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Solutions to the Exam MFE/3F Review Questions

Overview

Solutions to Review Questions

The Solutions to ActuarialBrew.com’s MFE/3F Review Questions provide full solutions to the over 650 exam-style practice questions contained in the Review Questions.

If you have access to the online normal distribution calculator or a spreadsheet when working the Review Questions, the SOA advises students to use 5 decimal places for both the inputs and the outputs of the normal distribution calculator. If you don’t have access to either one, we provide the old printed Normal Distribution Table at the end of the Review Questions for your convenience. Please note that your results from using the printed table may be different due to rounding.

Practice Questions

Each solution has the following key to indicate the question’s degree of difficulty. The more boxes that are filled in, the more difficult the question:

Easy: ☐ ☐ ☐ ☐ ☐

Very Difficult: ☐ ☐ ☐ ☐ ☐

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Errata

The errata for the MFE/3F Review Notes and the Solutions to the MFE/3F Review Questions can be found on our website at www.ActuarialBrew.com. Please let us know about any errata you find by emailing us at ExamMFESupport@ActuarialBrew.com.

Good Luck!
Chapter 9 – Solutions

Solution 1

C  Chapter 9, Put-call Parity

Since the underlying stock does not pay dividends, the value of a European call is equal to the value of the American call:

\[ C_{\text{Eur}}(K,T) = C_{\text{Amer}}(K,T) \]
\[ C_{\text{Eur}}(50,0.25) = 3.48 \]

We can use put-call parity to solve for the value of the European put option:

\[ C_{\text{Eur}}(K,T) + Ke^{-rT} = S_0 + P_{\text{Eur}}(K,T) \]
\[ 3.48 + 50e^{-0.08(0.25)} = 50 + P_{\text{Eur}}(50,0.25) \]
\[ P_{\text{Eur}}(50,0.25) = 2.49 \]

Solution 2

B  Chapter 9, Put-call Parity

The European call premium can be found using put-call parity:

\[ C_{\text{Eur}}(K,T) + Ke^{-rT} = S_0 - PV_{0,T}(\text{Div}) + P_{\text{Eur}}(K,T) \]
\[ C_{\text{Eur}}(48,0.5) + 48e^{-0.08(0.5)} = 50 - 5e^{-0.08(0.25)} + 4.38 \]
\[ C_{\text{Eur}}(48,0.5) = 3.36 \]

Solution 3

B  Chapter 9, Put-call Parity

We ignore the dividend occurring in 7 months, because it occurs after the call option expires.

The European call premium can be found using put-call parity:

\[ C_{\text{Eur}}(K,T) + Ke^{-rT} = S_0 - PV_{0,T}(\text{Div}) + P_{\text{Eur}}(K,T) \]
\[ C_{\text{Eur}}(80,0.5) + 80e^{-0.05(0.5)} = 75 - 4e^{-0.05(0.5)} + 10.37 \]
\[ C_{\text{Eur}}(80,0.5) = 3.41 \]

Solution 4

C  Chapter 9, Put-call Parity

This question may appear tricky since the options are euro-denominated. There is no other currency involved though, so we can use the standard put-call parity formula.
We can use put-call parity to solve for the risk-free rate of return:

\[ C_{\text{Eur}}(K,T) + Ke^{-rT} = S_0e^{-\delta T} + P_{\text{Eur}}(K,T) \]

\[ 7.53 + 80e^{-r(1)} = 75e^{-0.04(1)} + 10.07 \]
\[ e^{-r} = 0.93249 \]
\[ r = 0.0699 \]

**Solution 5**

**D**  Chapter 9, Put-call Parity

The dividend occurs after the expiration of the options, so we can ignore it. Furthermore, when there are no dividends prior to the expiration of an American call option, the option has the same value as a European call option.

We can use put-call parity to solve for the value of the call option:

\[ C_{\text{Eur}}(K,T) + Ke^{-rT} = S_0 + P_{\text{Eur}}(K,T) \]

\[ C_{\text{Eur}}(K,T) + 70e^{-0.07(0.5)} = 75 + 3.02 \]
\[ C_{\text{Eur}}(K,T) = 10.428 \]

**Solution 6**

**A**  Chapter 9, Put-call Parity

Since the options are at-the-money, the strike price is equal to the stock price. We can use put-call parity to find the price of the stock:

\[ C_{\text{Eur}}(K,T) + Ke^{-rT} = S_0 - PV_{0,T}(\text{Div}) + P_{\text{Eur}}(K,T) \]

\[ S_0 = C_{\text{Eur}}(K,T) - P_{\text{Eur}}(K,T) + Ke^{-rT} + PV_{0,T}(\text{Div}) \]

\[ S_0 = 1.34 + Ke^{-0.07(0.75)} + 3e^{-0.07(0.5)} \]
\[ S_0 = 1.34 + S_0 e^{-0.07(0.75)} + 3e^{-0.07(0.5)} \]
\[ S_0 = \frac{1.34 + 3e^{-0.07(0.5)}}{1 - e^{-0.07(0.75)}} = 30.44 \]

**Solution 7**

**D**  Chapter 9, Synthetic Stock

We can rearrange put-call parity so that it is a guide for replicating the stock:

\[ S_0 = C_{\text{Eur}}(K,T) + Ke^{-rT} + PV_{0,T}(\text{Div}) - P_{\text{Eur}}(K,T) \]
To replicate the stock, we must purchase the call, sell the put and lend the present value of the strike plus the present value of the dividends. The present value of the strike plus the present value of the dividends is:

\[ Ke^{-rT} + PV_{0,T}(Div) = 28e^{-0.09(0.75)} + 3e^{-0.09(0.5)} = 29.04 \]

**Solution 8**

**D** Chapter 9, Synthetic T-bills

As an alternative to the method below, we could solve for \( r \) and then use it to find the present value of $1,000.

We can rearrange put-call parity so that it is a guide for replicating a T-bill:

\[ Ke^{-rT} = S_0e^{-\delta T} - C_{Eur}(K,T) + P_{Eur}(K,T) \]

To create an asset that matures for the strike price of $50, we must purchase \( e^{-\delta T} \) shares of the stock, sell a call option, and buy a put option. The cost of doing this is:

\[
Ke^{-rT} = S_0e^{-\delta T} - C_{Eur}(K,T) + P_{Eur}(K,T)
= 52e^{-0.07(0.75)} - 6.56 + 3.61
= 46.3904
\]

Since it costs $46.3904 to create an asset that is guaranteed to mature for $50, it must cost 20 times as much to create an asset that is guaranteed to mature for $1,000:

\[ 20 \times 46.3904 = 927.81 \]

**Solution 9**

**A** Chapter 9, Synthetic Stock

To answer this question, we don’t need to know the risk-free rate of return or the time until the options expire.

We can rearrange put-call parity so that it is a guide for replicating the stock:

\[ S_0 = C_{Eur}(K,T) + Ke^{-rT} + PV_{0,T}(Div) - P_{Eur}(K,T) \]

To replicate the stock, we must purchase the call, sell the put and lend the present value of the strike plus the present value of the dividends. We can use the equation above to find the present value of the strike plus the present value of the dividends:

\[
52 = 6.01 + Ke^{-rT} + PV_{0,T}(Div) - 3.87
Ke^{-rT} + PV_{0,T}(Div) = 52 - 6.01 + 3.87 = 49.86
\]
Solution 10

E Chapter 9, Currency options

The current exchange rate is in the form of euros per dollar, which is the inverse of our usual form of dollars per euro.

The exchange rate in dollars per euro is:

\[ x_0 = \frac{1}{0.96} = 1.04167 \]

Put-call parity for currency options can be used to find the value of the call option:

\[
C_{Eur}(K,T) + Ke^{-rT} = x_0e^{-\gamma T} + P_{Eur}(K,T)
\]
\[
C_{Eur}(0.94,1) + 0.94e^{-0.06(1)} = 1.04167e^{-0.04(1)} + 0.005
\]
\[
C_{Eur}(0.94,1) = 0.12056
\]

Solution 11

A Chapter 9, Currency options

We treat the Swiss franc as the base currency for the first part of this question. But then at the end we must convert its value into dollars.

With the Swiss franc as the base currency, we have the following exchange rate in terms of francs per dollar:

\[ x_0 = \frac{1}{0.80} = 1.25 \]

Put-call parity for currency options can be used to find the value of the put option:

\[
C_{Eur}(K,T) + Ke^{-rT} = x_0e^{-\gamma T} + P_{Eur}(K,T)
\]
\[
0.127 + 1.15e^{-0.06(1)} = 1.25e^{-0.04(1)} + P_{Eur}(1.15,1)
\]
\[
P_{Eur}(1.15,1) = 0.00904
\]

Therefore the cost of the put option is 0.00904 Swiss francs. But the possible choices are all expressed in dollars, so we must convert the value to dollars at the current exchange rate. One Swiss franc costs $0.80, so 0.00904 Swiss francs must cost (in dollars):

\[ 0.00904 \times 0.80 = 0.00723 \]
Solution 12

C Chapter 9, Currency options

We treat the Iraqi dinar as the base currency.

Put-call parity for currency options can be used to find interest rate on the dinar:

\[ C_{Eur}(K,T) + Ke^{-rT} = x_0 e^{-\eta T} + P_{Eur}(K,T) \]

\[ 13.37 + 198e^{-0.5r} = 200e^{-0.08(0.5)} + 1.27 \]

\[ 198e^{-0.5r} = 180.05789 \]

\[ r = 0.19 \]

Solution 13

A Chapter 9, Options on Bonds

Here’s a quick refresher on compound interest. The annual effective interest rate is denoted by \( i \):

\[ 1 + i = e^r \]

\[ v = \frac{1}{1+i} \]

\[ a_{\overline{n}|} = \frac{1-v^n}{i} \]

The price of the bond is:

\[ B_0 = 120 a_{\overline{15}|} + 1,000e^{-0.10(15)} \]

\[ = 120 \frac{1-e^{-0.10(15)}}{e^{0.10}-1} + 1,000e^{-0.10(15)} \]

\[ = 120(7.3867) + 223.1302 \]

\[ = 1,109.5385 \]

Only one coupon occurs before the expiration of the options, and it occurs one year from now. Using put-call parity, we have:

\[ C_{Eur}(K,T) + Ke^{-rT} = B_0 - PV_{0,T}(Coupons) + P_{Eur}(K,T) \]

\[ 150 + 1,000e^{-0.10(1.25)} = 1,109.5385 - 120e^{-0.10(1)} + P_{Eur}(1,000,1.25) \]

\[ P_{Eur}(1,000,1.25) = 31.54 \]
Solution 14

E Chapter 9, Options on Bonds

The 6-month effective interest rate is:

\[ e^{0.08(0.5)} - 1 = 4.081\% \]

The price of the bond one month after it is issued is:

\[
B_0 = \left[ \frac{45e^{-0.04(24)}}{e^{0.04} - 1} + 1,000e^{-0.08(12)} \right] e^{0.08(1/12)}
\]

\[
= \left[ 45 \frac{e^{-0.04(24)}}{e^{0.04} - 1} + 1,000e^{-0.08(12)} \right] e^{0.08(1/12)}
\]

\[
= \left[ 45(15.1212) + 382.8929 \right] e^{0.08(1/12)}
\]

\[
= 1,063.3460 \times e^{0.08(1/12)}
\]

\[
= 1,070.4587
\]

Two coupons occur prior to the expiration of the option, the first of which occurs in 5 months and the second of which occurs in 11 months. Using put-call parity, we have:

\[
C_{Eur}(K,T) + Ke^{-rT} = B_0 - PV_{0,T}(Coupons) + P_{Eur}(K,T)
\]

\[
C_{Eur}(950,1) + 950e^{-0.08(1)} = 1,070.4587 - 45e^{-0.08(5/12)} - 45e^{-0.08(11/12)} + 25
\]

\[
C_{Eur}(950,1) = 133.16
\]

Solution 15

B Chapter 9, Options on Bonds

The price of the bond is equal to $1,000. Since the bond price is equal to its par value, its yield must be equal to its coupon rate. Therefore the yield is 7%. Since 7% is the only interest rate provided in the problem, we use 7% as the risk-free interest rate. The 7% interest rate is compounded twice per year since coupons are paid semi-annually. Therefore, the semiannual effective interest rate is 3.5%.

Using put-call parity for bonds, we have:

\[
Ke^{-rT} = B_0 - PV_{0,T}(Coupons) + P_{Eur}(K,T) - C_{Eur}(K,T)
\]

\[
K (1.035)^{-1.5} = 1,000 - \frac{35}{1.035} - 58.43
\]

\[
K = 955.83
\]
Solution 16

D  Chapter 9, Exchange Options

The first step is to pick one of the assets to be the underlying asset. You can choose either one. We chose Stock X below, so a call option costs $2.70. If we chose Stock Y to be the underlying asset, then the same option would be a put option.

Let’s assume that Stock X is the underlying asset and Stock Y is the strike asset. In that case, the option to give up Stock Y for Stock X is a call option:

\[ C_{Eur}(X_t, Y_t, 0.5) = 2.70 \]

We can now use put-call parity for exchange options:

\[ C_{Eur}(X_t, Y_t, 0.5) - P_{Eur}(X_t, Y_t, 0.5) = F_{Eur}^P(X_t) - F_{Eur}^P(Y_t) \]

\[ 2.70 - P_{Eur}(X_t, Y_t, 0.5) = \left[ 50 - 3e^{-0.06(2/12)} \right] - 51e^{-0.03(0.5)} \]

\[ P_{Eur}(X_t, Y_t, 0.5) = 5.91 \]

Solution 17

A  Chapter 9, Exchange Options

We didn’t need to know the risk-free rate of return. That was included in the question as a red herring.

The put option that allows its owner to give up Stock B in exchange for Stock A has Stock B as its underlying asset:

\[ P_{Eur}(B_t, A_t, 1) = 11.49 \]

Using put-call parity, we can find the value of the call option having Stock B as its underlying asset:

\[ C_{Eur}(B_t, A_t, 1) - P_{Eur}(B_t, A_t, 1) = F_{Eur}^P(B_t) - F_{Eur}^P(A_t) \]

\[ C_{Eur}(B_t, A_t, 1) - 11.49 = 67e^{-0.05(1)} - 70 \]

\[ C_{Eur}(B_t, A_t, 1) = 5.22 \]

We can describe the call option as a put option by switching the underlying asset and the strike asset:

\[ C_{Eur}(B_t, A_t, 1) = P_{Eur}(A_t, B_t, 1) \]

\[ P_{Eur}(A_t, B_t, 1) = 5.22 \]

Therefore, the value of a put option giving its owner the right to give up a share of Stock A in exchange for a share of Stock B is $5.22.
Solution 18

**Chapter 9, Exchange Options**

Let’s choose Stock X to be the underlying asset. In that case, Option A is a put option and Option B is a call option. Using put call parity, we have:

\[
C_{Eur}(S_t, Y_t, 5/12) - P_{Eur}(S_t, Y_t, 5/12) = F_{t,T}^P(X_t) - F_{t,T}^P(Y_t)
\]

Option B - Option A = \( F_{t,T}^P(X_t) - F_{t,T}^P(Y_t) \)

\[-7.76 = 0.04(5/12) - Y_0 e^{-0.04(5/12)} \]

\[Y_0 = 47.89\]

Solution 19

**Chapter 9, Exchange Options**

Let’s establish two portfolios. Portfolio X consists of 1 share of Stock A and 2 shares of Stock B. Portfolio Y consists of 1 share of Stock C and 1 share of Stock D.

The first call option described in the question has Portfolio X as its underlying asset:

\[C_{Eur}(X_t, Y_t, 1) = 10\]

We can find the prepaid forward prices for both portfolios:

\[F_{t,T}^P(X_t) = 40e^{-0.04(1)} + 2(50) = 138.4316\]

\[F_{t,T}^P(Y_t) = 60 + 75e^{-0.02(1)} = 133.5149\]

We can now use put-call parity to find the value of the corresponding put option:

\[C_{Eur}(X_t, Y_t, 1) - P_{Eur}(X_t, Y_t, 1) = F_{t,T}^P(X_t) - F_{t,T}^P(Y_t)\]

\[10 - P_{Eur}(X_t, Y_t, 1) = 138.4316 - 133.5149\]

\[P_{Eur}(X_t, Y_t, 1) = 5.0833\]

We can describe a put option as a call option by switching the underlying asset with the strike asset:

\[P_{Eur}(X_t, Y_t, 1) = C_{Eur}(Y_t, X_t, 1)\]

\[C_{Eur}(Y_t, X_t, 1) = 5.0833\]

Therefore, the call option giving its owner the right to acquire Portfolio Y in exchange for Portfolio X has a value of $5.0833.
Solution 20

B  Chapter 9, Exchange Options

This question is easier if we assume that Stock X is the underlying asset.

Let’s assume that Stock X is the underlying asset. The American option is then an American call option. Since Stock X does not pay dividends, the American call option is not exercised prior to maturity. Therefore the American call option has the same value as a European call option:

\[ C_{Eur}(X_t, Y_t, 7/12) = 10.22 \]

We are asked to find the value of the corresponding European put option. We can make use of put-call parity:

\[ C_{Eur}(X_t, Y_t, 7/12) - P_{Eur}(X_t, Y_t, 7/12) = F_{t,T}^P(X_t) - F_{t,T}^P(Y_t) \]
\[ 10.22 - P_{Eur}(X_t, Y_t, 7/12) = 100 - 100e^{-0.04(7/12)} \]
\[ P_{Eur}(X_t, Y_t, 7/12) = 7.91 \]

Solution 21

A  Chapter 9, Currency Options

We didn’t need to know the dollar-denominated or the euro-denominated rates of return. They were included in the question as red herrings.

The first option allows its owner to give up $1.25 in order to get €1.00.

The second option allows its owner to give up $1.00 in order to get €0.80.

Therefore, the value of the second option is 0.80 times the value of the first option:

\[ 0.80 \times \$0.083 = \$0.0664 \]

The value of the second option in euros is:

\[ \$0.0664 \times \frac{\text{€1}}{\$1.20} = \text{€0.0553} \]

Solution 22

E  Chapter 9, Currency Options

The first option allows its owner to give up €0.80 in order to get $1.00.

The second option allows its owner to give up €1.00 in order to get $1.25.

Therefore, the value of the second option is 1.25 times the value of the first option:

\[ 1.25 \times \text{€0.0898} = \text{€0.11225} \]

The value of the second option in dollars is:

\[ \text{€0.11225} \times \frac{\$1}{\text{€0.85}} = \$0.1321 \]
Solution 23

B Chapter 9, Currency Options

The first option allows its owner to give up €1.00 in order to get 1.60 francs.
The second option allows its owner to give up €0.625 in order to get 1.00 francs.
Therefore, the value of the second option is 0.625 times the value of the first option:

\[ 0.625 \times 0.0918 \text{francs} = 0.057375 \text{francs} \]

The value of the second option in euros is:

\[ 0.057375 \text{francs} \times \frac{1}{1.65 \text{francs}} = \€0.0348 \]

Solution 24

D Chapter 9, Currency Options

The pound-denominated put allows its owner to give up $1.00 in order to get £0.40.
The dollar-denominated put allows its owner to give up £1.00 in order to get $2.50.
We can find the value of a dollar-denominated call that allows its owner to give up $2.50
in order to get £1.00. The value of this call is 2.5 times the value of the pound-
denominated put. Below, the value of this call is expressed in dollars:

\[ C_\$ \left( \frac{1}{0.42}, 2.50, 1 \right) = 2.50 \times \£0.0133 \times \frac{\$1}{\£0.42} = \$0.07917 \]

We can use put-call parity to find the value of the dollar-denominated put:

\[ C_\$ \left( \frac{1}{0.42}, 2.50, 1 \right) + 2.50e^{-0.07} = \frac{1}{0.42}e^{-0.08} + P_\$ \left( \frac{1}{0.42}, 2.50, 1 \right) \]

\[ 0.07917 + 2.50e^{-0.07} = \frac{1}{0.42}e^{-0.08} + P_\$ \left( \frac{1}{0.42}, 2.50, 1 \right) \]

\[ P_\$ \left( \frac{1}{0.42}, 2.50, 1 \right) = 0.2123 \]

Solution 25

B Chapter 9, Currency Options

The euro-denominated call allows its owner to give up €0.625 in order to get $1.00.
The dollar-denominated call allows its owner to give up $1.60 in order to get €1.00.
We can find the value of a dollar-denominated put that allows its owner to give up €1.00
in order to get $1.60. The value of this put is 1.6 times the value of the euro-denominated
call. Below, the value of this put is expressed in dollars:

\[ P_\$ \left( \frac{1}{0.7}, 1.60, 1 \right) = 1.60 \times \€0.08 \times \frac{\$1}{\€0.7} = \$0.182857 \]
We can use put-call parity to find the value of the dollar-denominated call:

\[
C_S \left( \frac{1}{0.7}, 1.60, 0.5 \right) + 1.60e^{-0.07 \times 0.5} = \frac{1}{0.7} e^{-0.08 \times 0.5} + P_S \left( \frac{1}{0.7}, 1.60, 0.5 \right)
\]

\[
C_S \left( \frac{1}{0.7}, 1.60, 0.5 \right) + 1.60e^{-0.035} = \frac{1}{0.7} e^{-0.04} + 0.182857
\]

\[
C_S \left( \frac{1}{0.7}, 1.60, 0.5 \right) = 0.0104448
\]

The value of the Canadian dollar-denominated call is 0.0104448 Canadian dollars. The answer choices are expressed in euros though, so we must convert this amount into euros:

\[
$0.0104448 \times \frac{\varepsilon 0.7}{\$ 1} = \varepsilon 0.0073
\]

The value of the Canadian dollar-denominated call is 0.0073 euros.

**Solution 26**

B  Chapter 9, Currency Options

The put option gives its owner the right to give up €1 in order to get ¥100.

The call option gives its owner the right to give up €0.01 in order to get ¥1.

