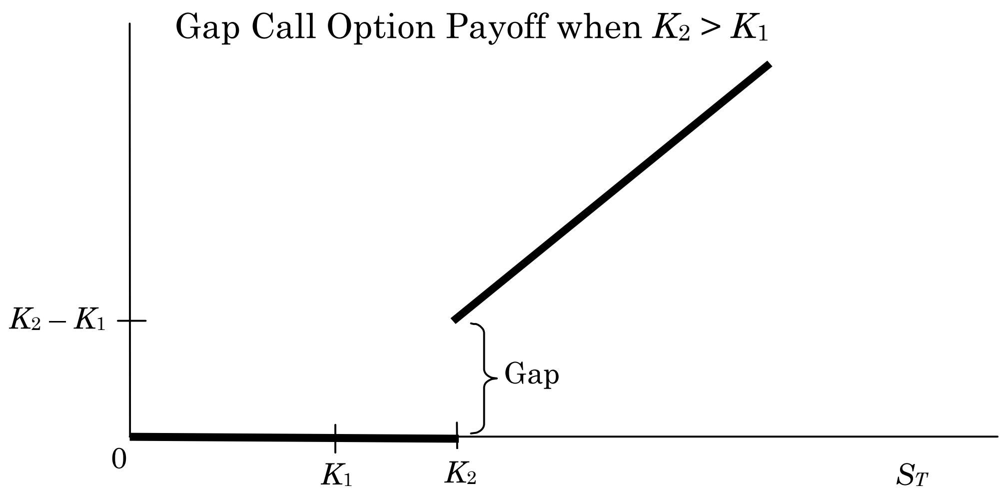
A gap call option has a payoff of:

$$\text{Gap call option payoff} = \begin{cases} S_T - K_1 & \text{if } S_T > K_2 \\ 0 & \text{if } S_T \le K_2 \end{cases}$$

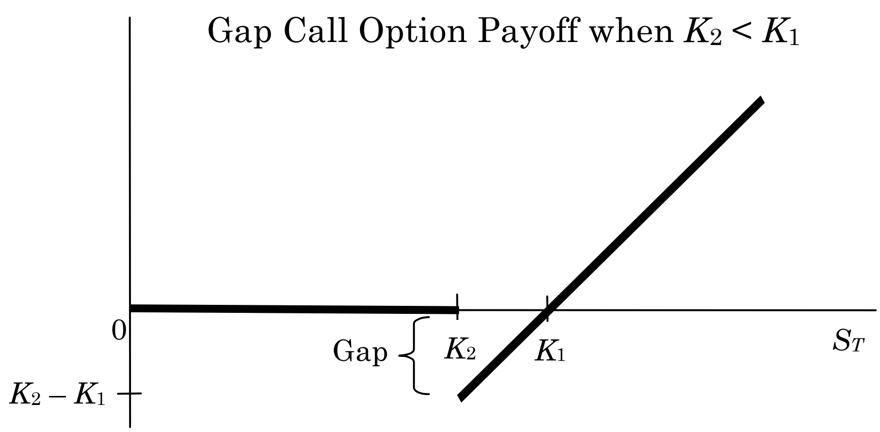
where:

$K_1 =$ Strike price
$K_2 =$ Trigger price

If we graph the payoff of a gap call option as a function of its final stock price, then we can see that there is a gap where $S_T = K_2$. In the graph below, there are no negative payoffs because the trigger price is greater than the strike price:



If the trigger price is less than the strike price for a gap call option, then negative payoffs are possible as shown below:



A gap put option has a nonzero payoff if the final stock price is less than the trigger price.

Gap Put Option

A gap put option has a payoff of:

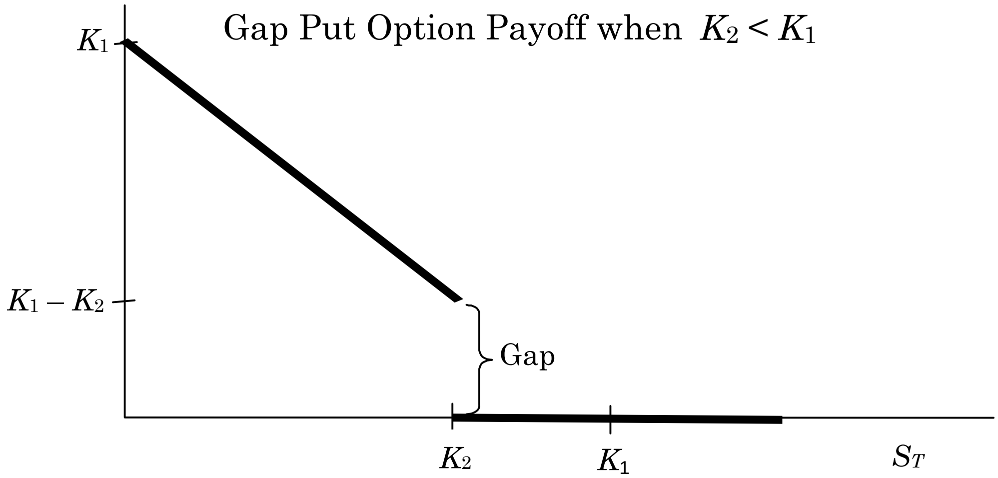
$$\text{Gap put option payoff} = \begin{cases} K_1 - S_T & \text{if } S_T < K_2 \\ 0 & \text{if } S_T \geq K_2 \end{cases}$$