Therefore, the value of the call option is 0.01 times the value of the put option:

\[
C_e \left( \frac{1}{110}, 0.01, T \right) = 0.01 \times ¥1.58 = ¥0.0158
\]

We can convert this value from yen into euros:

\[
C_e \left( \frac{1}{110}, 0.01, T \right) = ¥0.0158 \times \frac{\varepsilon 1}{¥110} = \varepsilon 0.000144
\]

**Solution 27**

C  Chapter 9, Options on Currencies

We can use put-call parity to determine the spot exchange rate:

\[
C_{Eur}(K,T) + Ke^{-rT} = x_0 e^{-\eta T} + P_{Eur}(K,T)
\]

\[
0.047 + 0.89e^{-0.05(0.5)} = x_0 e^{-0.035(0.5)} + 0.011
\]

\[
x_0 = 0.920
\]
Solution 28

C Chapter 9, Currency Options

The exchange rate can be expressed in two equivalent ways:

\[
\frac{\$0.008}{¥1} \text{ or } \frac{¥125}{\$1}
\]

The dollar-denominated call allows its owner to give up $0.008 in order to get ¥1.00.

The yen-denominated call allows its owner to give up ¥125.00 in order to get $1.00.

We can find the value of a yen-denominated put that allows its owner to give up $1.00 in order to get ¥125.00. The value of this put is 125 times the value of the dollar-denominated call. Below, the value of this put is expressed in yen:

\[
P_{¥}(125,125,1) = 125 \times \frac{¥125}{\$1} = ¥8.4375
\]

We can use put-call parity to find the value of the yen-denominated call:

\[
C_{¥}(125,125,1) + 125e^{-0.01} \times 125e^{-0.06} = 125e^{-0.06} + P_{¥}(125,125,1)
\]

\[
C_{¥}(125,125,1) + 125e^{-0.01} \times 8.4375 = 125e^{-0.06} + 8.4375
\]

\[
C_{¥}(125,125,1) = 2.4018
\]

Solution 29

B Chapter 9, Exchange Options

Neither the time to maturity nor the risk-free rate of return is needed to answer the question. They were provided as red herrings.

The prepaid forward price of Stock A is $22. The prepaid forward price of Stock B is $26. Stock A is the underlying asset, giving rise to the following generalized form of put-call parity:

\[
C_{Eur}(A_0, B_0, T) - P_{Eur}(A_0, B_0, T) = F_{0,T}^P(A) - F_{0,T}^P(B)
\]

\[
C_{Eur}(A_0, B_0, T) - P_{Eur}(A_0, B_0, T) = 22 - 26
\]

\[
C_{Eur}(A_0, B_0, T) - P_{Eur}(A_0, B_0, T) = -4
\]

Therefore the put price exceeds the call price by $4. By inspection, only Choice B has a put price that is $4 greater than the call price.
Solution 30

Chapter 9, Exchange Options, Options on Currencies

To answer this question, let’s use the yen as the base currency. In that case, the U.S. Dollar and the Canadian dollars are assets with prices denominated in the base currency. The U.S. dollar is the underlying asset for the options, and the Canadian dollar is the strike asset:

\[
C_{Eur}(S_t, Q_t, T - t) - P_{Eur}(S_t, Q_t, T - t) = F_{t,T}^P(S) - F_{t,T}^P(Q)
\]

\[
C_{Eur}(S_t, Q_t, 0.5) - P_{Eur}(S_t, Q_t, 0.5) = 120e^{-0.08(0.5)} - 105e^{-0.07(0.5)}
\]

\[
C_{Eur}(S_t, Q_t, 0.5) - P_{Eur}(S_t, Q_t, 0.5) = 13.91
\]

Solution 31

Chapter 9, Maximum and Minimum Option Prices

This question was written in terms of euros to make it look more confusing, but since there is no other currency involved, we don’t have to worry about exchange rates.

Unless arbitrage is available, it must be the case that:

\[
\max \left[ 0, F_{t,T}^P(S_t) - Ke^{-r(T-t)} \right] \leq C_{Eur}(S_t, K, T - t)
\]

In fact, the inequality above is not satisfied since:

\[
F_{t,T}^P(S_t) - Ke^{-r(T-t)} > C_{Eur}(S_t, K, T - t)
\]

\[
200 - 198e^{-0.19(0.5)} > 18
\]

\[
19.944 > 18
\]

The call option has too low of a price relative to the forward prices of the underlying asset and the strike asset. Therefore, we can earn arbitrage profits by buying the right side of the inequality and selling the left side of the inequality.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy call</td>
<td>-18</td>
<td>0</td>
</tr>
<tr>
<td>Sell stock</td>
<td>200</td>
<td>(-S_{0.5})</td>
</tr>
<tr>
<td>Lend the present value of the strike</td>
<td>(-198e^{-0.19(0.5)})</td>
<td>198</td>
</tr>
<tr>
<td>Total</td>
<td>1.944</td>
<td>198 (-S_{0.5})</td>
</tr>
</tbody>
</table>

The ending price of the stock is €199, so the cash flow at time 0 is 1.94, while the cash flow at time 0.5 is 0. The accumulated value of the arbitrage profits is:

\[
1.944e^{0.19(0.5)} = 2.14
\]
Solution 32
C  Chapter 9, Strike Price Grows Over Time

The solution below is lengthy, making this problem look harder than it really is. The solution is thorough to assist those that might have trouble with this question.

The strike price grows at the interest rate:

\[ 206.60 = 198e^{0.17(0.25)} \Rightarrow K_T \geq Ke^{r(T-t)} \]

Therefore, to preclude arbitrage it should be the case that:

\[ P(0.75) \geq P(0.5) \]

But the longer option costs less than the shorter option, indicating that arbitrage is possible:

\[ 3.35 < 4.03 \]

The arbitrageur buys the longer option for $3.35 and sells the shorter option for $4.03. The difference of $0.68 is lent at the risk-free rate of return.

The 6-month option

After 6 months, the stock price is $196. Therefore, the shorter option is exercised against the arbitrageur. The arbitrageur borrows $198 and uses it to buy the stock.

As a result, at the end of 9 months the arbitrageur owns the stock and must repay the borrowed funds. This position results in the following cash flow at the end of 9 months:

\[ S_{0.75} - 198e^{0.17(0.25)} = S_{0.75} - 206.60 = 205 - 206.60 = -1.60 \]

The 9-month option

The stock price at the end of 9 months is less than $206.60, so the 9-month put option has the following cash flow at the end of 9 months:

\[ 206.60 - S_{0.75} = 206.60 - 205 = 1.60 \]

Payoff table

We don’t necessarily recommend constructing the entire payoff table during the exam, but we’ve included it below to assist you in understanding this problem.

Below is the payoff table for the strategy.
### Time 0.75

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>$S_{0.75} &lt; 206.60$</th>
<th>$S_{0.75} \geq 206.60$</th>
<th>$S_{0.75} &lt; 198$</th>
<th>$S_{0.75} \geq 198$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell $P(0.5)$</td>
<td>4.03</td>
<td>$S_{0.75} - 206.60$</td>
<td>$0$</td>
<td>$S_{0.75} - 206.60$</td>
<td>$0$</td>
</tr>
<tr>
<td>Buy $P(0.75)$</td>
<td>$-3.35$</td>
<td>$206.60 - S_{0.75}$</td>
<td>$206.60 - S_{0.75}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Total</td>
<td>$0.68$</td>
<td>$0$</td>
<td>$206.60 - S_{0.75}$</td>
<td>$S_{0.75} - 206.60$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

The cash flow of the two options nets to zero after 9 months, since the stock price path makes the first column applicable.

The accumulated value of the arbitrage strategy is the difference between the cost of the two options, accumulated for 9 months:

$$(4.03 - 3.35)e^{0.17(0.75)} = 0.7725$$

### Solution 33

**E Chapter 9, Strike Price Grows Over Time**

The solution below is lengthy, making this problem look harder than it really is. The solution is thorough to assist those that might have trouble with this question.

The strike price grows at the interest rate:

$$207.63 = 198e^{0.19(0.25)} \quad \Rightarrow \quad K_T \leq K_te^{r(T-t)}$$

Therefore, to preclude arbitrage it should be the case that:

$$C(0.75) \geq C(0.5)$$

But the longer option costs less than the shorter option, indicating that arbitrage is possible:

$$22.00 < 23.50$$

The arbitrageur buys the longer option for $22.00 and sells the shorter option for $23.50. The difference of $1.50 is lent at the risk-free rate of return.

The 6-month option

After 6 months, the stock price is $278. Therefore, the shorter option is exercised against the arbitrageur. The arbitrageur borrows a share of stock and sells it for $198.

As a result, at the end of 9 months the arbitrageur owes the stock and has the accumulated value of the $198. This position results in the following cash flow at the end of 9 months:

$$-S_{0.75} + 198e^{0.19(0.25)} = -S_{0.75} + 207.63 = -205 + 207.63 = 2.63$$
The 9-month option
The stock price at the end of 9 months is less than $207.63, so the 9-month call option expires worthless.

Payoff table
*We don’t necessarily recommend constructing the entire payoff table during the exam, but we’ve included it below to assist you in understanding this problem.*

Below is the payoff table for the strategy.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0.75</th>
<th>$S_{0.75} &lt; 207.63$</th>
<th>$S_{0.75} \geq 207.63$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell C(0.5)</td>
<td>$S_{0.5} &lt; 198$</td>
<td>23.50</td>
<td>$-S_{0.75} + 207.63$</td>
</tr>
<tr>
<td>Buy C(0.75)</td>
<td>$S_{0.5} \geq 198$</td>
<td>-22.00</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>$S_{0.5} \geq 198$</td>
<td>1.50</td>
<td>$-S_{0.75} + 207.63$</td>
</tr>
</tbody>
</table>

After 9 months, the net cash flow from the two options is:

\[-S_{0.75} + 207.63 = -205 + 207.63 = 2.63\]

Adding this to the accumulated value of the $1.50 obtained at the outset, we have the accumulated arbitrage profits at the end of 9 months:

\[1.50e^{0.19(0.75)} + 2.63 = 4.36\]

**Solution 34**
C Chapter 9, Strike Price Grows Over Time

The option has the total return portfolio as its underlying asset, so it should be the case that:

\[P(\text{Port}_t, K_t, 1.5) \geq P(\text{Port}_t, K_t, 1.0)\]

But the longer option costs less than the shorter option, indicating that arbitrage is possible:

\[7.00 < 7.85\]

The arbitrageur buys the longer option for $7.00 and sells the shorter option for $7.85. The difference of $0.85 is lent at the risk-free rate of return.
The 1-year option
After 1 year, the stock price is $60. This means that the total return portfolio has a value of:

\[ Port_{1,0} = 60e^{0.03(1)} = 61.827 \]

The strike price is:

\[ K_1 = 61.2147e^{0.06(1)} = 65.000 \]

Therefore, the 1-year put option is exercised against the arbitrageur. The arbitrageur borrows $65 and uses it to buy the total return portfolio.

As a result, at the end of 1.5 years the arbitrageur owns the total return portfolio and must repay the borrowed funds. This position results in the following cash flow at the end of 1.5 years:

\[ Port_{1.5} - 65e^{0.06(0.5)} = Port_{1.5} - 66.980 = 66e^{0.03(1.5)} - 66.980 = 2.058 \]

The 1.5-year option
The stock price at the end of 1.5 years is $66. This means that the total return portfolio has a value of:

\[ 66e^{0.03(1.5)} = 69.038 \]

The strike price is:

\[ K_{1.5} = 61.2147e^{0.06(1.5)} = 66.980 \]

Therefore, the 1.5-year put option expires unexercised. This position results in zero cash flow at the end of 1.5 years.

Net accumulated cash flows
Buying the longer put option and selling the shorter put option results in a cash flow of 0.85 at time 0.

The 1-year option resulted in a cash flow of 2.058 at time 1.5 years.

The 1.5 year option resulted in a cash flow of 0 at time 1.5 years.

The accumulated net cash flows are:

\[ 0.85e^{0.06(1.5)} + 2.058 + 0 = 2.99 \]
Solution 35

A Chapter 9, Proposition 2

The prices of the options violate Proposition 2:

\[ P_{Eur}(K_2) - P_{Eur}(K_1) \leq K_2 - K_1 \]

because:

\[ 8.75 - 4 > (55 - 50)e^{-0.09(1)} \]

\[ 4.75 > 4.57 \]

Arbitrage is available using a put bull spread:

Buy \( P(50) \)

Sell \( P(55) \)

Here’s a tip for remembering which option to buy and which option to sell. Notice that Proposition 2 fails because the 55-strike put premium is too high relative to the 50-strike put premium. To earn arbitrage profits, we sell the asset that is priced “too high” and buy the asset that is priced “too low.” This is a recurring theme in arbitrage strategies.

This strategy produces the following payoff table:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>( S_1 &lt; 50 )</th>
<th>( 50 \leq S_1 \leq 55 )</th>
<th>( 55 &lt; S_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy ( P(50) )</td>
<td>-4.00</td>
<td>50 - ( S_1 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sell ( P(55) )</td>
<td>8.75</td>
<td>-(55 - ( S_1 ))</td>
<td>-(55 - ( S_1 ))</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>4.75</td>
<td>-5.00</td>
<td>-(55 - ( S_1 ))</td>
<td>0</td>
</tr>
</tbody>
</table>

If the final stock price is $48, then the accumulated arbitrage profits are:

\[ X = 4.75e^{0.09} - 5.00 = 0.19733 \]

If the final stock price is $52, then the accumulated arbitrage profits are:

\[ Y = 4.75e^{0.09} - (55 - 52) = 2.19733 \]

The ratio of \( X \) to \( Y \) is:

\[ \frac{X}{Y} = \frac{0.19733}{2.19733} = 0.09 \]
Solution 36

C Chapter 9, Proposition 3

The prices of the options violate Proposition 3:

\[
\frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}
\]

because:

\[
\frac{7 - 3}{55 - 50} > \frac{11 - 7}{61 - 55}
\]

Arbitrage is available using an asymmetric butterfly spread:

Buy \( \lambda \) of the 50-strike options
Sell 1 of the 55-strike options
Buy \((1 - \lambda)\) of the 61-strike options

where:

\[
\lambda = \frac{K_3 - K_2}{K_3 - K_1} = \frac{61 - 55}{61 - 50} = \frac{6}{11}
\]

In the payoff table below, we have scaled the strategy up by multiplying by 11:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>(S_1 &lt; 50)</th>
<th>(50 \leq S_1 \leq 55)</th>
<th>(55 \leq S_1 \leq 61)</th>
<th>(61 &lt; S_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 6 of (P(50))</td>
<td>-6(3.00)</td>
<td>6(50 - (S_1))</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Sell 11 of (P(55))</td>
<td>11(7.00)</td>
<td>-11(55 - (S_1))</td>
<td>-11(55 - (S_1))</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Buy 5 of (P(61))</td>
<td>-5(11.00)</td>
<td>5(61 - (S_1))</td>
<td>5(61 - (S_1))</td>
<td>5(61 - (S_1))</td>
<td>0.00</td>
</tr>
<tr>
<td>Total</td>
<td>4.00</td>
<td>0.00</td>
<td>6(S_1) - 300</td>
<td>305 - 5(S_1)</td>
<td>0.00</td>
</tr>
</tbody>
</table>

If the final stock price is $52, then the accumulated arbitrage profits are:

\[
X = 4.00e^{0.11} + 6S_1 - 300 = 4.00e^{0.11} + 6(52) - 300 = 16.4651
\]

If the final stock price is $60, then the accumulated arbitrage profits are:

\[
Y = 4.00e^{0.11} + 305 - 5S_1 = 4.00e^{0.11} + 305 - 5(60) = 9.4651
\]

The ratio of \(X\) to \(Y\) is:

\[
\frac{X}{Y} = \frac{16.4651}{9.4651} = 1.74
\]
Solution 37

E Chapter 9, Proposition 2

The prices of Option A and Option B violate Proposition 2:

\[ C_{Eur}(K_1) - C_{Eur}(K_2) \leq (K_2 - K_1)e^{-rt} \]

because:

\[ 14 - 9.25 > (55 - 50)e^{-0.07(1)} \]
\[ 4.75 > 4.66 \]

Arbitrage is available using a call bear spread:

Buy C(55)

Sell C(50)

This strategy produces the following payoff table:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>S₁ &lt; 50</th>
<th>50 ≤ S₁ ≤ 55</th>
<th>55 &lt; S₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy C(55)</td>
<td>-9.25</td>
<td>0.00</td>
<td>0.00</td>
<td>S₁ - 55</td>
</tr>
<tr>
<td>Sell C(50)</td>
<td>14.00</td>
<td>0.00</td>
<td>-(S₁ - 50)</td>
<td>-(S₁ - 50)</td>
</tr>
<tr>
<td>Total</td>
<td>4.75</td>
<td>0.00</td>
<td>-(S₁ - 50)</td>
<td>-5.00</td>
</tr>
</tbody>
</table>

If the final stock price is $52, then the accumulated arbitrage profits are:

\[ X = 4.75e^{0.07} - (S₁ - 50) = 4.75e^{0.07} - (52 - 50) = 4.75e^{0.07} - 2 = 3.0944 \]

If the final stock price is $60, then the accumulated arbitrage profits are:

\[ Y = 4.75e^{0.07} - 5 = 0.0944 \]

The ratio of \( X \) to \( Y \) is:

\[ \frac{X}{Y} = \frac{3.0944}{0.0944} = 32.77 \]

Solution 38

D Chapter 9, Proposition 1

The prices of Option A and Option B violate Proposition 1:

\[ P(K_2) \geq P(K_1) \]
This is because:

\[ P(127) < P(120) \]
\[ 10 < 12 \]

Therefore, arbitrage profits can be earned by buying the 127-strike put and selling the 120-strike put. This is a put bear spread.

**Solution 39**

C Chapter 9, Proposition 3

The prices of the options violate Proposition 3:

\[
\frac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2}
\]

This is because:

\[
\frac{C(45) - C(50)}{50 - 45} < \frac{C(50) - C(53)}{53 - 50}
\]
\[
\frac{7 - 5}{50 - 45} < \frac{5 - 3}{53 - 50}
\]
\[
\frac{2}{5} < \frac{2}{3}
\]

Arbitrage is available using an asymmetric butterfly spread:

Buy \( \lambda \) of the 45-strike options
Sell 1 of the 50-strike options
Buy \((1 - \lambda)\) of the 53-strike options

where:

\[
\lambda = \frac{K_3 - K_2}{K_3 - K_1} = \frac{53 - 50}{53 - 45} = \frac{3}{8}
\]

In the payoff table below, we have scaled the strategy up by multiplying by 8:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S &lt; 45</td>
<td>45 ≤ S &lt; 50</td>
</tr>
<tr>
<td>Buy 3 of C(45)</td>
<td>-3(7.00)</td>
<td>0.00</td>
</tr>
<tr>
<td>Sell 8 of C(50)</td>
<td>8(5.00)</td>
<td>0.00</td>
</tr>
<tr>
<td>Buy 5 of C(53)</td>
<td>-5(3.00)</td>
<td>0.00</td>
</tr>
<tr>
<td>Total</td>
<td>4.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
The cash flows at time 1 for each of the possible stock prices are in the table below:

<table>
<thead>
<tr>
<th>Answer Choice</th>
<th>Stock Price</th>
<th>Time 1 Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>40</td>
<td>0.00</td>
</tr>
<tr>
<td>B</td>
<td>46</td>
<td>$3(S_1 - 45) = 3(46 - 45) = 3.00$</td>
</tr>
<tr>
<td>C</td>
<td>50</td>
<td>$3(S_1 - 45) = 3(50 - 45) = 15.00$</td>
</tr>
<tr>
<td>D</td>
<td>52</td>
<td>$265 - 5S_1 = 265 - 5(52) = 5.00$</td>
</tr>
<tr>
<td>E</td>
<td>55</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The highest cash flow at time 1, $15.00, occurs if the final stock price is $50. This results in the highest arbitrage profits since the time 0 cash flow is the same for each future stock price.

**Solution 40**

A Chapter 9, Bid-Ask Prices

*This question might seem a bit unfair since bid and ask prices are not discussed in the assigned reading in the textbook. On the other hand, Problem 9.16 at the end of the textbook’s Chapter 9 uses bid and ask prices. Someone writing an exam question just might think that it is a clever idea for an exam question.*

The arbitrageur must pay the ask price to purchase assets. The arbitrageur receives the bid price when selling assets.

Let’s write put-call parity as follows:

$$S_0 = C_{Eur}(K,T) + Ke^{-rT} - P_{Eur}(K,T)$$

This shows that the purchase of a share of stock can be replicated by:

- buying a call option
- lending the present value of the strike price
- selling a put option

Let’s see if arbitrage profits can be earned by selling the stock and then replicating the purchase of the stock. The cash flows at time 0 are broken out below:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell Stock</td>
<td>99.75</td>
</tr>
<tr>
<td>Buy Call</td>
<td>-19.75</td>
</tr>
<tr>
<td>Lend $90e^{-0.059}$</td>
<td>-84.8436</td>
</tr>
<tr>
<td>Sell Put</td>
<td>5.00</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>0.1564</td>
</tr>
</tbody>
</table>

Selling the stock and replicating the purchase of the stock results in a time 0 cash flow of 0.1564. The cash flow at the end of the year will be zero regardless of the stock price. Therefore, the time 0 cash flow of 0.1564 is an arbitrage profit.
At this point, we have answered the question, but for thoroughness, we also consider whether arbitrage profits can be earned by buying the stock and replicating the sale of the stock.

The sale of a share of stock can be replicated by doing the opposite of the actions needed to replicate the purchase of a share of stock. The sale of a share of stock is replicated by:

- selling a call option
- borrowing the present value of the strike price
- buying a put option

Let’s see if arbitrage profits can be earned by buying the stock and then replicating the sale of the stock.

The cash flows at time 0 are broken out below:

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Stock</td>
<td>-100.00</td>
</tr>
<tr>
<td>Sell Call</td>
<td>19.50</td>
</tr>
<tr>
<td>Borrow $90e^{-0.06}$</td>
<td>84.7588</td>
</tr>
<tr>
<td>Buy Put</td>
<td>-5.10</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>-0.8412</td>
</tr>
</tbody>
</table>

Buying the stock and replicating the sale of the stock results in a time 0 cash flow that is negative, so it does not produce arbitrage profits.

Solution 41

We are told that the price of a European call option with a strike price of $54.08 has a value of $10.16. The payoff of the call option at time 2 is:

$$Max[0, S(2) - 54.08]$$

Once we express the payoff of the single premium deferred annuity in terms of the expression above, we will be able to obtain the price of the annuity.

The payoff at time 2 is:

$$\text{Time 2 Payoff} = P(1 - y\%) \times Max \left[ \frac{S(2)}{50}, 1.04^2 \right]$$

$$= P(1 - y\%) \times \frac{1}{50} Max \left[ S(2), 50 \times 1.04^2 \right]$$

$$= P(1 - y\%) \times \frac{1}{50} Max \left[ S(2), 54.08 \right]$$

$$= P(1 - y\%) \times \frac{1}{50} \left[ Max \left[ S(2) - 54.08, 0 \right] + 54.08 \right]$$
The current value of a payoff of \( \text{Max}[0, S(2) - 54.08] \) at time 2 is $10.16. The current value of $54.08 at time 2 is:

\[
54.08e^{-0.06 \times 2} = 47.9647
\]

Therefore, the current value of the payoff is:

\[
\text{Current value of payoff} = P(1 - y\%) \times \frac{1}{50} \{10.16 + 47.9647\}
\]

For the company to break even on the contract, the current value of the payoff must be equal to the single premium of \( P \):

\[
P(1 - y\%) \times \frac{1}{50} \{10.16 + 47.9647\} = P
\]

\[
(1 - y\%) \times 1.16249 = 1
\]

\[
y\% = 13.978\%
\]

**Solution 42**

**C** Chapter 9, Early Exercise

Early exercise should not occur if the interest on the strike price exceeds the value of the dividends obtained through early exercise:

\[
\text{No early exercise if: } K - Ke^{-r(T-t)} > PV_{t,T}(\text{div})
\]

The present value of the dividends is:

\[
PV_{t,T}(\text{div}) = 3 + 2e^{-0.10(0.75)} = 4.86
\]

The interest cost of paying the strike price early is shown in the rightmost column below:

<table>
<thead>
<tr>
<th>Option</th>
<th>Strike Price</th>
<th>( T )</th>
<th>( K - Ke^{-r(T-0)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>40</td>
<td>1.50</td>
<td>5.57</td>
</tr>
<tr>
<td>B</td>
<td>50</td>
<td>1.50</td>
<td>6.96</td>
</tr>
<tr>
<td>C</td>
<td>50</td>
<td>1.00</td>
<td>4.76</td>
</tr>
<tr>
<td>D</td>
<td>52</td>
<td>1.00</td>
<td>4.95</td>
</tr>
<tr>
<td>E</td>
<td>59</td>
<td>0.75</td>
<td>4.26</td>
</tr>
</tbody>
</table>

Only Option C and Option E have interest on the strike price that is less than the present value of the dividends of $4.86. Option E is not in the money though, because its strike price exceeds the stock price of $58, so it is not optimal to exercise Option E. Therefore, Option C is the only option for which early exercise might be optimal.
Solution 43

A Chapter 9, Put-Call Parity

Let each dividend amount be $D$. The first dividend occurs at the end of 2 months, and the second dividend occurs at the end of 5 months.

We can use put-call parity to find $D$:

\[
C_{Eur}(K,T) + Ke^{-rT} = S_0 - PV_{0,T}(Div) + P_{Eur}(K,T)
\]

\[
2.55 + 42e^{-0.05(0.5)} = 40 - De^{-0.05(2/12)} - De^{-0.05(5/12)} + 4.36
\]

\[
D[e^{-0.05(2/12)} + e^{-0.05(5/12)}] = 40 + 4.36 - 2.55 - 42e^{-0.05(0.5)}
\]

\[
1.97108 = 0.84698
\]

\[
D = 0.4297
\]

Solution 44

D Chapter 9, Early Exercise

For each put option, the choice is between having the exercise value now or having a 1-year European put option. Therefore, the decision depends on whether the exercise value is greater than the value of the European put option. The value of each European put option is found using put-call parity:

\[
C_{Eur}(K,T) + Ke^{-rT} = S_0e^{-rT} + P_{Eur}(K,T)
\]

\[
P_{Eur}(K,T) = C_{Eur}(K,T) + Ke^{-rT} - S_0e^{-rT}
\]

The values of each of the 1-year European put options are:

\[
P_{Eur}(25,1) = 21.93 + 25e^{-0.05(1)} - 50e^{-0.09(1)} = 0.01
\]

\[
P_{Eur}(50,1) = 3.76 + 50e^{-0.05(1)} - 50e^{-0.09(1)} = 5.62
\]

\[
P_{Eur}(75,1) = 0.21 + 75e^{-0.05(1)} - 50e^{-0.09(1)} = 25.86
\]

\[
P_{Eur}(100,1) = 0.01 + 100e^{-0.05(1)} - 50e^{-0.09(1)} = 49.44
\]

In the third and fourth columns of the table below, we compare the exercise value with the value of the European put options. The exercise value is $Max(K - S_0, 0)$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C$</th>
<th>Exercise Value</th>
<th>European Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>$25.00$</td>
<td>$21.93$</td>
<td>0</td>
<td>0.01</td>
</tr>
<tr>
<td>$50.00$</td>
<td>$3.76$</td>
<td>0</td>
<td>5.62</td>
</tr>
<tr>
<td>$75.00$</td>
<td>$0.21$</td>
<td>25</td>
<td>25.86</td>
</tr>
<tr>
<td>$100.00$</td>
<td>$0.01$</td>
<td>50</td>
<td>49.44</td>
</tr>
</tbody>
</table>
The exercise value is less than the value of the European put option when the strike price is $75 or less. When the strike price is $100, the exercise value is greater than the value of the European put option. Therefore, it is optimal to exercise the special put option with an exercise price of $100.

Solution 45

Chapter 9, Exchange Options

The first page of the study note, "Some Remarks on Derivatives Markets," tells us that "for each share of the stock the amount of dividends paid between time t and time t + dt is assumed to be $S(t)\delta dt \). Therefore, the continuously compounded dividend rate for Stock 1 is 7%, and the continuously compounded dividend rate for Stock 2 is 3%.

The claim has the following payoff at time 4:

$$\text{Max}[S_1(4), S_2(4)]$$

A portfolio consisting of a share of Stock 2 and the option to exchange Stock 2 for Stock 1 effectively gives its owner the maximum value of the two stocks. If the value of Stock 2 is greater than the value of Stock 1 at time 4, then the owner keeps Stock 2 and allows the exchange option to expire unexercised. If the value of Stock 1 is greater than the value of Stock 2, then the owner exercises the option, giving up Stock 2 for Stock 1.

Since Stock 2 has a continuously compounded dividend rate of 3%, the cost now of a share of Stock 2 at time 4 is:

$$F_{0.4}^{P}(S_2) = e^{-0.03(4)}S_0 = 75.39$$

The cost of an exchange option allowing its owner to exchange Stock 2 for Stock 1 at time 4 is $43.

Adding the costs together, we obtain the cost of the claim:

$$75.39 + 43 = 118.39$$

Solution 46

Chapter 9, Reverse Conversion

A T-bill with a current value of $55e^{-0.75r} + Xe^{-0.50r}$ can be replicated with a conversion by purchasing a share of stock, selling a call option, and purchasing a put option:

$$Ke^{-rT} + PV_{0,T}(Div) = S_0 - C_{Eur}(K,T) + P_{Eur}(K,T)$$

$$55e^{-0.75r} + Xe^{-0.50r} = 50 - 2.38 + 8.98$$

$$= 56.60$$
Doing the opposite creates a reverse conversion. Selling a share of stock, buying a call option, and selling a put option is a reverse conversion and provides us with $56.60 now and an obligation to pay \( 55 + Xe^{0.25r} \) in 9 months.

To borrow $1,000 now, we must sell:

\[
\frac{1,000}{56.60} = 17.67 \text{ shares of stock.}
\]

**Solution 47**

**B** Chapter 9, Strike Price Grows Over Time

The underlying asset is the total return portfolio.

The strike price grows at the interest rate:

\[
5,101 = 5,000e^{0.08(0.25)} \implies K_T \leq K_te^{r(T-t)}
\]

Therefore, to preclude arbitrage it should be the case that:

\[
C(0.75) \geq C(0.5)
\]

But the longer option costs less than the shorter option, indicating that arbitrage is possible:

\[
470 < 473
\]

The arbitrageur buys the longer option for $470 and sells the shorter option for $473. The difference of $3 is lent at the risk-free rate of return.

**The 6-month option**

After 6 months, the stock price is $49. Therefore, the value of the total return portfolio is:

\[
100 \times e^{0.5(0.06)} \times 49 = 5,049.23
\]

Since the strike price of the shorter option is $5,000, the shorter option is exercised against the arbitrageur. The arbitrageur sells the total return portfolio short for $5,000 and lends the $5,000 at the risk-free rate.

After 9 months, the stock price is $51. Therefore, the value of the total return portfolio is:

\[
100 \times e^{0.06(0.75)} \times S_{0.75} = 100 \times e^{0.06(0.75)} \times 51 = 5,334.74
\]

At the end of 9 months the arbitrageur owes the total return portfolio and owns the accumulated value of the $5,000. The position results in the following cash flow at the end of 9 months:

\[-5,334.74 + 5,000e^{0.08(0.25)} = -233.74\]
The 9-month option

The strike price of the 9-month option is $5,101, so the 9-month call option has the following cash flow at the end of 9 months:

\[100 \times e^{-0.06 \times 0.75} \times S_{0.75} - 5,101 = 5,334.74 - 5,101.00 = 233.74\]

The net cash flow

Since the 6-month option and the 9-month options have offsetting cash flows at the end of 9 months, the accumulated value of the arbitrage strategy is the difference between the cost of the two options, accumulated for 9 months:

\[(473 - 470)e^{0.08 \times 0.75} = 3.19\]

Solution 48

Let’s begin by noting the prepaid forward price of the stock and the present value of the strike price:

\[F_{t,T}^P(S) = e^{-\delta(T-t)} S = e^{-0.08 \times 0.5} S = S e^{-0.04}\]

\[Ke^{-r(T-t)} = 100e^{-0.08 \times 0.5} = 96.08\]

For the European call option, the bounds are:

\[\text{Max} \left[ 0, F_{t,T}^P(S) - Ke^{-r(T-t)} \right] \leq C_{Eur}(S_t, K, T-t) \leq F_{t,T}^P(S)\]

\[\text{Max} \left[ 0, Se^{-0.04} - 96.08 \right] \leq C_{Eur}(S_t, K, T-t) \leq Se^{-0.04}\]

This describes Graph I, so the European call option corresponds to Graph I.

For the American call option, the bounds are:

\[\text{Max} \left[ 0, F_{t,T}^P(S) - Ke^{-r(T-t)}, S - K \right] \leq C_{Amer}(S_t, K, T-t) \leq S_t\]

\[\text{Max} \left[ 0, Se^{-0.04} - 96.08, S - 100 \right] \leq C_{Amer}(S_t, K, T-t) \leq S\]

\[\text{Max} \left[ 0, S - 100 \right] \leq C_{Amer}(S_t, K, T-t) \leq S\]

This describes Graph II, so the American call option corresponds to Graph II.

For the European put option, the bounds are:

\[\text{Max} \left[ 0, Ke^{-r(T-t)} - F_{t,T}^P(S) \right] \leq P_{Eur}(S_t, K, T-t) \leq Ke^{-r(T-t)}\]

\[\text{Max} \left[ 0, 96.08 - Se^{-0.04} \right] \leq P_{Eur}(S_t, K, T-t) \leq 96.08\]

This describes Graph IV, so the European put option corresponds to Graph IV.
For the American put option, the bounds are:
\[
\begin{align*}
\Max & \left[ 0, Ke^{-r(T-t)} - F^P_{t,T}(S), K - S \right] \leq P_{\text{Amer}}(S_t, K, T - t) \leq K \\
\Max & \left[ 0, 96.08 - Se^{-0.04}, 100 - S \right] \leq P_{\text{Amer}}(S_t, K, T - t) \leq 100 \\
\Max & \left[ 0, 100 - S \right] \leq P_{\text{Amer}}(S_t, K, T - t) \leq 100
\end{align*}
\]
This describes Graph III, so the American put option corresponds to Graph III.

**Solution 49**

**C** Chapter 9, Propositions 2 and 3

Choice A is true. Proposition 2 can be used to determine the following relationship for the first derivative of a call price:
\[
\begin{align*}
\frac{C(K_1) - C(K_2)}{K_2 - K_1} & \leq 1 \\
\frac{C(K_2) - C(K_1)}{K_2 - K_1} & \leq 1 \\
\Rightarrow & - \frac{dC}{dK} \leq 1 \\
0 & \leq 1 + \frac{dC}{dK}
\end{align*}
\]
Choice B is true, because Proposition 2 implies that the first derivative of a put price is positive, and the first derivative of a call option is negative.

Choice C is false because Proposition 3 implies that the second derivative of the call price is positive:
\[
\begin{align*}
\frac{C(K_1) - C(K_2)}{K_2 - K_1} & \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2} \\
\frac{C(K_2) - C(K_1)}{K_2 - K_1} & \leq \frac{C(K_3) - C(K_2)}{K_3 - K_2} \\
0 & \leq \frac{C(K_3) - C(K_2)}{K_3 - K_2} - \frac{C(K_2) - C(K_1)}{K_2 - K_1} \\
\Rightarrow & 0 \leq \frac{d^2C}{dK^2}
\end{align*}
\]
Choice D and Choice E are true because the second derivative of a European put price is equal to the second derivative of a European call price. We can show this by taking the derivative of both sides of the put-call parity expression:

\[
\frac{d^2C}{dK^2} = \frac{d^2P}{dK^2}
\]

Solution 50

A Chapter 9, Option Payoffs

A butterfly spread involves options with 3 different strike prices. Part one of the investor's position is a symmetric butterfly spread with strike prices $2, $3 and $4. Part two of the investor's position is a symmetric butterfly spread with strike prices $4, $5 and $6. The payoff when buying a butterfly spread is never less than zero. In this case, each butterfly spread was sold, which means the payoffs are less than zero. The payoff of the butterfly spread looks the same whether the position is composed of calls or puts. Part one of the position's payoff is shown as the bold line below at the left, and part two of the position's payoff is shown as the bold line below at the right.

When combined, the investor's position looks like Choice A.

Solution 51

D Chapter 9, Strike Price Grows Over Time

The prices of the options decrease as time to maturity increases. Therefore, if the strike price increases at a rate that is less than the risk-free rate, then arbitrage is available. Option B expires 0.5 years after Option A, so let's accumulate Option A's strike price for 0.5 years at the risk-free rate:

\[
50e^{0.06 \times 0.5} = 51.5227
\]

Since the strike price of Option B is $52, which is greater than $51.5227, arbitrage is not indicated by the prices of Option A and Option B.
Option C expires 0.5 years after Option B, so let’s accumulate Option B’s strike price for 0.5 years at the risk-free rate:

\[ 52e^{0.06 \times 0.5} = 53.5836 \]

Since the strike price of Option C is $53, the strike price grows from time 1 to time 1.5 at a rate that is less than the risk-free rate of return. Consequently, arbitrage can be earned by purchasing Option C and selling Option B (i.e., buy low and sell high).

The arbitrageur buys the 2-year option for $7.50 and sells the 1.5-year option for $7.70. The difference of $0.20 is lent at the risk-free rate of return.

The 1.5-year option

After 1.5 years, the stock price is $52.50. Therefore, the 1.5-year option is exercised against the arbitrageur. The arbitrageur borrows a share of stock and sells it for the strike price of $52. As a result, at the end of 2 years the arbitrageur owes the share of stock and has the accumulated value of the $52. This position results in the following cash flow at the end of 2 years:

\[ -52.50 + 52e^{0.06 \times 0.5} = 1.0836 \]

The 2-year option

The stock price of $52.50 at the end of 2 years is less than the strike price of the 2-year option, which is $53. Therefore, the 2-year call option expires worthless, and the resulting cash flow is zero.

The net cash flow

The net cash flow at the end of 2 years is the sum of the accumulated value of the $0.20 that was obtained by establishing the position, the $1.0836 resulting from the 1.5-year option, and the $0.00 resulting from the 2-year option:

\[ 0.20e^{0.06 \times 2} + 1.0836 + 0.00 = 1.3091 \]

Solution 52

Let \( X \) be the number of puts with a strike price of $60 that are sold for Jill’s portfolio. The fact that the net cost of establishing the portfolio is zero allows us to solve for \( X \):

\[ -P(45) + 3P(55) + 1 - P(60) \times X = 0 \]
\[ -4 + 3 \times 9 + 1 - 12X = 0 \]
\[ X = 2 \]
An arbitrage strategy does not allow the cash flow at expiration to be negative. But suppose that only the $60-strike put option is in-the-money at expiration. Since Jill is short the $60-strike put option, this results in a negative cash flow. In particular, if the stock price at expiration is between $55 and $59, then the payoff from Jill’s strategy will be negative. For example, if the stock price at expiration is $57, the payoff from Jill’s strategy will be:

\[-\text{[45-strike payoff]} + 3\text{[55-strike payoff]} - 2\text{[60-strike payoff]} + \text{[Proceeds from loan]}\]

\[0 + 0 - 2(60 - 57) + 1 = -5\]

Since Jill’s strategy can result in a negative payoff at expiration, Jill’s strategy is not arbitrage.

Let $Y$ be the number of calls with a strike price of $60 that are purchased for Sabrina’s portfolio. The fact that the net cost of establishing the portfolio is zero allows us to solve for $Y$:

\[C(45) - 3C(55) + 1 + C(60) \times Y = 0\]

\[12 - 3 \times 7 + 1 + 4Y = 0\]

\[Y = 2\]

The table below shows that regardless of the stock price at time $T$, Sabrina’s payoff is positive. Therefore, Sabrina is correct.

<table>
<thead>
<tr>
<th>Sabrina’s Portfolio</th>
<th>Time $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Transaction</strong></td>
<td><strong>Time 0</strong></td>
</tr>
<tr>
<td>Buy 1 of $C(45)$</td>
<td>12.00</td>
</tr>
<tr>
<td>Sell 3 of $C(55)$</td>
<td>7.00</td>
</tr>
<tr>
<td>Buy 2 of $C(60)$</td>
<td>4.00</td>
</tr>
<tr>
<td>Lend $1$</td>
<td>1.00</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0.00</td>
</tr>
</tbody>
</table>

Let $Z$ be the number of calls purchased and puts sold with a strike price of $55 for Kelly’s portfolio. Since the net cost of establishing the portfolio is zero, we can solve for $Z$:

\[2[C(60) - P(60)] + 1[C(45) - P(45)] + 2 - Z[C(55) - P(55)] = 0\]

\[2[4 - 12] + [12 - 4] + 2 - Z[7 - 9] = 0\]

\[-16 + 8 + 2 + 2Z = 0\]

\[Z = 3\]
In evaluating Kelly’s portfolio, we can make use of the fact that purchasing a call option and selling a put option is equivalent to purchasing a prepaid forward on the stock and borrowing the present value of the strike price. We can see this by writing put-call parity as:

\[ C_{Eur}(K,T) - P_{Eur}(K,T) = F_{0,T}^P(S) - Ke^{-rT} \]

Therefore, purchasing a call option and selling a put option results in a payoff of:

\[ S_T - K \]

Since Kelly purchases offsetting amounts of puts and calls for any given strike price, we can use this result to evaluate her payoffs.

**Kelly’s Portfolio**

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 2 of C(60) &amp; sell 2 of P(60)</td>
<td>2(12.00 − 4.00)</td>
<td>2(S_T − 60)</td>
</tr>
<tr>
<td>Buy 1 of C(45) &amp; sell 1 of P(45)</td>
<td>4.00 − 12.00</td>
<td>S_T − 45</td>
</tr>
<tr>
<td>Sell 3 of C(55) &amp; buy 3 of P(55)</td>
<td>3(7.00 − 9.00)</td>
<td>−3(S_T − 55)</td>
</tr>
<tr>
<td>Lend $2 &amp; buy 3 of P(55)</td>
<td>−2.00</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>0.00</td>
<td>2</td>
</tr>
</tbody>
</table>

Kelly’s portfolio is certain to have a positive payoff at time \( T \), so Kelly is correct.