where:

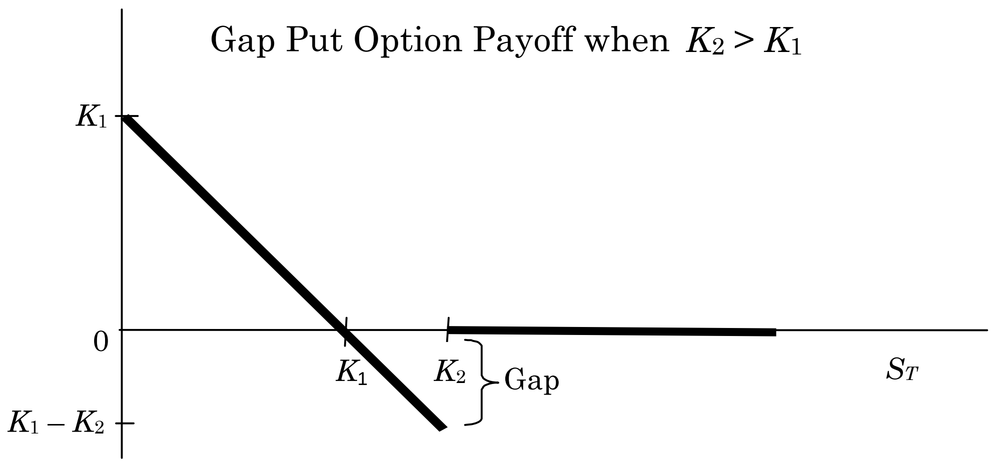
K_1 = Strike price

K_2 = Trigger price

If we graph the payoff of a gap put option as a function of its final stock price, once again we see that there is a gap where $S_T = K_2$. There are no negative payoffs in the graph below because the trigger price is less than the strike price:

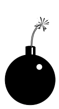


If the trigger price is greater than the strike price for a gap put option, then negative payoffs are possible:



Because negative payoffs are possible, gap options can have negative premiums.

A gap option must be exercised even if it results in a negative payoff, so perhaps it shouldn't really be called an "option."



Don't confuse gap options with knock-in barrier options! There are two differences:

1. *For barrier options, the barrier can be reached prior to expiration, but for gap options, only the final stock price is compared to the trigger.*
2. *Barrier options never have negative payoffs.*

The pricing formulas for gap calls and gap puts are similar to the Black-Scholes formulas for ordinary calls and puts, but there are two differences:

1. K_1 is substituted for K in the primary formula.
2. K_2 is substituted for K in the formula for d_1.

Pricing Formulas for Gap Options

The prices of gap options are:

$$\text{GapCall}(S, K_1, K_2, T) = Se^{-\delta T}N(d_1) - K_1 e^{-rT}N(d_2)$$

$$\text{GapPut}(S, K_1, K_2, T) = K_1 e^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

where:

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{K_2 e^{-rT}}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and:

$K_1 =$ Strike price $\qquad K_2 =$ Trigger price

As mentioned in Chapter 7, the value of σ used to calculate d_1 and d_2 is always positive.

From the formula above, we can see that the price of a gap option is linearly related to its strike price. This means that if we know the prices of two gap options that are identical other than their strike prices, then we can use linear interpolation (or extrapolation) to find the price of yet another gap option with a different strike price.

Example: The following gap call options have the same underlying asset:

Option Type	Strike	Trigger	Maturity	Price
Gap Call Option 1	46	50	1 year	5.75
Gap Call Option 2	51	50	1 year	?
Gap Call Option 3	53	50	1 year	3.05

Calculate the price of Gap Call Option 2.

Solution: The price of Gap Call Option 2 can be found using linear interpolation:

$$5.75 + \left(\frac{51-46}{53-46}\right)(3.05 - 5.75) = 3.82$$

The price of Gap Call Option 2 is \$3.82.