**Solution 53**

E Chapter 9, Minimum of 2 Assets

The European option has a payoff of:

\[ Max[17 - Min(S_1(1), S_2(1)), 0] \]

We recognize the European option as a put option on the minimum of the two stocks. Let’s call the underlying asset \( X \): \( X(1) = Min[S_1(1), S_2(1)] \)

The price of the option is $0.67:

\[ P_{Eur}(X,17,1) = 0.67 \]

We are told that a claim that pays the minimum of \( X(1) \) and 17 has a current value of $15.50:

\[ F_{0,1}^P(\text{Min}[S_1(1), S_2(1), 17]) = F_{0,1}^P(\text{Min}[X(1), 17]) = 15.50 \]
We can express the value of the claim as the value of the strike asset minus the value of the put option:

\[
F_{t,T}^P (\text{Min}[X_{T,T}, Q_{T,T}]) = F_{t,T}^P (Q) - P_{\text{Eur}} (X_t, Q_t, T - t)
\]

\[
F_{0,1}^P (\text{Min}[X(1), 17]) = 17e^{-r} - 0.67
\]

\[
15.50 = 17e^{-r} - 0.67
\]

\[
16.17 = 17e^{-r}
\]

\[
r = 0.05
\]

**Solution 54**

**A** Chapter 9, Maximum of 2 Assets

The European option has a payoff of:

\[
\text{Max} [\text{Max}(S_1(1), S_2(1)) - 22, 0]
\]

We recognize the European option as a call option on the maximum of the two stocks. Let’s call the underlying asset \(X\):

\[
X(1) = \text{Max}[S_1(1), S_2(1)]
\]

The price of the option is $2.83:

\[
C_{\text{Eur}} (X, 22, 1) = 2.83
\]

We are told that a claim that pays the maximum of \(X(1)\) and 22 has a current value of $23.34:

\[
F_{0,1}^P (\text{Max}[S_1(1), S_2(1), 22]) = F_{0,1}^P (\text{Max}[X(1), 22]) = 23.34
\]

We can express the value of the claim as the value of the call option plus the value of the strike asset:

\[
F_{t,T}^P (\text{Max}[X_{T,T}, Q_{T,T}]) = C_{\text{Eur}} (X_t, Q_t, T - t) + F_{t,T}^P (Q)
\]

\[
F_{0,1}^P (\text{Max}[X(1), 22]) = 2.83 + 22e^{-r}
\]

\[
23.34 = 2.83 + 22e^{-r}
\]

\[
20.51 = 22e^{-r}
\]

\[
r = 0.0701
\]
Solution 55

D  Chapter 9, Maximum of 2 Assets

The European option has a payoff of:

\[
\text{Max} \left[ \text{Max} \left( S_1(3), S_2(3) \right) - 25, 0 \right]
\]

We recognize the European option as a call option on the maximum of the two stocks. Let’s call the underlying asset \( X \):

\[
X(3) = \text{Max} \left[ S_1(3), S_2(3) \right]
\]

We are told that a claim that pays the maximum of \( X(3) \) and 25 has a current value of $25.50:

\[
F_{0,3}^P \left( \text{Max} \left[ S_1(3), S_2(3), 25 \right] \right) = F_{0,3}^P \left( \text{Max} \left[ X(3), 25 \right] \right) = 25.50
\]

We can express the value of the claim as the value of the call option on \( X \) plus the value of the strike asset:

\[
F_{t,T}^P \left( \text{Max} \left[ X_T, Q_T \right] \right) = C_{\text{Eur}} \left( X_t, Q_t, T-t \right) + F_{t,T}^P \left( Q \right)
\]

\[
F_{0,3}^P \left( \text{Max} \left[ X_3, 25 \right] \right) = C_{\text{Eur}} \left( X_3, 25e^{-3 \times 0.08}, 3 \right) + 25e^{-3 \times 0.08} = 25.50
\]

\[
C_{\text{Eur}} \left( X_3, 25e^{-3 \times 0.08}, 3 \right) = 5.8343
\]

Solution 56

B  Chapter 9, Maximum of 2 Assets

The European option has a payoff of:

\[
\text{Max} \left[ 25 - \text{Min} \left( 2X(5), 3Y(5) \right), 0 \right]
\]

We recognize the European option as a put option on the minimum of the two stocks. Let’s call the underlying asset \( Z \):

\[
Z(5) = \text{Min} \left( 2X(5), 3Y(5) \right)
\]

The rainbow option pays the minimum of \( Z(5) \) and 25, and it has a current value of $12.67:

\[
F_{0,5}^P \left( \text{Min} \left[ 2X(5), 3Y(5), 25 \right] \right) = F_{0,5}^P \left( \text{Min} \left[ Z(5), 25 \right] \right) = 12.67
\]
We can express the value of the claim as the value of the prepaid forward price of the strike asset minus the value of the European put option:

\[
P_{t,T}^F \left( \text{Min}[Z_T, Q_T] \right) = F_{t,T}^P(Q) - P_{Eur}(Z_t, Q_t, T - t)
\]

\[
P_{0.5}^F \left( \text{Min}[Z(5), 25] \right) = 25e^{-5 \times 0.05} - P_{Eur}(Z_t, 25e^{-5 \times 0.05}, 5)
\]

\[
12.67 = 19.4700 - P_{Eur}(Z_t, 25e^{-5 \times 0.05}, 5)
\]

\[
P_{Eur}(Z_t, 25e^{-5 \times 0.05}, 5) = 6.80
\]

**Solution 57**

**D** Chapter 9, Bounds on Option Prices

From Propositions 1 and 2, we see that Statement I is true:

Proposition 1: \( P(K_2) \geq P(K_1) \) for \( K_1 < K_2 \)

\[\Rightarrow 0 \leq P(85) - P(80)\]

Proposition 2: \( P_{Eur}(K_2) - P_{Eur}(K_1) \leq (K_2 - K_1)e^{-rT} \) for \( K_1 < K_2 \)

\[\Rightarrow P_{Eur}(85) - P_{Eur}(80) \leq (85 - 80)e^{-rT}\]

We make use of put-call parity for Statements II and III:

\[C(K) + K^{-rT} = S + P(K)\]

When the strike price for a call is increased, its price goes down, so the first inequality in Statement II is true:

\[P(75) - C(75) + S = 75e^{-rT} \Rightarrow P(75) - C(80) + S \geq 75e^{-rT}\]

When the strike price for a put is decreased, its price goes down, so the second inequality in Statement II is true:

\[P(80) - C(80) + S = 80e^{-rT} \Rightarrow P(75) - C(80) + S \leq 80e^{-rT}\]

As we saw in Statement II (directly above), the first inequality in Statement III is false:

\[P(80) - C(80) + S = 80e^{-rT} \Rightarrow P(75) - C(80) + S \leq 80e^{-rT}\]

The second inequality in Statement III is true (but Statement III is still false):

\[P(80) - C(80) + S = 80e^{-rT} \Rightarrow P(75) - C(80) + S \leq 80e^{-rT} \Rightarrow P(75) - C(80) + S \leq 85\]

**Solution 58**

**D** Chapter 9, Early Exercise of an American Call

If it is optimal to exercise an American call prior to maturity, then the early exercise takes place just before a dividend payment. Therefore the call is not exercised at time 0, and Choice A is not the correct answer.
The call is not exercised early if the present value of the interest on the strike exceeds the present value of the dividends:

\[ K - Ke^{-r(T-t)} > PV_{t,T}(div) \Rightarrow \text{Don't exercise early} \]

Let's consider time 0.25:

\[ 86 - 86e^{-0.09(1-0.25)} > 2 + 1.9e^{-0.09(0.75-0.25)} \]
\[ 5.6134 > 3.8164 \]

Since the present value of the interest on the strike exceeds the present value of the dividends, the call option is not exercised at time 0.25, so Choice B is not the correct answer.

Let's consider time 0.75:

\[ 86 - 86e^{-0.09(1-0.75)} > 1.9 \]
\[ 1.9134 > 1.9 \]

Since the present value of the interest on the strike exceeds the present value of the dividends, the call option is not exercised at time 0.75, so Choice C is not the correct answer.

Since the call was exercised, it must have been exercised at maturity, at the end of 12 months. Therefore, the correct answer is Choice D.

**Solution 59**

**B**  Chapter 9, Put-Call Parity

We can rewrite the payoff of the first option with parentheses in the first Max function:

\[ \max[0, S(1) - (K - 1)] - \max[0, S(1) - K] \]

We can now see that the special option consists of a long position in a call with a strike price of \((K - 1)\) and a short position in a call with a strike price of \(K\):

\[ \hat{C}(K) = C(K - 1) - C(K) \]

Likewise, the second option consists of a long position in a put with a strike price of \(K\) and a short position in a put with a strike price of \((K - 1)\):

\[ \hat{P}(K) = P(K) - P(K - 1) \]
We can add the value of the two special options together and use put-call parity to show that the value of the sum does not depend on $K$:

$$
\hat{C}(K) + \hat{P}(K) = C(K - 1) - C(K) + P(K) - P(K - 1)
= [C(K - 1) - P(K - 1)] + [P(K) - C(K)]
= \left[ S(1) - (K - 1)e^{-r} \right] + \left[ Ke^{-r} - S(1) \right]
= 1e^{-r} = e^{-0.22} = 0.8025
$$

We can now find $\hat{P}(109)$:

$$
\hat{C}(K) + \hat{P}(K) = 0.8025
\hat{C}(109) + \hat{P}(109) = 0.8025
0.3 + \hat{P}(109) = 0.8025
\hat{P}(109) = 0.8025 - 0.3 = 0.5025
$$

**Solution 60**

A Chapter 9, Put-Call Parity

To answer this question, we use put-call parity:

$$
C(K) - P(K) = F_{0.2}^P(S) - Ke^{-2r}
$$

Let’s subtract the value of Set 2 from the value of Set 1 and use put-call parity to simplify:

$$
[C(48) - P(54)] - [P(48) - C(54)] = 11.91 - (-10.53)
[C(48) - P(48)] + [C(54) - P(54)] = 22.44
[F_{0.2}^P(S) - 48e^{-2r}] + [F_{0.2}^P(S) - 54e^{-2r}] = 22.44
2F_{0.2}^P(S) - 102e^{-2r} = 22.44
$$

Subtracting Set 4 from Set 3 and again using put-call parity to simplify, we obtain:

$$
[C(55) - P(47)] - [P(55) - C(47)] = 10.30 - [P(55) - C(47)]
[C(55) - P(55)] + [C(47) - P(47)] = 10.30 - [P(55) - C(47)]
[F_{0.2}^P(S) - 55e^{-2r}] + [F_{0.2}^P(S) - 47e^{-2r}] = 10.30 - [P(55) - C(47)]
2F_{0.2}^P(S) - 102e^{-2r} = 10.30 - [P(55) - C(47)]
22.44 = 10.30 - [P(55) - C(47)]
[P(55) - C(47)] = -12.14
$$
Solution 61

A Chapter 9, Application of Option Pricing Concepts

We are told that the price of a European put option with a strike price of $54.08 has a value of $8.96. The payoff of the put option at time 2 is:

\[ \text{Max}[0, 54.08 - S(2)] \]

Once we express the payoff of the single premium deferred annuity in terms of the expression above, we will be able to obtain the price of the annuity.

The payoff at time 2 is:

\[
\begin{align*}
\text{Time 2 Payoff} &= P(1 - y\%) \times \text{Max}\left[ \frac{S(2)}{50}, 1.04^2 \right] \\
&= P(1 - y\%) \times \frac{1}{50} \text{Max}\left[ S(2), 50 \times 1.04^2 \right] \\
&= P(1 - y\%) \times \frac{1}{50} \text{Max}\left[ S(2), 54.08 \right] \\
&= P(1 - y\%) \times \frac{1}{50} \left( \text{Max}[0, 54.08 - S(2)] + S(2) \right)
\end{align*}
\]

The current value of a payoff of \[ \text{Max}[0, 54.08 - S(2)] \] at time 2 is $8.96. The current value of \( S(2) \) is its prepaid forward price:

\[
P_{0,2}^P(S) = e^{-2\delta} S(0) = e^{-2 \times 0.07} \times 50 = 43.4679
\]

Therefore, the current value of the payoff is:

\[
\text{Current value of payoff} = P(1 - y\%) \times \frac{1}{50} \{8.96 + 43.4679\}
\]

For the company to break even on the contract, the current value of the payoff must be equal to the single premium of \( P \):

\[
P(1 - y\%) \times \frac{1}{50} \{8.96 + 43.4679\} = P
\]

\[
(1 - y\%) \times 1.04856 = 1
\]

\[
y\% = 4.631\%
\]
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Chapter 10 – Solutions

Solution 1
C Chapter 10, One-Period Binomial Tree

If the stock price moves up, then the call option pays $25. If the stock price moves down, then the call pays $0:

\[
\begin{array}{c c c}
75 & 25 = \text{Max}[75 - 50, 0] \\
51 & V \\
40 & 0 = \text{Max}[40 - 50, 0]
\end{array}
\]

In this case, \( u = \frac{75}{51} \) and \( d = \frac{40}{51} \).

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{0.07(1)} - 40/51}{75/51 - 40/51} = \frac{51e^{0.07(1)} - 40}{75 - 40} = 0.41994
\]

The value of the call option is:

\[
V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.07(1)} \left[ 0.41994(25) + 0 \right] = 9.79
\]

Solution 2
B Chapter 10, One-Period Binomial Tree

If the stock price moves up, then the put option pays $0. If the stock price moves down, then the put pays $10:

\[
\begin{array}{c c c}
75 & 0 = \text{Max}[50 - 75, 0] \\
51 & V \\
40 & 10 = \text{Max}[50 - 40, 0]
\end{array}
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{51e^{0.07(1)} - 40}{75 - 40} = 0.41994
\]

The value of the put option is:

\[
V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.07(1)} \left[ 0.41994(0) + (1 - 0.41994)10 \right] = 5.41
\]
Solution 3
D  Chapter 10, Delta

If the stock price moves up, then the call option pays $25. If the stock price moves down, then the call pays $0:

\[
\begin{array}{c|c|c}
75 & 25 \\
51 & \\
40 & 0 \\
\end{array}
\]

To replicate the option, the investor must purchase delta shares of the underlying stock:
\[
\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u - d)} = e^{-0.03(1)} \frac{25 - 0}{75 - 40} = 0.693
\]

Solution 4
A  Chapter 10, Replication

If the stock price moves up, then the call option pays $25. If the stock price moves down, then the call pays $0:

\[
\begin{array}{c|c|c}
75 & 25 \\
51 & \\
40 & 0 \\
\end{array}
\]

To replicate the option, the investor must lend $B$:
\[
B = e^{-rh} \frac{u V_d - d V_u}{u - d} = e^{-0.07(1)} \frac{75(0) - 40(25)}{75 - 40} = -26.64
\]

Lending $-26.64$ is equivalent to borrowing $26.64$.

Solution 5
B  Chapter 10, One-Period Binomial Tree

We can use the binomial option pricing model to price any option that has defined values at the end of each period. We aren’t limited to pricing just calls and puts.

If the stock price moves up, then the option pays $75^2$. If the stock price moves down, then the option pays $40^2$:

\[
\begin{array}{c|c|c}
75 & 5,625 \\
51 & \\
40 & 1,600 \\
\end{array}
\]

The risk-neutral probability of an upward movement is:
\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{51e^{(0.07-0.03)(1)} - 40}{75 - 40} = 0.37375
\]
The value of the option is:

\[
V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.07(1)} [0.37375(5,625) + (1 - 0.37375)(1,600)]
\]

\[= 2,894.48 \]

**Solution 6**

E  Chapter 10, Arbitrage

The first step is to find the values of \( u \) and \( d \):

\[
u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.08 - 0.05)(1) + 0.27 \sqrt{1}} = 1.34986
\]

\[
d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.08 - 0.05)(1) - 0.27 \sqrt{1}} = 0.78663
\]

The possible stock prices are therefore:

\[
65u = 65(1.34986) = 87.7408
\]

\[
65d = 65(0.78663) = 51.1308
\]

If the stock price moves up, then the option pays $0. If the stock price moves down, then the option pays 11.8692:

\[
\begin{array}{c|c|c}
65 & 87.7408 & 0.0000 \\
51.1308 & V & 11.8692 \\
\end{array}
\]

To replicate the option, the investor must purchase delta shares of the underlying stock:

\[
\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u - d)} = e^{-0.05(1)} \frac{0.000 - 11.8692}{87.7408 - 51.1308} = -0.3084
\]

To replicate the option, the investor must lend \( B \):

\[
B = e^{-rh} \frac{uV_d - dV_u}{u - d} = e^{-0.08(1)} \frac{1.34986(11.8692) - 0.78663(0.000)}{1.34986 - 0.78663} = 26.259
\]

The purchase of the option can therefore be replicated by selling 0.3084 shares of stock and lending $26.259.

Therefore, the price of the put option should be:

\[
V = S_0 \Delta + B = 65(-0.3084) + 26.259 = 6.213
\]

But the market price of the option is $6.00. Therefore an arbitrageur can earn an arbitrage profit by purchasing the option for $6.00 and synthetically selling it for $6.213.

The synthetic sale of the option requires that the arbitrageur buy 0.3084 shares of stock and borrow $26.259.
Solution 7

C Chapter 10, Graphical Interpretation of the Binomial Formula

Delta is the slope of the line that passes through \((S_d, C_d)\) and \((S_u, C_u)\):

\[
\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u - d)} = \frac{C_u - C_d}{S_u - S_d}
\]

Although we are not given the value of \(S_u\), we can still find the slope of the line because we have two points on the line:

\((0, -26.58)\) and \((40.53, 0)\)

The “rise-over-run” for the two points is:

\[
\Delta = \frac{0 - (-26.58)}{40.53 - 0.00} = \frac{26.58}{40.53} = 0.656
\]

Solution 8

D Chapter 10, Graphical Interpretation of the Binomial Formula

Delta is the slope of the line that passes through \((S_d, C_d)\) and \((S_u, C_u)\):

\[
\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u - d)} = \frac{C_u - C_d}{S_u - S_d}
\]

Although we are not given the value of \(S_d\), we can still find the slope of the line because we have two points on the line:

\((0, -32.38)\) and \((88.62, 26.62)\)

The “rise-over-run” for the two points is:

\[
\Delta = \frac{26.62 - (-32.38)}{88.62 - 0.00} = \frac{59.00}{88.62} = 0.6658
\]

The intercept at the vertical axis is the value of the replicating portfolio if the stock price is zero at the end of 1 year:

\[
\Delta \times 0 + Be^{0.09} = -32.38 \quad B = -29.5931
\]

The option can be replicated by purchasing 0.6658 shares of stock and borrowing $29.5931:

\[
C = S_0\Delta + B = 60(0.6658) - 29.5931 = 10.35
\]
Solution 9

E  Chapter 10, Risk-Neutral Probability

To find the price using the risk-free rate of return, we must use risk-neutral probabilities, not realistic probabilities. The 45% probability of an upward movement is a red herring in this question.

The values of $u$ and $d$ are:

\[
\begin{align*}
    u &= e^{(r\cdot\delta)h+\sigma \sqrt{h}} = e^{(0.05-0.02)(1)+0.30\sqrt{1}} = 1.39097, \\
    d &= e^{(r\cdot\delta)h-\sigma \sqrt{h}} = e^{(0.05-0.02)(1)-0.30\sqrt{1}} = 0.76338.
\end{align*}
\]

If the stock price moves up, then the option pays $0. If the stock price moves down, then the option pays $14.5634:

\[
\begin{align*}
    97.3678 & \quad 0.0000 \\
    70 & \quad V \\
    53.4366 & \quad 14.5634
\end{align*}
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r\cdot\delta)h-d}}{u-d} = \frac{e^{(0.05-0.02)(1)-0.76338}}{1.39097-0.76338} = 0.42556
\]

The value of the put option is:

\[
V = e^{-rh} \left[ (p^*)V_u + (1-p^*)V_d \right] = e^{-0.05(1)} \left[ 0.42556(0.00) + (1-0.42556)(14.5634) \right] = 7.96
\]

Solution 10

E  Chapter 10, Arbitrage

Unless the following inequality is satisfied, arbitrage is available:

\[d < e^{(r\cdot\delta)h} < u\]

Let’s consider each of the possible choices:

\[
\begin{array}{ccc}
    d & e^{(r\cdot\delta)h} & u \\
    A & 0.872 & 1.013 & 1.176 \\
    B & 0.805 & 0.995 & 1.230 \\
    C & 0.996 & 1.002 & 1.008 \\
    D & 0.783 & 1.000 & 1.278 \\
    E & 0.920 & 1.105 & 1.100
\end{array}
\]
The final row violates the inequality because the risk-free investment outperforms the risky asset in both scenarios. Therefore, the model described in Choice E permits arbitrage.

**Solution 11**

**B** Chapter 10, One-Period Binomial Tree

This question can be answered using put-call parity:

\[ C_{Eur}(K,T) + Ke^{-rT} = S_0e^{-\delta T} + P_{Eur}(K,T) \]

\[ 2.58 + 55e^{-0.02(0.5)} = 50e^{-0.02(0.5)} + P_{Eur}(55,0.5) \]

\[ P_{Eur}(55,0.5) = 5.92 \]

**Solution 12**

**B** Chapter 10, Two-Period Binomial Tree

The values of \( u \) and \( d \) are:

\[ u = e^{(r-\delta)h+\sigma \sqrt{h}} = e^{(0.08-0.03)(1)+0.23\sqrt{1}} = 1.32313 \]

\[ d = e^{(r-\delta)h-\sigma \sqrt{h}} = e^{(0.08-0.03)(1)-0.23\sqrt{1}} = 0.83527 \]

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.08-0.03)(1)} - 0.83527}{1.32313 - 0.83527} = 0.44275 \]

The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>American Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>126.0484</td>
<td>52.0484</td>
</tr>
<tr>
<td>95.2653</td>
<td>24.1392</td>
</tr>
<tr>
<td>79.5723</td>
<td>11.0375</td>
</tr>
<tr>
<td>50.2327</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The tree of prices for the call option is found by working from right to left. The rightmost column is found as follows:

\[ Max[0, 126.0484 - 74] = 52.0484 \]

\[ Max[0, 79.5723 - 74] = 5.5723 \]

\[ Max[0, 50.232 - 74] = 0.0000 \]
The prices after 1 year are found using the risk-neutral probabilities:

\[
e^{-0.08(1)} \left[ (0.44275)(52.0484) + (1 - 0.44275)(5.5723) \right] = 24.1392
\]

\[
e^{-0.08(1)} \left[ (0.44275)(5.5723) + (1 - 0.44275)(0.0000) \right] = 2.2775
\]

There is only one opportunity to call the option early, and this occurs after 1 year if the stock price has risen. Early exercise is not optimal since:

\[
95.2653 - 74 = 21.2653 \quad \text{and} \quad 21.2653 < 24.1392
\]

The current price of the option is:

\[
e^{-0.08(1)} \left[ (0.44275)(24.1392) + (1 - 0.44275)(2.2775) \right] = 11.0375
\]

**Solution 13**

**D** Chapter 10, Two-Period Binomial Tree

The values of \( u \) and \( d \) are:

\[
u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.08 - 0.03)(1) + 0.23 \sqrt{1}} = 1.32313
\]

\[
d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.08 - 0.03)(1) - 0.23 \sqrt{1}} = 0.83527
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h}}{u - d} = \frac{e^{(0.08 - 0.03)(1) - 0.83527}}{1.32313 - 0.83527} = 0.44275
\]

The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>American Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>126.0484</td>
<td>52.0484</td>
</tr>
<tr>
<td>95.2653</td>
<td>24.1392</td>
</tr>
<tr>
<td>72.0000</td>
<td>79.5723</td>
</tr>
<tr>
<td>60.1395</td>
<td>11.0375</td>
</tr>
<tr>
<td>50.2327</td>
<td>2.2775</td>
</tr>
<tr>
<td>50.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

We are told that after 1 year, the stock price has increased. Therefore, the new stock price is $95.2653. At this point the value of delta is:

\[
\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u - d)} = e^{-0.03(1)} \frac{52.0484 - 5.5723}{126.0484 - 79.5723} = 0.9704
\]

Therefore, the investor must hold 0.9704 shares at the end of 1 year in order to replicate the call option.
Solution 14

B Chapter 10, Two-Period Binomial Tree

The values of $u$ and $d$ are:

$$
u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.08 - 0.03)(1) + 0.23\sqrt{1}} = 1.32313$$

$$d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.08 - 0.03)(1) - 0.23\sqrt{1}} = 0.83527$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.08 - 0.03)(1) - 0.83527}}{1.32313 - 0.83527} = 0.44275$$

The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>European Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>126.0484</td>
<td>0.0000</td>
</tr>
<tr>
<td>95.2653</td>
<td>0.0000</td>
</tr>
<tr>
<td>79.5723</td>
<td>6.2891</td>
</tr>
<tr>
<td>72.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>79.5723</td>
<td>6.2891</td>
</tr>
<tr>
<td>60.1395</td>
<td>12.2260</td>
</tr>
<tr>
<td>50.2327</td>
<td>23.7673</td>
</tr>
</tbody>
</table>

The put option can be exercised only if the stock price falls to $50.2327. At that point, the put option has a payoff of $23.7673. Although the entire table of put prices is filled in above, we can directly find the value of the put option as follows:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1 - p^*)^{n-j} V(S_0u^j d^{n-j}, K, hn)$$

$$= e^{-0.08(1)(2)} (1 - 0.44275)^2 23.7673 + 0 + 0 = 6.289$$

Solution 15

D Chapter 10, Two-Period Binomial Tree

The values of $u$ and $d$ are:

$$u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.08 - 0.03)(1) + 0.23\sqrt{1}} = 1.32313$$

$$d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.08 - 0.03)(1) - 0.23\sqrt{1}} = 0.83527$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.08 - 0.03)(1) - 0.83527}}{1.32313 - 0.83527} = 0.44275$$
The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
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<tbody>
<tr>
<td>126.0484</td>
<td>0.0000</td>
</tr>
<tr>
<td>95.2653</td>
<td>0.0000</td>
</tr>
<tr>
<td>72.0000</td>
<td>7.1299</td>
</tr>
<tr>
<td>60.1395</td>
<td>13.8605</td>
</tr>
<tr>
<td>50.2327</td>
<td>23.7673</td>
</tr>
</tbody>
</table>

If the stock price initially moves down, then the resulting put price is $13.8605. This price is in bold type above to indicate that it is optimal to exercise early at this node:

\[ 74 - 60.1395 = 13.8605 \]

The exercise value of 13.8605 is greater than the value of holding the option, which is:

\[ e^{-0.08(1)} \left[ (0.44275)0.0000 + (1 - 0.44275)(23.7673) \right] = 12.2260 \]

The current value of the American option is:

\[ e^{-0.08(1)} \left[ (0.44275)0.0000 + (1 - 0.44275)(13.8605) \right] = 7.1299 \]

**Solution 16**

**B Chapter 10, Three-Period Binomial Tree**

The values of \( h, u \) and \( d \) are:

\[ h = 9/12 \times 1/3 = 1/4 \]
\[ u = e^{(r-\delta)h + \sigma \sqrt{h}} = e^{(0.10-0.08)(0.25)+0.32\sqrt{0.25}} = 1.17939 \]
\[ d = e^{(r-\delta)h-\sigma \sqrt{h}} = e^{(0.10-0.08)(0.25)-0.32\sqrt{0.25}} = 0.85642 \]

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r-\delta)h-d}}{u-d} = \frac{e^{(0.10-0.08)(0.25) - 0.85642}}{1.17939-0.85642} = 0.46009 \]

The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>180.4548</th>
<th>European Call</th>
<th>80.4548</th>
</tr>
</thead>
<tbody>
<tr>
<td>153.0065</td>
<td>130.3071</td>
<td>30.8676</td>
<td>31.0371</td>
</tr>
<tr>
<td>129.7332</td>
<td>111.055</td>
<td>17.1419</td>
<td>13.9271</td>
</tr>
<tr>
<td>110.0000</td>
<td>95.1525</td>
<td>6.2465</td>
<td>0.0000</td>
</tr>
<tr>
<td>94.2057</td>
<td>80.6792</td>
<td>69.0949</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Although we have shown all of the prices in the tree above, we can also solve more directly for the price of the call option \((n = 3)\) as follows:

\[
V(S_0, K, 0) = e^{-rhn} \sum_{j=0}^{n} \binom{n}{j} \left( p^* \right)^j \left( 1 - p^* \right)^{n-j} V(S_0u^j d^{n-j}, K, hn)
\]

\[
= e^{-0.10(0.75)} \left[ 0 + 3(0.46009)^2 (1 - 0.46009)(31.0371) + (0.46009)^3 (80.4548) \right]
\]

\[= 17.14 \]

Solution 17
A Chapter 10, Three-Period Binomial Tree

The values of \(u\) and \(d\) are:

\[
u = e^{(r-\delta)h+\sigma \sqrt{h}} = e^{(0.10-0.08)(0.25)+0.32\sqrt{0.25}} = 1.17939
\]

\[
d = e^{(r-\delta)h-\sigma \sqrt{h}} = e^{(0.10-0.08)(0.25)-0.32\sqrt{0.25}} = 0.85642
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r-\delta)h-d}}{u-d} = \frac{e^{(0.10-0.08)(0.25) - 0.85642}}{1.17939 - 0.85642} = 0.46009
\]

The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>180.4548</th>
<th>American Call</th>
<th>80.4548</th>
</tr>
</thead>
<tbody>
<tr>
<td>153.0065</td>
<td>129.7332</td>
<td>131.0371</td>
<td>31.1192</td>
</tr>
<tr>
<td>129.7332</td>
<td>111.0555</td>
<td>17.2548</td>
<td>13.9271</td>
</tr>
<tr>
<td>110.0000</td>
<td>94.2057</td>
<td>95.1525</td>
<td>6.2495</td>
</tr>
<tr>
<td>94.2057</td>
<td>80.6792</td>
<td>69.0949</td>
<td>0.0000</td>
</tr>
<tr>
<td>80.6792</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
</tbody>
</table>

At each node, the value of holding the option must be compared with the value obtained by exercising it. If the stock price reaches $153.0065, then it is optimal to exercise the option early because:

\[
153.0065 - 100 > e^{-0.10(0.25)} \left[ (0.46009) \times 80.4548 + (1 - 0.46009) \times 31.0371 \right]
\]

\[
53.0065 > 52.4458
\]

Working from right to left, we calculate the current value of the option to be $17.25.
Solution 18

A  Chapter 10, Three-Period Binomial Tree

The values of $u$ and $d$ are:

$$u = e^{(r \delta)h + \sigma \sqrt{h}} = e^{(0.10 - 0.08)(0.25) + 0.32\sqrt{0.25}} = 1.17939$$

$$d = e^{(r \delta)h - \sigma \sqrt{h}} = e^{(0.10 - 0.08)(0.25) - 0.32\sqrt{0.25}} = 0.85642$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r \delta)h - d}}{u - d} = \frac{e^{(0.10 - 0.08)(0.25) - 0.85642}}{1.17939 - 0.85642} = 0.46009$$

The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>180.4548</th>
<th>European Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>153.0065</td>
<td>131.0371</td>
<td>1.3442</td>
<td>0.0000</td>
</tr>
<tr>
<td>129.7332</td>
<td>111.1055</td>
<td>6.3222</td>
<td>2.5526</td>
</tr>
<tr>
<td>110.0000</td>
<td>94.2057</td>
<td>10.8606</td>
<td>4.8475</td>
</tr>
<tr>
<td>80.6792</td>
<td>95.1525</td>
<td>18.4494</td>
<td>30.9051</td>
</tr>
<tr>
<td>69.0949</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Although we have shown all of the prices in the trees above, we can also solve more directly for the price of the European put option as follows:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \left( \frac{n}{j} \right) (p^*)^j (1 - p^*)^{n-j} V(S_0 u^j d^{n-j}, K, hn)$$

$$= e^{-0.10(0.75)} \left[ (1 - 0.46009)^3 (30.9051) + 3(0.46009)(1 - 0.46009)^2 (4.8475) + 0 + 0 \right]$$

$$= 6.3222$$

Solution 19

C  Chapter 10, Three-Period Binomial Tree

The values of $u$ and $d$ are:

$$u = e^{(r \delta)h + \sigma \sqrt{h}} = e^{(0.10 - 0.08)(0.25) + 0.32\sqrt{0.25}} = 1.17939$$

$$d = e^{(r \delta)h - \sigma \sqrt{h}} = e^{(0.10 - 0.08)(0.25) - 0.32\sqrt{0.25}} = 0.85642$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r \delta)h - d}}{u - d} = \frac{e^{(0.10 - 0.08)(0.25) - 0.85642}}{1.17939 - 0.85642} = 0.46009$$
The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>180.4548</th>
<th>American Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>153.0065</td>
<td>131.0371</td>
<td>1.3442</td>
<td>0.0000</td>
</tr>
<tr>
<td>129.7332</td>
<td>111.1055</td>
<td>6.5638</td>
<td>2.5526</td>
</tr>
<tr>
<td>110.0000</td>
<td>95.1525</td>
<td>11.3195</td>
<td>4.8475</td>
</tr>
<tr>
<td>94.2057</td>
<td>69.0949</td>
<td>19.3208</td>
<td>30.9051</td>
</tr>
</tbody>
</table>

At each node, the value of holding the option must be compared with the value obtained by exercising it. If the stock price reaches $80.6792, then it is optimal to exercise the option early because:

\[
100 - 80.6792 > e^{-0.10(0.25)} \left[ (0.46009) \times 4.8475 + (1 - 0.46009) \times 30.9051 \right] \approx 19.3208 > 18.4494
\]

Working from right to left, we calculate the current value of the option to be $6.5638.

Solution 20

C Chapter 10, Three-Period Binomial Tree

The values of \( u \) and \( d \) are:

\[
u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.10-0.08)(0.25)+0.32\sqrt{0.25}} = 1.17939
\]

\[
d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.10-0.08)(0.25)-0.32\sqrt{0.25}} = 0.85642
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r-\delta)h-d}}{u-d} = \frac{e^{(0.10-0.08)(0.25)-0.85642}}{1.17939-0.85642} = 0.46009
\]

The stock price tree and its corresponding tree of option prices are:

| Stock     | | American Put | |
|-----------||--------------|---|
| 129.7332  | | 131.0371     | |
| 110.0000  | 111.1055     | | 0.0000 |
| 94.2057   | 95.1525      | | 4.8475 |

We have left much of the tree blank because there is no need to fill in all of the cells to answer this question. The amount that must be lent at the risk-free rate of return is:

\[
B = e^{-rh} \frac{uV_d - dV_u}{u-d} = e^{-0.10(0.25)} \frac{1.17939(4.8475) - 0.85642(0.0000)}{1.17939-0.85642} = 17.26
\]
Solution 21
C Chapter 10, Three-Period Binomial Tree

The values of $u$ and $d$ are:

$u = e^{(r - \delta)h + \sigma\sqrt{h}} = e^{(0.04 - 0.03)(1/3) + 0.30\sqrt{1/3}} = 1.19308$

$d = e^{(r - \delta)h - \sigma\sqrt{h}} = e^{(0.04 - 0.03)(1/3) - 0.30\sqrt{1/3}} = 0.84377$

The risk-neutral probability of an upward movement is:

$p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.04 - 0.03)(1/3) - 0.84377}}{1.19308 - 0.84377} = 0.45681$

The stock price tree and its corresponding tree of option prices are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>American Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>128.1096</td>
<td>152.8451</td>
<td>107.3772</td>
</tr>
<tr>
<td>76.4475</td>
<td>64.0758</td>
<td>54.0654</td>
</tr>
</tbody>
</table>

At each node, the value of holding the option must be compared with the value obtained by exercising it. If the stock price reaches $64.0758, then it is optimal to exercise the option early because:

$86 - 64.0758 > e^{-0.04(1/3)} \left[ (0.45681)9.5525 + (1 - 0.45681)31.9346 \right]$

$21.9242 > 21.4227$

When the stock price is $75.9396, the value of the put option is:

$e^{-0.04(1/3)} \left[ (0.45681)5.1201 + (1 - 0.45681)21.9242 \right] = 14.0593$

Solution 22
B Chapter 10, Three-Period Binomial Tree

The values of $u$ and $d$ are:

$u = e^{(r - \delta)h + \sigma\sqrt{h}} = e^{(0.04 - 0.03)(1/3) + 0.30\sqrt{1/3}} = 1.19308$

$d = e^{(r - \delta)h - \sigma\sqrt{h}} = e^{(0.04 - 0.03)(1/3) - 0.30\sqrt{1/3}} = 0.84377$

The risk-neutral probability of an upward movement is:

$p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.04 - 0.03)(1/3) - 0.84377}}{1.19308 - 0.84377} = 0.45681$
The stock price tree and its corresponding tree of option prices are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>152.8451</th>
<th>American Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>128.1096</td>
<td>107.3772</td>
<td>108.0955</td>
<td>2.7444</td>
</tr>
<tr>
<td>90.0000</td>
<td>90.6020</td>
<td>8.7728</td>
<td>5.1201</td>
</tr>
<tr>
<td>75.9396</td>
<td>76.4475</td>
<td>14.0593</td>
<td>9.5525</td>
</tr>
<tr>
<td>64.0758</td>
<td>54.0654</td>
<td>31.9346</td>
<td></td>
</tr>
</tbody>
</table>

We are told that during the first period, the stock price decreases. Therefore, the new stock price is $75.9396. At this point the value of delta is:

\[ \Delta = e^{-\delta h} \frac{V_u - V_d}{S(u - d)} = e^{-0.03(1/3)} \frac{5.1201 - 21.9242}{90.6020 - 64.0758} = -0.627 \]

**Solution 23**

**B Chapter 10, Four-Period Binomial Tree**

The values of \( u \) and \( d \) are:

\[ u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.07)(0.25) + 0.30\sqrt{0.25}} = 1.18235 \]

\[ d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.07)(0.25) - 0.30\sqrt{0.25}} = 0.87590 \]

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.07)(0.25) - 0.87590}}{1.18235 - 0.87590} = 0.46257 \]

Since the stock does not pay dividends, the price of the American call option is equal to the price of an otherwise equivalent European call option. Therefore, we do not need to calculate the possible values of the stock at times 0.25, 0.50, and 0.75. Furthermore, since the option expires out of the money if the final price is less than $131, it makes sense to begin by calculating the highest possible final prices:

\[ S_{uuuu} = 90u^4 = 90(1.18235)^4 = 175.8814 \]

\[ S_{uuud} = 90u^3d = 90(1.18235)^3(0.87590) = 130.2961 \]

The portions of the stock price tree that we need are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>175.8814</th>
<th>—</th>
<th>130.2961</th>
</tr>
</thead>
<tbody>
<tr>
<td>90.0000</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
We did not complete the stock price tree above because the option is in-the-money at the end of the year only if the final stock price is $175.8814. Since it is not in-the-money if the stock price is $130.2961, it clearly will not be in-the-money at any lower stock price. Since the end of the year payoff is zero for all of the nodes except the uppermost one, it is relatively quick to find the value of the option:

\[
V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1 - p^*)^{n-j} V(S_0 u^j d^{n-j}, K, hn) \\
= e^{-0.07(1)(0.46257)^4(175.8814 - 131)} = 1.9159
\]

Solution 24

E Chapter 10, Three-Period Binomial Tree

The values of \( h, u \) and \( d \) are provided in the question:

\[
h = 1/4 \times 1/3 = 1/12 \\
u = 1.20 \\
d = 0.90
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.03 - 0.05)(1/12)} - 0.90}{1.20 - 0.90} = 0.32778
\]

The tree of stock prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>69.1200</th>
<th>European Call</th>
<th>36.1200</th>
</tr>
</thead>
<tbody>
<tr>
<td>69.1200</td>
<td>57.6000</td>
<td>24.4429</td>
<td></td>
</tr>
<tr>
<td>48.0000</td>
<td>51.8400</td>
<td>14.7663</td>
<td></td>
</tr>
<tr>
<td>40.0000</td>
<td>43.2000</td>
<td>7.9074</td>
<td></td>
</tr>
<tr>
<td>36.0000</td>
<td>38.8800</td>
<td>4.5924</td>
<td></td>
</tr>
<tr>
<td>32.4000</td>
<td>29.1600</td>
<td>1.9225</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
</tbody>
</table>

Although we have shown all of the prices in the trees above, we can also solve more directly for the price of the European option as follows:

\[
V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1 - p^*)^{n-j} V(S_0 u^j d^{n-j}, K, hn) \\
= e^{-0.03(3/12)} [0 + 3(0.32778)(1 - 0.32778)^2(5.88) + 3(0.32778)^2(1 - 0.32778)(18.84) + (0.32778)^3(36.12)] \\
= 7.9074
\]
Solution 25
D Chapter 10, Three-Period Binomial Tree

The values of $h, u$ and $d$ are provided in the question:

- $h = 1/4 \times 1/3 = 1/12$
- $u = 1.20$
- $d = 0.90$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.03-0.05)(1/12)} - 0.90}{1.20 - 0.90} = 0.32778$$

The tree of stock prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>69.1200</th>
<th>57.6000</th>
<th>48.0000</th>
<th>40.0000</th>
<th>36.0000</th>
<th>32.4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Call</td>
<td>36.1200</td>
<td>24.6000</td>
<td>15.0000</td>
<td>10.2000</td>
<td>8.0052</td>
<td>4.6242</td>
</tr>
</tbody>
</table>

Early exercise is optimal at the three nodes shown in bold type. After two up jumps, the option exercise value of 24.60 exceeds the option expected present value of 24.4429, which we calculated in the prior solution. After one up jump and one down jump, the option exercise value of 10.20 exceeds the option expected present value of 10.1028, which we also calculated in the prior solution. After one up jump, the option exercise value of 15.00 exceeds the option expected present value of:

$$e^{-0.03(1/12)} \left[ (0.32778)(24.6000) + (1 - 0.32778)(10.2000) \right] = 14.8828$$

The value of the American call option is:

$$e^{-0.03(1/12)} \left[ (0.32778)(15.0000) + (1 - 0.32778)(4.6242) \right] = 8.0052$$

Solution 26
D Chapter 10, Multiple-Period Binomial Tree

It isn’t very reasonable to expect us to work through an 8-period binomial tree during the exam, so there must be a shortcut to the answer. As it turns out, the option is in-the-money at only one final node. That makes it fairly easy to find the correct answer.

The value of $h$ is 1/12 since the intervals are monthly periods. The values of $u$ and $d$ are:

- $u = e^{(r-\delta)h + \sigma \sqrt{h}} = e^{(0.07-0.02)(1/12) + 0.35 \sqrt{1/12}} = 1.11094$
- $d = e^{(r-\delta)h - \sigma \sqrt{h}} = e^{(0.07-0.02)(1/12) - 0.35 \sqrt{1/12}} = 0.90767$
The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.07-0.02)(1/12)} - 0.90767}{1.11094 - 0.90767} = 0.47476 \]

If the stock price increases each month, then at the end of 8 months, the price is:

\[ 130u^8 = 130(1.11094)^8 = 301.6170 \]

If the stock price increases for 7 of the months and decreases for 1 of the months, then the price is:

\[ 130u^7d = 130(1.11094)^7(0.90767) = 246.4319 \]

Since the price of $246.4319 is out-of-the-money, all other lower prices are also out-of-the-money. This means that the only price at which the option expires in-the-money is the highest possible price, which is $301.6170. Consequently, the value of the option is:

\[ V(S_0,K,0) = e^{-r(hn)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1-p^*)^{n-j} V(S_0u^jd^{n-j},K,hn) \]

\[ = e^{-0.07(8/12)}[0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + (0.47476)^8(301.6170 - 247)] \]

\[ = 0.1345 \]

Solution 27

C Chapter 10, Multiple-Period Binomial Tree

If we work through the entire binomial tree, this is a very time-consuming problem. Even using the direct method takes a lot of time since the put option provides a positive payoff at multiple final nodes. But once we notice that the value of the corresponding call option can be calculated fairly quickly, we can use put-call parity to find the value of the put option.

The value of \( h \) is 1/12 since the intervals are monthly periods. The values of \( u \) and \( d \) are:

\[ u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)+0.35\sqrt{1/12}} = 1.11094 \]

\[ d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)-0.35\sqrt{1/12}} = 0.90767 \]

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.07-0.02)(1/12)} - 0.90767}{1.11094 - 0.90767} = 0.47476 \]

If the stock price increases each month, then at the end of 8 months, the price is:

\[ 130u^8 = 130(1.11094)^8 = 301.6170 \]
If the stock price increases for 7 of the months and decreases for 1 of the months, then the price is:

\[130u^7d = 130(1.11094)^7(0.90767) = 246.4319\]

Consider the corresponding call option with a strike price of $247. Since the price of $246.4319 is out-of-the-money, all other lower prices are also out-of-the-money. This means that the only price at which the call option expires in-the-money is the highest possible price, which is $301.6170.