For a given strike price, the trigger price that produces the maximum gap option price is the strike price. When the strike price and the trigger price are equal, the gap option is the same as an ordinary option. Increasing the trigger price above the strike price causes the value of the gap option to fall, and decreasing the trigger price below the strike price also causes the value of the gap option to fall.

Example: Which of the three gap call options has the highest value?

Gap Call Option	Strike K_1	Trigger K_2
A	50	40
B	50	50
C	50	60

Solution: Option B has the same value as an ordinary European call. Option B pays only if $S_T > 50$.

Option A has the same payoffs as Option B if $S_T > 50$. Option A also has payoffs if $40 < S_T < 50$, and these payoffs are negative. Therefore, Option A is worth less than Option B.

Option C has the same payoffs as Option B if $S_T > 60$. If $50 < S_T \leq 60$, then Option B makes a positive payoff while Option C has a payoff of zero. Therefore, Option C is worth less than Option B.

Since Option A and Option C are both worth less than Option B, the option with the highest value is Option B.

The example above deals with gap call options, but similar reasoning applies for gap put options. Increasing the trigger price of a gap put option above its strike price introduces negative payoffs. Decreasing the trigger price of a gap put option below its strike price cuts off some of the positive payoffs.

If a gap call option and a gap put option both have a trigger price of K_2 and a strike price of K_1, then the purchase of a gap call option and the sale of a gap put option has the same payoff regardless of whether the final stock price is below or above the trigger price:

$$S_T < K_2: \quad \text{Payoff} = \text{GapCall Payoff} - \text{GapPut Payoff} = 0 - (K_1 - S_T) = S_T - K_1$$

$$S_T > K_2: \quad \text{Payoff} = \text{GapCall Payoff} - \text{GapPut Payoff} = (S_T - K_1) - 0 = S_T - K_1$$

What if $S_T = K_2$? In that case the payoff is zero:

$$S_T = K_2: \quad \textit{Payoff} = \textit{GapCall Payoff} - \textit{GapPut Payoff} = (S_T - K_1) - (K_1 - S_T) = 0$$

But in the Black-Scholes framework, the stock price distribution is continuous, meaning that the probability of any single stock price is zero. Therefore, under the Black-Scholes framework, we can ignore the possibility that $S_T = K_2$.

The current value of a payoff of $S_T - K_1$ is the prepaid forward price of the stock minus the present value of the strike price, and this can also be expressed as the value of the gap call minus the value of the gap put:

$$GapCall - GapPut = Se^{-\delta T} - K_1 e^{-rT}$$

In the Key Concept below, this equation is rearranged into the familiar form for put-call parity.

Put-Call Parity for Gap Calls and Gap Puts

Under the Black-Scholes framework, if a gap call option and a gap put option have the same strike price of K_1 and also have the same trigger price, then the gap call price plus the present value of the strike price is equal to the prepaid forward price of the stock plus the gap put price:

$$GapCall + K_1 e^{-rT} = Se^{-\delta T} + GapPut$$

The delta of a gap call is the partial derivative of its price with respect to the stock price:

$$GapCall = Se^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)$$

$$\Delta_{GapCall} = \frac{\partial(GapCall)}{\partial S} = e^{-\delta T} N(d_1) + Se^{-\delta T} N'(d_1)\frac{\partial d_1}{\partial S} - K_1 e^{-rT} N'(d_2)\frac{\partial d_2}{\partial S}$$

$$= e^{-\delta T} N(d_1) + Se^{-\delta T} \frac{\mathrm{e}^{-0.5 d_1^2}}{\sqrt{2\pi}} \times \frac{1}{S\sigma\sqrt{T}} - K_1 e^{-rT} \frac{\mathrm{e}^{-0.5 d_2^2}}{\sqrt{2\pi}} \times \frac{1}{S\sigma\sqrt{T}}$$

$$= e^{-\delta T} N(d_1) + \frac{Se^{-\delta T}\, \mathrm{e}^{-0.5 d_1^2} - K_1 e^{-rT}\, \mathrm{e}^{-0.5 d_2^2}}{S\sigma\sqrt{2\pi T}}$$

Using put-call parity for gap options, we can find the delta of a gap put in terms of the delta of the corresponding gap call:

$$GapCall + K_1 e^{-rT} = Se^{-\delta T} + GapPut$$

$$\Delta_{GapCall} + 0 = e^{-\delta T} + \Delta_{GapPut}$$

$$\Delta_{GapPut} = \Delta_{GapCall} - e^{-\delta T}$$