Calculating the price of the put option is daunting, but the corresponding call option price can be found fairly easily. The value of the call option is:

\[
V(S_0, K, 0) = e^{-r(h)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1 - p^*)^{n-j} V(S_0u^j d^{n-j}, K, hn) \\
= e^{-0.07(8/12)}[0 + 0 + 0 + 0 + 0 + 0 + 0 + (0.47476)^8(301.6170 - 247)] \\
= 0.1345
\]

Now we can use put-call parity to find the value of the corresponding put option:

\[
C_{Eur}(K, T) + Ke^{-rT} = S_0e^{-\delta T} + P_{Eur}(K, T) \\
0.1345 + 247e^{-0.07(8/12)} = 130e^{-0.02(8/12)} + P_{Eur}(K, T) \\
P_{Eur}(K, T) = 107.59
\]

Solution 28

B Chapter 10, Put-Call Parity

This problem is fairly easy if we use put-call parity.

The value of \( h \) is 1/12 since the intervals are monthly periods. The values of \( u \) and \( d \) are:

\[
u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)+0.35\sqrt{1/12}} = 1.11094 \\
d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)-0.35\sqrt{1/12}} = 0.90767
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.07-0.02)(1/12)} - 0.90767}{1.11094 - 0.90767} = 0.47476
\]

If the stock price increases each month, then at the end of 4 months, the price is:

\[130u^4 = 130(1.11094)^4 = 198.0157\]

Since the other possible prices at the end of 4 months will also be below $200, all of the nodes are in-the-money.

But none of the nodes are in-the-money for the corresponding European call option. This means that the value of the corresponding European call option is zero.
We can use put-call parity to find the value of the corresponding put option:

\[
C_{Eur}(K,T) + Ke^{-rT} = S_0 e^{-\delta T} + P_{Eur}(K,T)
\]

\[
0 + 200e^{-0.07(4/12)} = 130e^{-0.02(4/12)} + P_{Eur}(K,T)
\]

\[
P_{Eur}(K,T) = 66.25
\]

**Solution 29**

**C  Chapter 10, Replication**

The values of \(u\) and \(d\) are:

\[
u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.07 - 0.02)(1/12) + 0.35\sqrt{1/12}} = 1.11094
\]

\[
d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.07 - 0.02)(1/12) - 0.35\sqrt{1/12}} = 0.90767
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.07 - 0.02)(1/12)} - 0.90767}{1.11094 - 0.90767} = 0.47476
\]

If the stock price increases during each of the first 3 months, then the price is:

\[
130u^3 = 130(1.11094)^3 = 178.2422
\]

The relevant portions of the stock price tree and the option price tree are shown below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>European Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>198.0157</td>
<td>1.9843</td>
</tr>
<tr>
<td>178.2422</td>
<td>—</td>
</tr>
<tr>
<td>161.786</td>
<td>38.2141</td>
</tr>
</tbody>
</table>

The amount that must be lent at the risk-free rate of return is:

\[
B = e^{-rh} \frac{uV_d - dV_u}{u - d} = e^{-0.07(1/12)} \frac{1.11094(38.2141) - 0.90767(1.9843)}{1.11094 - 0.90767} = 198.84
\]

**Solution 30**

**C  Chapter 10, Option on a Stock Index**

The value of \(h\) is 1 since the intervals are annual periods. The values of \(u\) and \(d\) are:

\[
u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.05 - 0.04)(1) + 0.34\sqrt{1}} = 1.41907
\]

\[
d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.05 - 0.04)(1) - 0.34\sqrt{1}} = 0.71892
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.05 - 0.04)(1)} - 0.71892}{1.41907 - 0.71892} = 0.41581
\]
The tree of stock index prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Stock Index</th>
<th>285.7651</th>
<th>European Call</th>
<th>190.7651</th>
</tr>
</thead>
<tbody>
<tr>
<td>201.3753</td>
<td>103.1124</td>
<td></td>
<td></td>
</tr>
<tr>
<td>141.9068</td>
<td>144.7735</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0000</td>
<td>102.0201</td>
<td></td>
<td></td>
</tr>
<tr>
<td>71.8924</td>
<td>73.3447</td>
<td></td>
<td></td>
</tr>
<tr>
<td>51.6851</td>
<td>37.1577</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Although we have shown all of the prices in the trees above, we can also solve more directly for the price of the European option as follows:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1 - p^*)^{n-j} V(S_0 u^j d^{n-j}, K, hn)$$

$$= e^{-0.05(3)} [0 + 0 + 3(0.41581)^2 (1 - 0.41581)(49.7735) + (0.41581)^3 (190.7651)]$$

$$= 24.7855$$

Solution 31

A Chapter 10, Option on a Stock Index

The value of $h$ is 1 since the intervals are annual periods. The values of $u$ and $d$ are:

$$u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.05 - 0.04)(1) + 0.34 \sqrt{1}} = 1.41907$$

$$d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.05 - 0.04)(1) - 0.34 \sqrt{1}} = 0.71892$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.05 - 0.04)(1)} - 0.71892}{1.41907 - 0.71892} = 0.41581$$

The tree of stock index prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Stock Index</th>
<th>285.7651</th>
<th>European Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>201.3753</td>
<td>0.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>141.9068</td>
<td>144.7735</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0000</td>
<td>102.0201</td>
<td></td>
<td></td>
</tr>
<tr>
<td>71.8924</td>
<td>73.3447</td>
<td></td>
<td></td>
</tr>
<tr>
<td>51.6851</td>
<td>37.1577</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Although we have shown all of the prices in the trees above, we can also solve more directly for the price of the European option as follows:

\[
V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1 - p^*)^{n-j} V(S_0u^j d^{n-j}, K, hn)
\]

\[
= e^{-0.05(3)}[(1 - 0.41581)^3(57.8423) + 3(0.41581)(1 - 0.41581)^2(21.6553) + 0 + 0]
\]

\[
= 17.8608
\]

**Solution 32**

**A** Chapter 10, Option on a Stock Index

The values of \(u\) and \(d\) are:

\[
u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.05 - 0.04)(1) + 0.34 \sqrt{1}} = 1.41907
\]

\[
d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.05 - 0.04)(1) - 0.34 \sqrt{1}} = 0.71892
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{0.05(1) - 0.71892}}{1.41907 - 0.71892} = 0.41581
\]

The tree of stock index prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Stock Index</th>
<th>285.7651</th>
<th>American Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>201.3753</td>
<td></td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>141.9068</td>
<td>144.7735</td>
<td>18.6657</td>
<td></td>
</tr>
<tr>
<td>102.0201</td>
<td>18.6657</td>
<td></td>
<td></td>
</tr>
<tr>
<td>71.8924</td>
<td>73.3447</td>
<td>28.8298</td>
<td></td>
</tr>
<tr>
<td>51.6851</td>
<td>37.1577</td>
<td>43.3149</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>71.8924</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>51.6851</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If the stock price reaches $51.6851 after 2 years, then early exercise is optimal since the exercise value of the option exceeds the expected present value of the option at that node.

We have:

\[
e^{-0.05} [(0.41581)(12.0338) + (1 - 0.41581)(43.3149)] = 28.8298
\]

The value of the American put option is:

\[
e^{-0.05} [(0.41581)(6.6872) + (1 - 0.41581)(28.8298)] = 18.6657
\]
Solution 33

The value of $h$ is $1/4$ since the intervals are quarterly periods. The values of $u$ and $d$ are:

$$u = e^{(r - r_f) h + \sigma \sqrt{h}} = e^{(0.05 - 0.09)(0.25) + 0.15 \sqrt{0.25}} = 1.06716$$

$$d = e^{(r - r_f) h - \sigma \sqrt{h}} = e^{(0.05 - 0.09)(0.25) - 0.15 \sqrt{0.25}} = 0.91851$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r - \delta) h} - d}{u - d} = \frac{e^{0.05 - 0.09}(0.25) - 0.91851}{1.06716 - 0.91851} = 0.48126$$

The tree of euro prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Euros</th>
<th>1.4584</th>
<th>European Call</th>
<th>0.3584</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.3666</td>
<td></td>
<td>0.2499</td>
</tr>
<tr>
<td>1.2806</td>
<td>1.2552</td>
<td>0.1565</td>
<td>0.1552</td>
</tr>
<tr>
<td>1.2000</td>
<td>1.1762</td>
<td>0.0924</td>
<td>0.0738</td>
</tr>
<tr>
<td>1.1022</td>
<td>1.0804</td>
<td>0.0351</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.0124</td>
<td>0.9299</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Although we have shown all of the prices in the trees above, we can also solve more directly for the price of the European option as follows:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1 - p^*)^{n-j} V(S_0 u^j d^{n-j}, K, hn)$$

$$= e^{-0.05(0.75)} [0 + 0 + 3(0.48126)^2 (1 - 0.48126)(0.1552) + (0.48126)^2(0.3584)]$$

$$= 0.0924$$

Solution 34

The value of $h$ is $1/4$ since the intervals are quarterly periods. The values of $u$ and $d$ are:

$$u = e^{(r - r_f) h + \sigma \sqrt{h}} = e^{(0.05 - 0.09)(0.25) + 0.15 \sqrt{0.25}} = 1.06716$$

$$d = e^{(r - r_f) h - \sigma \sqrt{h}} = e^{(0.05 - 0.09)(0.25) - 0.15 \sqrt{0.25}} = 0.91851$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r - \delta) h} - d}{u - d} = \frac{e^{0.05 - 0.09}(0.25) - 0.91851}{1.06716 - 0.91851} = 0.48126$$
The tree of euro prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Euros</th>
<th>1.4584</th>
<th>American Call</th>
<th>0.3584</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3666</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2806</td>
<td>1.2552</td>
<td>0.2666</td>
<td>0.1806</td>
</tr>
<tr>
<td>1.2000</td>
<td>1.1762</td>
<td>0.1044</td>
<td>0.0762</td>
</tr>
<tr>
<td>1.1022</td>
<td>1.0804</td>
<td>0.0362</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.0124</td>
<td>0.9299</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The three nodes that are in bold type in the American call option tree above indicate that early exercise is optimal at those nodes since the option exercise value exceeds the option expected present value at those nodes.

The value of the American call option is:

\[
e^{-0.05(0.25)} \left[ (0.48126)(0.1806) + (1 - 0.48126)(0.0362) \right] = 0.1044
\]

**Solution 35**

**D** Chapter 10, Options on Currencies

The dollar call option is an option to buy a dollar. Since it is yen-denominated, the strike price is denominated in yen. Since it is at-the-money, the strike price is 118 yen.

Since the call is yen-denominated, we treat the yen as the base currency.

Unless told otherwise, we assume that the price of a yen-denominated option is denominated in yen. Therefore, the answer choices are in yen, not dollars.

The value of \( h \) is 1/3 since the intervals are 1/3 of a year. The values of \( u \) and \( d \) are:

\[
u = e^{(r_y - r_s)h + \sigma \sqrt{h}} = e^{(0.01 - 0.06)(1/3) + 0.11\sqrt{1/3}} = 1.04796
\]

\[
d = e^{(r_y - r_s)h - \sigma \sqrt{h}} = e^{(0.01 - 0.06)(1/3) - 0.11\sqrt{1/3}} = 0.92295
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r_y - r_s)h} - d}{u - d} = \frac{e^{(0.01 - 0.06)(1/3) - 0.92295}}{1.04796 - 0.92295} = 0.48413
\]

The tree of prices for a dollar and the tree of option prices are below. Both trees are expressed in yen:

<table>
<thead>
<tr>
<th>Dollar</th>
<th>135.8037</th>
<th>American Call</th>
<th>17.8037</th>
</tr>
</thead>
<tbody>
<tr>
<td>129.5891</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>123.6588</td>
<td>119.6048</td>
<td>5.9901</td>
<td>1.6048</td>
</tr>
<tr>
<td>118.0000</td>
<td>114.1315</td>
<td>3.0824</td>
<td>0.7744</td>
</tr>
<tr>
<td>108.9086</td>
<td>105.3382</td>
<td>0.3736</td>
<td>0.0000</td>
</tr>
<tr>
<td>100.5177</td>
<td>92.7733</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

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The node that is in bold type in the American call option tree above indicates that early exercise is optimal since the option exercise value exceeds the option expected present value at that node.

The value of the American call option is:

\[ e^{-0.01(1/3)} \left[ (0.48413)(5.9901) + (1 - 0.48413)(0.3736) \right] = 3.0824 \]

**Solution 36**

**E Chapter 10, Options on Currencies**

The dollar put option is an option to sell a dollar. Since it is yen-denominated, the strike price is denominated in yen. Since it is at-the-money, the strike price is 118 yen. Since the put is yen-denominated, we treat the yen as the base currency.

The value of \( h \) is \( 1/3 \) since the intervals are \( 1/3 \) of a year. The values of \( u \) and \( d \) are:

\[
\begin{align*}
    u &= e^{(r_y - r_d)h + \sigma\sqrt{h}} = e^{(0.01 - 0.06)(1/3) + 0.11\sqrt{1/3}} = 1.04796 \\
    d &= e^{(r_y - r_d)h - \sigma\sqrt{h}} = e^{(0.01 - 0.06)(1/3) - 0.11\sqrt{1/3}} = 0.92295
\end{align*}
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r_y - r_d)h} - d}{u - d} = \frac{e^{(0.01 - 0.06)(1/3)} - 0.92295}{1.04796 - 0.92295} = 0.48413
\]

The tree of prices for a dollar and the tree of option prices are below. Both trees are expressed in yen:

<table>
<thead>
<tr>
<th>Dollar</th>
<th>135.8037</th>
<th>American Put</th>
<th>129.5891</th>
</tr>
</thead>
<tbody>
<tr>
<td>123.6588</td>
<td>119.6048</td>
<td>3.3472</td>
<td>118.0000</td>
</tr>
<tr>
<td>114.1315</td>
<td>105.3382</td>
<td>12.9513</td>
<td>108.9086</td>
</tr>
<tr>
<td>100.5177</td>
<td>92.7733</td>
<td>19.0800</td>
<td>92.9845</td>
</tr>
</tbody>
</table>

It is not rational to exercise this American put option early.

The value of the American put option is:

\[ e^{-0.01(1/3)} \left[ (0.48413)(3.3472) + (1 - 0.48413)(12.9513) \right] = 8.2741 \]
Solution 37

The pound call is an option to buy a pound. Since it is franc-denominated, the strike price is denominated in francs. Since the call is franc-denominated, we treat the franc as the base currency. For this problem, the subscript _f_ denotes franc.

The value of _h_ is 1 since the intervals are annual periods. The values of _u_ and _d_ are:

\[ u = e^{(r_f - r_k)h + \sigma \sqrt{h}} = e^{(0.05 - 0.07)(1) + 0.12\sqrt{1}} = 1.10517 \]
\[ d = e^{(r_f - r_k)h - \sigma \sqrt{h}} = e^{(0.05 - 0.07)(1) - 0.12\sqrt{1}} = 0.86936 \]

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r_f - r_k)h} - d}{u - d} = \frac{e^{(0.05 - 0.07)(1)} - 0.86936}{1.10517 - 0.86936} = 0.47004 \]

The tree of prices for a pound and the tree of option prices are below. Both trees are expressed in francs:

<table>
<thead>
<tr>
<th>Pound</th>
<th>American Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9069</td>
<td>0.5469</td>
</tr>
<tr>
<td>2.6303</td>
<td>0.2703</td>
</tr>
<tr>
<td>2.3800</td>
<td>0.1209</td>
</tr>
<tr>
<td>2.0691</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.7988</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The node that is in bold type in the American call option tree above indicates that early exercise is optimal at that node since the option exercise value exceeds the option expected present value at that node.

The value of the American call option is:

\[ e^{-0.05(1)} \left[ (0.47004)(0.2703) + (1 - 0.47004)(0.000) \right] = 0.1209 \]
Solution 38

**D** Chapter 10, Options on Futures Contracts

Since there is only one period, it doesn’t make any difference whether the call option is a European option or an American option.

The lease rate was thrown in as a red herring. It could be used to calculate the current price of gold, but we don’t need to know the current price of gold. But if you are curious, it is:

\[ S_0 = F_{0,T}e^{-(r-\delta)T} = 600e^{-0.07-0.04}(1) = 582.27 \]

The values of \( u_F \) and \( d_F \) are:

\[
\begin{align*}
  u_F &= e^{\sigma \sqrt{h}} = e^{0.12\sqrt{1}} = 1.12750 \\
  d_F &= e^{-\sigma \sqrt{h}} = e^{-0.12\sqrt{1}} = 0.88692
\end{align*}
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{1 - d_F}{u_F - d_F} = \frac{1 - 0.88692}{1.12750 - 0.88692} = 0.47004
\]

The futures price tree and the call option tree are below:

\[
\begin{array}{c|c|c}
  F_{0,1} & F_{1,1} & \text{Call Option} \\
  \hline
  676.4981 & 56.4981 & \\
  600.0000 & 24.7608 & 0.0000 \\
  532.1523 & & \\
\end{array}
\]

If the futures price rises to $676.4981, then the call option has a payoff of:

\[ 676.4981 - 620 = 56.4981 \]

The price of the call option is:

\[
e^{-0.07(1)}\left[(0.47004)(56.4981) + (1 - 0.47004)(0.0000)\right] = 24.7608
\]
Solution 39

A Chapter 10, Options on Futures Contracts

The value of $h$ is $1/3$ since the intervals are $1/3$ of a year. The values of $u_F$ and $d_F$ are:

$$u_F = e^{\sigma \sqrt{h}} = e^{0.30 \sqrt{1/3}} = 1.18911$$
$$d_F = e^{-\sigma \sqrt{h}} = e^{-0.30 \sqrt{1/3}} = 0.84097$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{1 - d_F}{u_F - d_F} = \frac{1 - 0.84097}{1.18911 - 0.84097} = 0.45681$$

The futures price tree and the call option tree are below:

<table>
<thead>
<tr>
<th>$F_{0,1}$</th>
<th>$F_{1,1}$</th>
<th>$F_{1,1}$</th>
<th>$F_{1,1}$</th>
<th>American Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,681.3806</td>
<td>1,413.9825</td>
<td>1,189.1099</td>
<td>122.4206</td>
<td>681.3806</td>
</tr>
<tr>
<td>1,189.1099</td>
<td>1,189.1099</td>
<td>229.5336</td>
<td>0.0000</td>
<td>189.1099</td>
</tr>
<tr>
<td>1,000.0000</td>
<td>1,000.0000</td>
<td>122.4206</td>
<td>0.0000</td>
<td>84.3943</td>
</tr>
<tr>
<td>840.9651</td>
<td>840.9651</td>
<td>37.6628</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>707.2224</td>
<td>594.7493</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

If the futures price reaches $1,413.9825$, then it is optimal to exercise the call option early since the option exercise value exceeds the expected present value of the option at that node.

The value of the American option is:

$$e^{-0.07(1/3)} \left[ (0.45681)(229.5336) + (1 - 0.45681)(37.6628) \right] = 122.4206$$

Solution 40

A Chapter 10, Options on Futures Contracts

The values of $u_F$ and $d_F$ are:

$$u_F = e^{\sigma \sqrt{h}} = e^{0.30 \sqrt{1/3}} = 1.18911$$
$$d_F = e^{-\sigma \sqrt{h}} = e^{-0.30 \sqrt{1/3}} = 0.84097$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{1 - d_F}{u_F - d_F} = \frac{1 - 0.84097}{1.18911 - 0.84097} = 0.45681$$
The futures price tree and the call option tree are below:

$$F_{0,1} \quad F_{\frac{1}{3},1} \quad F_{\frac{2}{3},1} \quad F_{1,1} \quad \text{American Call}$$

1,681.3806 \quad 1,413.9825 \quad 681.3806

1,189.1099 \quad 1,189.1099 \quad 229.5336 \quad 189.1099

1,000.0000 \quad 1,000.0000 \quad 122.4206 \quad 84.3943 \quad 0.0000

840.9651 \quad 840.9651 \quad 37.6628 \quad 0.0000 \quad 0.0000

707.2224 \quad 594.7493

If the futures price reaches $1,413.9825, then it is optimal to exercise the call option early since the option exercise value exceeds the expected present value of the option at that node.

The number of futures contracts that the investor must be long is:

$$\Delta = \frac{V_u - V_d}{F(u_F - d_F)} = \frac{229.5336 - 37.6628}{1,189.1099 - 840.9651} = 0.5511$$

**Solution 41**

**E Chapter 10, Two-Period Binomial Model**

Since the stock does not pay dividends, the price of the American call option is equal to the price of an otherwise equivalent European call option.

The stock price tree and the associated option payoffs at the end of 2 years are:

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>Call Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>75.6877</td>
<td>27.6877</td>
</tr>
<tr>
<td>58.3605</td>
<td></td>
</tr>
<tr>
<td>45.0000</td>
<td>50.7386</td>
</tr>
<tr>
<td>39.1230</td>
<td>34.0135</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.06 - 0.00)1} - 0.8694}{1.2969 - 0.8694} = 0.45014$$

The value of the call option is:

$$V(S_0, K, 0) = e^{-r(hn)} \sum_{i=0}^{n} \binom{n}{i} (p^*)^{(n-i)} (1 - p^*)^i V(S_0 u^{n-i} d^i, K, hn)$$

$$= e^{-0.06(2)} \left[ (0.45014)^2 (27.6877) + 2(0.45014)(1 - 0.45014)(2.7386) + 0 \right]$$

$$= 6.1783$$
Solution 42

Chapter 10, Three-Period Binomial Model for Currency

When the type of binomial model is not specified in the question, we should assume that we are to use the textbook’s standard binomial model. Question 5 on the SOA’s sample exam is an example of a question where we are expected to assume that the textbook’s standard binomial model applies.

The values of \( u \) and \( d \) are:

\[
\begin{align*}
  u &= e^{(r - r_f)h + \sigma \sqrt{h}} = e^{(0.10 - 0.04)0.25 + 0.32 \sqrt{0.25}} = 1.19125 \\
  d &= e^{(r - r_f)h - \sigma \sqrt{h}} = e^{(0.10 - 0.04)0.25 - 0.32 \sqrt{0.25}} = 0.86502
\end{align*}
\]

The risk-neutral probability of an upward movement is:

\[
\begin{align*}
p^* &= \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.10 - 0.04)(0.25) - 0.86502}}{1.19125 - 0.86502} = 0.46009
\end{align*}
\]

The tree of pound prices and the tree of option prices are below:

<table>
<thead>
<tr>
<th>Pound</th>
<th>2.6962</th>
<th>2.2634</th>
<th>1.9000</th>
<th>1.6435</th>
<th>1.4217</th>
<th>1.2298</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Put</td>
<td>0.0000</td>
<td>0.0988</td>
<td>0.2629</td>
<td>0.4151</td>
<td>0.6283</td>
<td>0.8202</td>
</tr>
<tr>
<td>3.2119</td>
<td>2.3323</td>
<td>1.9579</td>
<td>1.6936</td>
<td>1.4217</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If the exchange rate falls to 1.4217 at the end of 6 months, then early exercise is optimal. The value of the American put option at time 0 is 0.2629.

Solution 43

Chapter 10, Risk-Neutral Probability

When the type of binomial model is not specified in the question, then we should assume that we are to use the textbook’s standard binomial model.

The values of \( u \) and \( d \) are:

\[
\begin{align*}
  u &= e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.085 - 0.022)(0.25) + 0.28 \sqrt{0.25}} = 1.16853 \\
  d &= e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.085 - 0.022)(0.25) - 0.28 \sqrt{0.25}} = 0.88316
\end{align*}
\]

The risk-neutral probability of an upward movement is:

\[
\begin{align*}
p^* &= \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.085 - 0.022)(0.25) - 0.88316}}{1.16853 - 0.88316} = 0.46506
\end{align*}
\]
The risk-neutral probability of a downward movement is:
\[(1 - p^*) = 1 - 0.46506 = 0.53494\]

**Solution 44**

**D**  Chapter 10, Replication

The call option pays 11 in the up-state and 0 in the down-state:
\[V_u = \text{Max}[38 - 27, 0] = 11\]
\[V_d = \text{Max}[21 - 27, 0] = 0\]

The dividend rate is zero, so the number of shares needed to replicate the option is:
\[\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u-d)} = e^{-0.07 \times 1} \frac{11 - 0}{38 - 21} = 0.6471\]

**Solution 45**

**B**  Chapter 10, One-Period Binomial Tree

A straddle consists of a long call and a long put on the same stock, where both options have the same strike price and the same expiration date. The question describes the payoff of a straddle, so it is not necessary to know the definition of a straddle to answer this question.

If the stock price moves up, then the straddle pays $25. If the stock price moves down, then the straddle pays $2:
\[
\begin{align*}
95 & \quad 25 = |70 - 95| \\
75 & \quad V \\
68 & \quad 2 = |70 - 68|
\end{align*}
\]

In this case, \( u = 95/75 \) and \( d = 68/75 \).

The risk-neutral probability of an upward movement is:
\[p^* = e^{(r-\delta)h - d} = e^{(0.07-0.00) \times 1 - \frac{68}{75}} = \frac{95/75 - 68/75}{95/75 - 68/75} = 0.46067\]

The value of the straddle is:
\[
V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.07(1)} \left[ 0.46067(25) + (1 - 0.46067)(2) \right] = 11.74
\]
Solution 46

Chapter 10, Replication

The end-of-year payoffs of the call and put options in each scenario are shown in the table below. The rightmost column is the payoff resulting from buying the put option and selling the call option.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>End of Year Price of Stock A</th>
<th>End of Year Price of Stock B</th>
<th>C(A) Payoff</th>
<th>P(B) Payoff</th>
<th>P(B) – C(A) Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>$80</td>
<td>$0</td>
<td>45</td>
<td>35</td>
<td>-10</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>$30</td>
<td>$0</td>
<td>0</td>
<td>35</td>
<td>35</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>$0</td>
<td>$400</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We need to determine the cost of replicating the payoffs in the rightmost column above. We can replicate those payoffs by determining the proper amount of Stock A, Stock B, and the risk-free asset to purchase.

Let’s define the following variables:

- \( A \) = Number of shares of Stock A to purchase
- \( B \) = Number of shares of Stock B to purchase
- \( C \) = Amount to lend at the risk-free rate

We have 3 equations and 3 unknown variables:

- Scenario 1: \( 80A + 0B +Ce^{0.08} = -10 \)
- Scenario 2: \( 30A + 0B +Ce^{0.08} = 35 \)
- Scenario 3: \( 0 + 400B +Ce^{0.08} = 0 \)

Subtracting the second equation from the first equation allows us to find \( A \):

\[
50A = -45
\]

\( A = -0.9 \)

We can put this value of \( A \) into the first equation to find the value of \( C \):

\[
80(-0.9) + Ce^{0.08} = -10
\]

\[
Ce^{0.08} = 62
\]

\[
C = 57.2332
\]

We can use this value of \( C \) in the third equation to find the value of \( B \):

\[
400B + 57.2332e^{0.08} = 0
\]

\( B = -0.155 \)
The cost now of replicating the payoffs resulting from buying the put and selling the call is equal to the cost of establishing a position consisting of $A$ shares of Stock A, $B$ shares of Stock B, and $C$ lent at the risk-free rate:

$$50A + 50B + C = 50 \times (-0.9) + 50 \times (-0.155) + 57.2332 = 4.48$$

**Solution 47**

Chapter 10, Replication

The end-of-year payoffs of the call and put options in each scenario are shown in the table below. The rightmost column is the payoff resulting from buying the call option and selling the put option.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>End of Year Price of Stock A</th>
<th>End of Year Price of Stock B</th>
<th>$P_A(45)$ Payoff</th>
<th>$C_B(40)$ Payoff</th>
<th>$C_B(40) - P_A(45)$ Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>$30$</td>
<td>$0$</td>
<td>15</td>
<td>0</td>
<td>$-15$</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>$40$</td>
<td>$50$</td>
<td>5</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>$50$</td>
<td>$40$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We need to determine the cost of replicating the payoffs in the rightmost column above. We can replicate those payoffs by determining the proper amount of Stock A, Stock B, and the risk-free asset to purchase.

Let’s define the following variables:

$A = \text{Number of shares of Stock A to purchase}$  
$B = \text{Number of shares of Stock B to purchase}$  
$C = \text{Amount to lend at the risk-free rate}$

We have 3 equations and 3 unknown variables:

Scenario 1:  
$$30A + 0B + Ce^{0.05} = -15$$

Scenario 2:  
$$40A + 50B + Ce^{0.05} = 5$$

Scenario 3:  
$$50A + 40B + Ce^{0.05} = 0$$

The two equations below are obtained by subtracting the first equation above from the second and third equations above:

$$10A + 50B = 20$$
$$20A + 40B = 15$$

We can solve for $B$ by subtracting twice the first equation above from the second equation:

$$-60B = -25$$
$$B = \frac{5}{12}$$
We can put this value of $B$ into one of the two equations found earlier in order to find the value of $A$:

\[
10A + 50B = 20
\]
\[
10A + 50 \times \frac{5}{12} = 20
\]
\[
A = -\frac{1}{12}
\]

We can now find the value of $C$ using one of the 3 original equations:

\[
30A + 0B + Ce^{0.05} = -15
\]
\[
30 \times \frac{-1}{12} + 0 + Ce^{0.05} = -15
\]
\[
C = -11.89
\]

Since $C$ is negative, the payoffs resulting from buying the call and selling the put are replicated by borrowing $11.89, which is equivalent to lending $-11.89$.

The cost now of replicating the payoffs resulting from buying the call and selling the put is equal to the cost of establishing a position consisting of $A$ shares of Stock A, $B$ shares of Stock B, and $C$ lent at the risk-free rate. Since the price of Stock A is $40 and the price of Stock B is $30, we have:

\[
40A + 30B + C = 40 \times \frac{-1}{12} + 30 \times \frac{5}{12} - 11.89 = -2.72
\]

**Solution 48**

The risk-free asset can be replicated with the three stocks. Let’s find the cost of replicating an asset that is certain to pay a fixed amount at time 1. We can choose any fixed amount, and in this solution we use $100. We begin by defining the following variables:

\[
A = \text{Number of shares of Stock A to purchase}
\]
\[
B = \text{Number of shares of Stock B to purchase}
\]
\[
C = \text{Number of shares of Stock C to purchase}
\]

We have 3 equations and 3 unknown variables. The $100 on the right side of the equations below is the fixed amount certain to be paid by our risk-free asset:

- Scenario 1: \[200A + 0B + 0C = 100\]
- Scenario 2: \[50A + 0B + 100C = 100\]
- Scenario 3: \[0A + 300B + 50C = 100\]
The solution to these equations is:
\[ A = 0.5 \]
\[ C = 0.75 \]
\[ B = \frac{5}{24} \]

The cost now of replicating an asset that is certain to pay $100 at time 1 is equal to the cost of establishing a position consisting of \( A \) shares of Stock A, \( B \) shares of Stock B, and \( C \) shares of Stock C. Since the price of Stock A is $100, the price of Stock B is $50, and the price of Stock C is $40, the cost of the risk-free asset is:
\[
100A + 50B + 40C = 100 \times 0.5 + 50 \times \frac{5}{24} + 40 \times 0.75 = 90.4167
\]

Since \( t \) is one year, we can now solve for \( r \):
\[
90.4167e^{rt} = 100
\]
\[
r = 0.1007
\]

**Solution 49**

**E Chapter 10, Replication**

The end-of-year payoffs of the call and put options in each scenario are shown in the table below. The rightmost column is the payoff resulting from buying the put option and selling the call option.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>End of Year Price of Stock A</th>
<th>End of Year Price of Stock B</th>
<th>( P_A(100) ) Payoff</th>
<th>( C_B(200) ) Payoff</th>
<th>( P_A(100) - C_B(200) ) Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$200</td>
<td>$0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$50</td>
<td>$0</td>
<td>50</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>$25</td>
<td>$300</td>
<td>75</td>
<td>100</td>
<td>-25</td>
</tr>
</tbody>
</table>

We need to determine the cost of replicating the payoffs in the rightmost column above. We can replicate those payoffs by determining the proper amount of Stock A, Stock B, and the risk-free asset to purchase.

Let’s define the following variables:
\[
A = \text{Number of shares of Stock A to purchase} \\
B = \text{Number of shares of Stock B to purchase} \\
C = \text{Amount to lend at the risk-free rate}
\]
Since Stock A pays a $10 dividend at time 0.5, each share of Stock A that is purchased provides its holder with the final price of Stock A at time 1 plus the accumulated value of $10. Since Stock B pays continuously compounded dividends of 5%, each share of stock B purchased at time 0 grows to $e^{0.05}$ shares of Stock B at time 1.

We have 3 equations and 3 unknown variables:

Scenario 1: \[
200 + 10e^{0.08(0.5)}A + 0B + Ce^{0.08} = 0
\]

Scenario 2: \[
50 + 10e^{0.08(0.5)}A + 0B + Ce^{0.08} = 50
\]

Scenario 3: \[
25 + 10e^{0.08(0.5)}A + 300e^{0.05}B + Ce^{0.08} = -25
\]

Subtracting the first equation from the second equation allows us to solve for $A$:

\[
A = \frac{50}{50 + 10e^{0.04} - 200 - 10e^{0.04}} = -\frac{1}{3}
\]

The equation associated with Scenario 1 can now be used to find $C$:

\[
C = \left[200 + 10e^{0.08(0.5)}\right]e^{-0.08} = \left[200 + 10e^{0.08(0.5)}\right] \frac{1}{3} e^{-0.08} = 64.7437
\]

We use the equation associated with Scenario 3 to find $B$:

\[
-\left[25 + 10e^{0.08(0.5)}\right] \frac{1}{3} + 300e^{0.05}B + 64.7437e^{0.08} = -25
\]

\[
B = -0.2642
\]

The cost now of replicating the payoffs resulting from buying the put and selling the call is equal to the cost of establishing a position consisting of $A$ shares of Stock A, $B$ shares of Stock B, and $C$ lent at the risk-free rate. Since the price of Stock A is $100 and the price of Stock B is $75, we have:

\[
100A + 75B + C = 100 \times \frac{-1}{3} + 75 \times (-0.2642) + 64.7437 = 11.5931
\]
Solution 50

A Chapter 10, Delta

The delta of the put option is $-0.3275$. The delta of the corresponding call option satisfies the following formula:

\[ \Delta_{Call} - \Delta_{Put} = e^{-\delta T} \]
\[ \Delta_{Call} - (-0.3275) = e^{-0.07(0.75)} \]
\[ \Delta_{Call} = 0.6214 \]

Solution 51

A Chapter 10, Replication

Since we are given the value of the put option and the delta of the put option, we can find the amount that must be lent to replicate the put option:

\[ V_{Put} = S_0 \Delta_{Put} + B_{Put} \]
\[ 3.89 = 52(-0.3275) + B_{Put} \]
\[ B_{Put} = 20.92 \]

The put option is replicated by lending $20.92$, so $X = 20.92$. We can now solve for $B_{Call}$:

\[ B_{Put} - B_{Call} = Ke^{-rT} \]
\[ 20.92 - B_{Call} = 50e^{-0.10(0.75)} \]
\[ B_{Call} = 25.47 \]

Therefore $25.47$ must be borrowed to replicate the call option.

Solution 52

E Chapter 10, Greeks in the Binomial Model

The values of $u$ and $d$ are:

\[ u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.11-0.04)(1)+0.32\sqrt{1}} = 1.47698 \]
\[ d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.11-0.04)(1)-0.32\sqrt{1}} = 0.77880 \]

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.11-0.04)(1)} - 0.77880}{1.47698 - 0.77880} = 0.42068 \]
The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>87.2589</th>
<th>59.0792</th>
<th>40.0000</th>
<th>31.1520</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Put</td>
<td>0.0000</td>
<td>0.0000</td>
<td>5.6299</td>
<td>10.8480</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.0000</td>
<td></td>
</tr>
</tbody>
</table>

If the stock price initially moves down, then the resulting put price is $10.8480. This price is in bold type above to indicate that it is optimal to exercise early at this node:

\[ 42 - 31.1520 = 10.8480 \]

The exercise value of $10.8480$ is greater than the value of holding the option, which is:

\[ e^{-0.11(1)} \left[ (0.42068)0.0000 + (1 - 0.42068)(17.7388) \right] = 9.2060 \]

The current value of the American option is:

\[ e^{-0.11(1)} \left[ (0.42068)0.0000 + (1 - 0.42068)(10.8480) \right] = 5.6299 \]

We need to calculate 3 values of delta:

\[ \Delta(S,0) = e^{-\delta h} \frac{V_u - V_d}{S_u - S_d} = e^{-0.04 \times 1} \frac{0 - 10.8480}{59.0792 - 31.1520} = -0.3732 \]

\[ \Delta(S_u,h) = e^{-\delta h} \frac{V_{uu} - V_{ud}}{S_u^2 - S_{ud}} = e^{-0.04 \times 1} \frac{0.0000 - 0.0000}{87.2589 - 46.0110} = 0.0000 \]

\[ \Delta(S_d,h) = e^{-\delta h} \frac{V_{ud} - V_{dd}}{S_{ud} - S_d^2} = e^{-0.04 \times 1} \frac{0.0000 - 17.7388}{46.0110 - 24.2612} = -0.7836 \]

We can now calculate gamma:

\[ \Gamma(S,0) = \Gamma(S_u,h) = \frac{\Delta(S_u,h) - \Delta(S_d,h)}{S_u - S_d} = \frac{0 - (-0.7836)}{59.0792 - 31.1520} = 0.0281 \]

We can now estimate theta:

\[ \theta(S,0) = \frac{V_{ud} - V - (S_u - S)\Delta(S,0) - \frac{(S_d - S)^2}{2} \Gamma(S,0)}{2h} \]

\[ = \frac{0 - 5.6299 - (46.0110 - 40.0000)(-0.3732) - \frac{(46.0110 - 40.0000)^2}{2}(0.0281)}{2 \times 1} \]

\[ = -1.9467 \]
Solution 53

B Chapter 10, Greeks in the Binomial Model

The question does not tell us when the call option expires, so we cannot assume that it expires in 6 months.

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.10 - 0.03)0.25} - 61.60/70}{82.38/70 - 61.60/70} = 0.4637 \]

The time 0 value of the option is:

\[ C_0 = e^{-rh} [(p^*)C_u + (1 - p^*)C_d] = e^{-0.10 \times 0.25} \times [0.4637 \times 14.91 + (1 - 0.4637) \times 2.72] \]
\[ = 8.1658 \]

The delta-gamma-theta approximation is:

\[ C_{ud} = C_0 + (Su-S)\Delta(S,0) + \frac{(Su-S)^2}{2} \Gamma(S,0) + 2h\theta(S,0) \]
\[ = 8.1658 + (72.49 - 70)(0.5822) + \frac{(72.49 - 70)^2}{2}(0.0233) + 2(0.25)(-7.3361) \]
\[ = 6.0197 \]

Solution 54

B Chapter 10, Greeks in the Binomial Model

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.10 - 0.03)0.25} - 61.60/70}{82.38/70 - 61.60/70} = 0.4637 \]

The time 0 value of the option is:

\[ C_0 = e^{-rh} [(p^*)C_u + (1 - p^*)C_d] = e^{-0.10 \times 0.25} \times [0.4637 \times 14.91 + (1 - 0.4637) \times 2.72] \]
\[ = 8.1658 \]

The delta-gamma-theta approximation is:

\[ C_u \approx C_0 + (Su-S)\Delta(S,0) + \frac{(Su-S)^2}{2} \Gamma(S,0) + h\theta(S,0) \]
\[ = 8.1658 + (82.38 - 70)(0.5822) + \frac{(82.38 - 70)^2}{2}(0.0233) + (0.25)(-7.3361) \]
\[ = 15.32 \]

The difference between the estimate and the actual value of $14.91 (which can be read from the tree of option prices) is:

\[ 15.32 - 14.91 = 0.41 \]
### Solution 55

**E** Chapter 10, Three-Period Binomial Tree

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.10-0.065)\times1} - \frac{210}{300}}{\frac{375}{300} - \frac{210}{300}} = 0.6102
\]

The tree of prices for the American call option is shown below:

<table>
<thead>
<tr>
<th>Stock Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Call</td>
</tr>
<tr>
<td>168.7500</td>
</tr>
<tr>
<td>98.6519</td>
</tr>
<tr>
<td>57.4945</td>
</tr>
<tr>
<td>8.5744</td>
</tr>
<tr>
<td>0.0000</td>
</tr>
</tbody>
</table>

Early exercise is optimal if the stock price increases to $468.75 at the end of 2 years. This is indicated by the bolding of that node in the tree above.

The price of the option is:

\[
e^{-0.10\times1} \left[ 0.6102 \times 98.6519 + (1 - 0.6102) \times 8.5744 \right] = 57.4945
\]

### Solution 56

**A** Chapter 10, Greeks in the Binomial Model

We need to calculate the two possible values of delta at the end of 1 year:

\[
\Delta(Su,h) = e^{-\delta h} \frac{V_{uu} - V_{ud}}{Su^2 - Sud} = e^{-0.065\times1} \frac{168.75 - 15.5292}{468.75 - 262.50} = 0.6961
\]

\[
\Delta(Sd,h) = e^{-\delta h} \frac{V_{ud} - V_{dd}}{Sud - Sd^2} = e^{-0.065\times1} \frac{15.5292 - 0.0000}{262.50 - 147.00} = 0.1260
\]

We can now calculate gamma:

\[
\Gamma(S,0) \approx \Gamma(S_h,h) = \frac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd} = \frac{0.6961 - 0.1260}{375 - 210} = 0.003455
\]

### Solution 57

**A** Chapter 11, Three-Period Binomial Tree

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.10-0.065)\times1} - \frac{210}{300}}{\frac{375}{300} - \frac{210}{300}} = 0.6102
\]
The tree of prices for the American put option is shown below:

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>0.0000</th>
<th>13.8384</th>
<th>37.6195</th>
<th>85.0000</th>
<th>148.0000</th>
<th>192.1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>American put</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

It is not necessary to calculate all of the prices in the tree above to answer this question, but we included the full tree for the sake of completeness.

Early exercise is optimal if the stock price falls to $210 at the end of 1 year or falls to $147 at the end of 2 years. This is indicated by the bolding of those two nodes in the tree above.

The only year-2 node at which early exercise occurs is the bottom node. But to reach that node, the stock must first pass through the year-1 bottom node. Since it will be exercised at the end of year 1 if the stock price reaches $210, there is no possibility of the stock being exercised at the end of year 2. Therefore, \( p_2 = 0 \).

**Solution 58**

A Chapter 10, Options on Futures Contracts

We are given that the ratio of the factors applicable to the futures price is:

\[
\frac{u_F}{d_F} = \frac{4}{3}
\]

The formula for the risk-neutral probability of an up move can be used to find \( u_F \) and \( d_F \):

\[
p^* = \frac{1 - d_F}{u_F - d_F}
\]

\[
p^* = \frac{1}{d_F} \cdot \frac{d_F}{u_F - d_F}
\]

\[
\frac{1}{3} = \frac{1}{d_F} - 1
\]

\[
d_F = 0.9
\]

\[
u_F = \frac{4}{3} \times d_F = 1.2
\]
The tree of futures prices is therefore:

<table>
<thead>
<tr>
<th>Futures Prices</th>
<th>115.2000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>96.0000</td>
</tr>
<tr>
<td></td>
<td>80.0000</td>
</tr>
<tr>
<td></td>
<td>86.4000</td>
</tr>
<tr>
<td></td>
<td>72.0000</td>
</tr>
<tr>
<td></td>
<td>64.8000</td>
</tr>
</tbody>
</table>

The tree of prices for the European put option is:

<table>
<thead>
<tr>
<th>European Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>8.5399</td>
<td>0.0000</td>
</tr>
<tr>
<td>13.1342</td>
<td>20.2000</td>
</tr>
</tbody>
</table>

The price of the European put is:

\[ e^{-0.05 \times 0.5} \left[ \frac{1}{3} \times 0.0000 + \frac{2}{3} \times 13.1342 \right] = 8.5399 \]

The tree of prices for the American put option is:

<table>
<thead>
<tr>
<th>American Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>8.5399</td>
<td>0.0000</td>
</tr>
<tr>
<td>13.1342</td>
<td>20.2000</td>
</tr>
</tbody>
</table>

The price of the American put is:

\[ e^{-0.05 \times 0.5} \left[ \frac{1}{3} \times 0.0000 + \frac{2}{3} \times 13.1342 \right] = 8.5399 \]

Early exercise of the American option is never optimal, so the prices of the European option and the American option are the same.

The price of the American put option exceeds the price of the European put option by:

\[ 8.5399 - 8.5399 = 0.0000 \]

**Solution 59**

C Chapter 10, Options on Futures Contracts

We are given that the ratio of the factors applicable to the futures price is:

\[ \frac{d_F}{u_F} = 0.6 \]
The formula for the risk-neutral probability of an up move can be used to find $u_F$ and $d_F$:

$$p^* = \frac{1 - d_F}{u_F - d_F}$$

$$p^* = \frac{1}{u_F} - \frac{d_F}{u_F}$$

$$\frac{1}{6} = \frac{1}{u_F} - 0.6$$

$$u_F = 1.5$$

$$d_F = 0.6 \times u_F = 0.9$$

The tree of futures prices is therefore:

<table>
<thead>
<tr>
<th>Futures Prices</th>
<th>157.5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>105.0000</td>
<td></td>
</tr>
<tr>
<td>70.0000</td>
<td>94.5000</td>
</tr>
<tr>
<td>63.0000</td>
<td>56.7000</td>
</tr>
</tbody>
</table>

The tree of prices for the European call option is:

<table>
<thead>
<tr>
<th>European Call</th>
<th>92.5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>38.8178</td>
<td></td>
</tr>
<tr>
<td>10.1370</td>
<td>29.5000</td>
</tr>
<tr>
<td>4.7714</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The price of the European call is:

$$e^{-0.06 \times 0.5} \left[ \frac{1}{6}(38.8178) + \frac{5}{6}(4.7714) \right] = 10.1370$$

The tree of prices for the American call option is:

<table>
<thead>
<tr>
<th>American Call</th>
<th>92.5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>40.0000</td>
<td></td>
</tr>
<tr>
<td>10.3283</td>
<td>29.5000</td>
</tr>
<tr>
<td>4.7714</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The price of the American call is:

$$e^{-0.06 \times 0.5} \left[ \frac{1}{6}(40.0000) + \frac{5}{6}(4.7714) \right] = 10.3283$$

If the futures price moves up at the end of 6 months, then early exercise of the American option is optimal, and this is indicated by the bolding of that node above.
The price of the American call option exceeds the price of the European call option by:

$$10.3283 - 10.1370 = 0.1912$$

**Solution 60**

### C Chapter 10, American Put Option

The values of $u$ and $d$ are:

$$u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.06 - 0.00)0.5 + 0.35\sqrt{0.5}} = 1.3198$$

$$d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.06 - 0.00)0.5 - 0.35\sqrt{0.5}} = 0.8045$$

The risk-free probability of an upward movement is:

$$p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.06 - 0.00)0.5} - 0.8045}{1.3198 - 0.8045} = 0.4384$$

The stock price tree is:

98.99  
75  
60.34

Let’s use trial and error. We begin with the middle strike price of $93. The value of immediate exercise is:

$$93 - 75 = 18$$

The value of holding on to the option is:

$$(93 - 60.34) \times e^{-0.06 \times 0.5 \times (1 - 0.4384)} = 17.80$$

Since 17.80 < 18.00, the investor will exercise immediately if the strike price is $93.

If the strike price is higher than $93, then the put option will be even more in-the-money, so the option is sure to be exercised at strike prices that are greater than or equal to $93.

Next, let’s try a strike price of $92. The value of immediate exercise is:

$$92 - 75 = 17$$

The value of holding on to the option is:

$$(92 - 60.34) \times e^{-0.06 \times 0.5 \times (1 - 0.4384)} = 17.25$$

Since 17.25 > 17.00, the investor will not exercise immediately if the strike price is $92.

If the strike price is less than $92, then the option is even less in-the-money, meaning that it is even less likely to be exercised early.

Therefore, the smallest integer-valued strike price for which the investor will exercise the put option at the beginning of the period is $93.
Alternate Solution
The strike price \( K \), for which an investor will exercise the put option at the beginning of the period must be at least $75, since otherwise the payoff to immediate exercise would be zero. Since we are seeking the lowest strike price that results in immediate exercise, let’s begin by determining whether there is a strike price that is greater than $75 but less than $98.99 that results in immediate exercise.

If there is such a strike price, then the value of exercising now must exceed the value of holding the option:

\[
0.06(0.5) - 0.06(0.5) - 75 (60.34)(1) > 75 (60.34)(1 - 0.4384)
\]

\[
75 - 0.5450 32.88 0.4550 42.12 92.56
\]

\[
K > 93.
\]

The smallest integer that satisfies this inequality is $93.

Solution 61
A Chapter 10, American Call Option

The stock does not pay dividends. Therefore, the general rule is that an American call option on the stock will not be exercised early.

But we must select one of the answer choices, so let’s consider the unusual situation of a strike price of zero. If \( K = 0 \), then the exercise value is equal to the $75. Furthermore, the value of holding the option is $75 as well, since the option delivers the same future cash flows as ownership of one share of stock.

Therefore, the owner of the option is indifferent to exercising the option when the strike price is 0. That is the only possible strike price at which the investor will exercise the call option, so Choice A is the best answer to this question.

Referring back to the solution to Question 60, some students might be tempted to solve the inequality below to obtain an answer:

\[
75 - K \geq (98.99 - K) \times 0.4384e^{-0.06(0.5)}
\]

\[
75 - K \geq (98.99 - K) \times 0.4255
\]

\[
0.5745K \geq -32.8832
\]

\[
K \leq 57.2363
\]
Does the solution above suggest that the correct answer is $57? No, because the inequality above is only valid if $K \geq 60.34$, which is obviously contradicted by a solution of $K = 57.235$. If the strike price is less than $60.34$, then both 6-month nodes produce a payoff, and the inequality becomes:

$$
\begin{align*}
75 - K \geq (98.99 - K) \times 0.4384e^{-0.06(0.5)} + (60.34 - K) \times (1 - 0.4384)e^{-0.06(0.5)} \\
75 - K \geq (98.99 - K) \times 0.4255 + (60.34 - K) \times 0.5450 \\
75 - K \geq 42.1168 - 0.4255K + 32.8832 - 0.5450K \\
K \leq 0
\end{align*}
$$

Solution 62

E Chapter 10, American Put Option

The values of $u$ and $d$ are:

$$
\begin{align*}
u &= e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.06-0.08)0.5 + 0.35\sqrt{0.5}} = 1.2681 \\
d &= e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.06-0.08)0.5 - 0.35\sqrt{0.5}} = 0.7730
\end{align*}
$$

The risk-free probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h - d}}{u - d} = \frac{e^{(0.06-0.08)0.5 - 0.7730}}{1.2681 - 0.7730} = 0.4384$$

The stock price tree is:

\begin{align*}
95.10 \\
75 \\
57.97
\end{align*}

Let’s use trial and error. We begin with the middle strike price of $96. The value of immediate exercise is:

$$93 - 75 = 21$$

The value of holding on to the option is:

$$(96 - 95.10) \times e^{-0.06 \times 0.5} \times 0.4384 + (96 - 57.97) \times e^{-0.06 \times 0.5} \times (1 - 0.4384) = 21.10$$

Since 21.00 < 21.10, the investor will not exercise immediately if the strike price is $96.

If the strike price is less than $96, then the option is even less in-the-money, meaning that it is even less likely to be exercised early. Therefore, for our next guess, we’ll try a higher strike price.

Next, let’s try a strike price of $99. The value of immediate exercise is:

$$99 - 75 = 24$$

The value of holding on to the option is:

$$(99 - 95.10) \times e^{-0.06 \times 0.5} \times 0.4384 + (99 - 57.97) \times e^{-0.06 \times 0.5} \times (1 - 0.4384) = 24.01$$
Since $24.00 < 24.01$, the investor will not exercise immediately if the strike price is $99.

By the process of elimination, we know that the correct answer must be Choice E, which is $100. Let’s verify this. The value of immediate exercise is:

$$100 - 75 = 25$$

The value of holding on to the option is:

$$(100 - 95.10) \times e^{-0.06 \times 0.5} \times 0.4384 + (100 - 57.97) \times e^{-0.06 \times 0.5} \times (1 - 0.4384) = 24.99$$

Since $25.00 > 24.99$, the option is exercised immediately if the strike price is $100.

Therefore, the smallest integer-valued strike price for which the investor will exercise the put option at the beginning of the period is $100.

Alternate Solution

The strike price $K$, for which an investor will exercise the put option at the beginning of the period must be at least $75$, since otherwise the payoff to immediate exercise would be zero. Since we are seeking the lowest strike price that results in immediate exercise, let’s begin by determining whether there is a strike price that is greater than $75$ but less than $95.10$ that results in immediate exercise.

If there is strike price that is less than $95.10$, then the value of exercising now must exceed the value of holding the option:

$$K - 75 > (K - 57.97)(1 - p^*)e^{-0.06(0.5)}$$

$$K - 75 > (K - 57.97)(1 - 0.4384)e^{-0.06(0.5)}$$

$$K - 75 > 0.5450K - 31.59$$

$$0.4550K > 43.41$$

$$K > 95.39$$

The inequality above suggests that the strike price must be greater than $95.39$, but the original inequality was predicated on the assumption that $K < 95.10$. Clearly, the strike price cannot simultaneously be greater than $95.39$ and less than $95.10$. Therefore, the assumption that the strike price is less than $95.10$ is incorrect.

We now consider the possibility that the strike price is greater than $95.10$. Once again, the value of exercising now exceeds the value of holding the option:

$$K - 75 > (K - 95.10)(p^*)e^{-0.06(0.5)} + (K - 57.97)(1 - p^*)e^{-0.06(0.5)}$$

$$K - 75 > (K - 95.10)(0.4384)e^{-0.06(0.5)} + (K - 57.97)(1 - 0.4384)e^{-0.03}$$

$$K - 75 > 0.9704K - 72.06$$

$$0.0296K > 2.94$$

$$K > 99.50$$

The smallest integer that satisfies this inequality is $100$. 

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Solution 63

B Chapter 10, Arbitrage in the Binomial Model

If the stock price moves up, then the option pays $\text{Max}[0, 110 - 100] = 10$. If the stock price moves down, then the option pays $\text{Max}[0, 80 - 100] = 0$:

\[
\begin{array}{ccc}
\text{110} & 10 \\
\text{100} & V \\
\text{80} & 0
\end{array}
\]

Let’s use the replication method to find the value of the option:

\[
\begin{align*}
\Delta &= e^{-\delta_h} \frac{V_u - V_d}{S(u - d)} = e^{-0.10(1)} \frac{10 - 0}{110 - 80} = 0.3016 \\
B &= e^{-r_h} \frac{uV_d - dV_u}{u - d} = e^{-0.04(1)} \frac{110(0) - 80(10)}{110 - 80} = -25.6211
\end{align*}
\]

If no arbitrage is available, then the price of the option is:

\[
V = S_0 \Delta + B = 100(0.3016) - 25.6211 = 4.5402
\]

The actual option price of $4.00 is less than $4.5402$, so the option’s price is too low. Arbitrage is obtained by buying the option (buy low) and also replicating the sale of the option. The sale of the option is replicated by selling $\Delta$ shares and borrowing $B$. In this case, $B$ is negative, so borrowing $B$ is equivalent to lending 25.6211.

To summarize, arbitrage is obtained by:

- buying the option for $4.00
- selling 0.3016 shares of stock
- lending $25.6211$ at the risk-free rate.

These actions are described in Choice B.

Solution 64

C Chapters 10 and 13, Theta in the Binomial Model

The formula for theta is:

\[
\theta(S, 0) = \frac{V_{ud} - V - (Sud - S)\Delta(S, 0) - (Sud - S)^2 \Gamma(S, 0)}{2h}
\]

Since we have $S = Sud = 120$, this simplifies to:

\[
\theta(S, 0) = \frac{V_{ud} - V}{2h}
\]

The risk-neutral probability of an upward move is:

\[
p^* = \frac{60e^{0.08} - 48}{75 - 48} = 0.6295
\]
Since the put option is an American option, we solve for its value from right to left. The completed tree is shown below.

0.0000
0.0000
1.4035 0.0000
4.9195 4.1039
12.0000 12.0000
21.6000
29.2800

The bolded entries in the table above indicate where early exercise is optimal.

The value of theta can now be found:

\[
\theta(S,0) = \frac{V_{ud} - V}{2h} = \frac{4.1039 - 4.9195}{2 \times 1} = -0.4078
\]
Chapter 11 – Solutions

Solution 1

Chapter 11, State Prices

The question tells us that the stock price might decrease to $X, which implies that:

\[ X < 100 \]

The stock price tree and the call option tree are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>150.0000</td>
<td>50.0000</td>
</tr>
<tr>
<td>100.0000</td>
<td>23.7200</td>
</tr>
<tr>
<td>( X )</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

We can use the price of the call option to obtain the state price associated with an upward movement:

\[ 23.72 = 50Q_u \]
\[ Q_u = 0.4744 \]

The risk-free rate of return can now be used to obtain the state price associated with a downward movement:

\[ e^{-r_t} = Q_u + Q_d \]
\[ e^{-0.10 \times 1} = 0.4744 + Q_d \]
\[ Q_d = 0.4304 \]

We can use the stock’s price and the state prices to obtain the value of \( X \):

\[ 100 = 150Q_u + XQ_d \]
\[ 100 = 150 \times 0.4744 + 0.4304 \times X \]
\[ X = 67.0016 \]

The strike price of the put option is:

\[ K = 1.6X = 1.6 \times 67.0016 = 107.2026 \]

The put option pays off only if the stock’s price falls to 67.0016, so the current value of the put option is:

\[ 0Q_u + (107.2026 - 67.0016)Q_d = 0 + 40.2010 \times 0.4304 = 17.3040 \]

Alternate Solution

We don’t have to use state prices to answer this question.
Since the call option has a payoff of 50 if the stock price increases and 0 if the stock price falls, we can use the call price to obtain the risk-neutral probability of an upward movement:

\[ 23.72 = e^{-0.10} \left[ 50p^* + 0(1 - p^*) \right] \]
\[ p^* = 0.5243 \]

We can now use the current stock price to solve for \( X \):

\[ 100 = e^{-0.10} \left[ 150p^* + X(1 - p^*) \right] \]
\[ 100 = e^{-0.10} \left[ 150 \times 0.5243 + X(1 - 0.5243) \right] \]
\[ X = 67.0016 \]

The strike price of the put option is:

\[ K = 1.6X = 1.6 \times 67.0016 = 107.2026 \]

The put option pays off only if the stock price falls to 67.0016, so the current value of the put option is:

\[ e^{-0.10} \left[ 0 \times 0.5243 + (107.2026 - 67.0016)(1 - 0.5243) \right] = 17.3040 \]

**Solution 2**

D Chapter 11, Early Exercise of Perpetual Options

The option should be exercised early if and only if:

\[ \delta S > rK \]

This is equivalent to:

\[ S > \frac{rK}{\delta} \]

This condition is satisfied when:

\[ S > \frac{(0.14)(30)}{0.11} \]
\[ S > 38.1818 \]

Therefore, the lowest price (rounded up to the nearest penny) for which early exercise is optimal is $38.19.

**Solution 3**

C Chapter 11, Early Exercise of Perpetual Options

The option should be exercised early if and only if:

\[ S > \frac{rK}{\delta} \]
This condition is satisfied when:

\[ S > \frac{(0.07)(68)}{0.05} \]
\[ S > 95.20 \]

The option is called when the stock price reaches $95.20.

The volatility of the stock is 0%, so the return on the stock is the risk-free rate of return. The stock pays dividends at a rate of 5%, so for it to earn the risk-free rate of 7% it must increase in price at a rate of 2%:

\[ r - \delta = 0.07 - 0.05 = 0.02 \]

Another way to see this is to consider that in the binomial model, \( u \) and \( d \) are equal:

\[ u = e^{(r-\delta)h+\sigma \sqrt{h}} = e^{(0.07-0.05)h+0.00 \sqrt{h}} = e^{0.02h} \]
\[ d = e^{(r-\delta)h-\sigma \sqrt{h}} = e^{(0.07-0.05)h-0.00 \sqrt{h}} = e^{0.02h} \]

Regardless of the size of \( h \), the stock price increases at a continuously compounded rate of 2% per year.

The time until the stock price reaches $95.20 can be found by solving the following equation for \( T \):

\[ 70e^{0.02T} = 95.20 \]
\[ e^{0.02T} = \frac{95.20}{70} \]
\[ 0.02T = \ln \left( \frac{95.20}{70} \right) \]
\[ T = \ln \left( \frac{95.20}{70} \right) \times \frac{1}{0.02} \]
\[ T = 15.37 \]

**Solution 4**

The option should be exercised early if and only if:

\[ S > \frac{rK}{\delta} \]

This condition is satisfied when:

\[ S > \frac{(0.09)(48)}{0.06} \]
\[ S > 72.00 \]

The option is called when the stock price reaches $72.00.
The volatility of the stock is 0%, so the return on the stock is the risk-free rate of return. The stock pays dividends at a rate of 6%, so for it to earn the risk-free rate of 9% it must increase in price at a rate of 3%:
\[ r - \delta = 0.09 - 0.06 = 0.03 \]
The time until the stock price reaches $72.00 can be found by solving the following equation for \( T \):
\[
50e^{0.03T} = 72 \\
0.03T = \ln \left( \frac{72}{50} \right) \\
T = \ln \left( \frac{72}{50} \right) \times \frac{1}{0.03} \\
T = 12.1548
\]
The option is exercised in 12.1548 years when the stock price is $72.00. At that time, the exercise value is:
\[ 72 - 48 = 24 \]
Since the volatility is zero, the American call option is certain to pay out $24 in 12.1548 years. Since the payoff is certain, its present value is obtained by discounting at the risk-free rate of return:
\[ 24e^{-12.1548 \times 0.09} = 24e^{-12.1548(0.09)} = 8.0376 \]

Solution 5

C Chapter 11, Exercise Boundaries

Answer A is not correct because an increase in the volatility of the stock increases the value of the implicit insurance, thereby making early exercise less attractive.

Answer B is not correct because an increase in the volatility of the stock increases the value of the implicit insurance, thereby making early exercise less attractive.

Answer C is correct because an increase in the volatility of the stock increases the value of the implicit insurance, making early exercise less attractive and increasing the exercise boundary that must be reached in order for early exercise to be optimal.

Answer D is not correct because as an American call option ages, the early-exercise criteria become less stringent, which means that the exercise boundary decreases.

Answer E is not correct because as an American put option ages, the early-exercise criteria become less stringent, which means that the exercise boundary increases.
Solution 6

E Chapter 11, Exercise Boundaries

For each option, there are 3 effects to consider: the change in volatility, the new stock price, and the passage of time.

A call option is exercised only if the stock’s price is greater than the exercise boundary, and a put option is exercised only if the stock’s price is less than the exercise boundary.

Let’s consider the options in order.

American call option on Stock X

- Higher volatility increases the exercise boundary: Less likely exercise
- Unchanged price does not affect likelihood of exercise: No effect
- Passage of time decreases the exercise boundary: More likely exercise
  
  Net effect: Indeterminate

American call option on Stock Y

- Lower volatility decreases the exercise boundary: More likely exercise
- Lower price decreases the likelihood of exercise: Less likely exercise
- Passage of time decreases the exercise boundary: More likely exercise
  
  Net effect: Indeterminate

American call option on Stock Z

- Higher volatility increases the exercise boundary: Less likely exercise
- Lower price decreases the likelihood of exercise: Less likely exercise
- Passage of time decreases the exercise boundary: More likely exercise
  
  Net effect: Indeterminate

American put option on Stock X

- Higher volatility decreases the exercise boundary: Less likely exercise
- Unchanged price does not affect likelihood of exercise: No effect
- Passage of time increases the exercise boundary: More likely exercise
  
  Net effect: Indeterminate

American put option on Stock Y

- Lower volatility increases the exercise boundary: More likely exercise
- Lower price increases the likelihood of exercise: More likely exercise
- Passage of time increases the exercise boundary: More likely exercise
  
  Net effect: More likely exercise

The only option which we can say for sure is more likely to be exercised now is the American put option on Stock Y. Since we are told that only one of the options is optimal to exercise now, that option must be the put option on Stock Y.
Solution 7

D Chapter 11, Risk-Neutral Pricing

The values of $u$ and $d$ are:

\[ u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.08 - 0.00)(1) + 0.30\sqrt{\frac{1}{2}}} = 1.46228 \]
\[ d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.08 - 0.00)(1) - 0.30\sqrt{\frac{1}{2}}} = 0.80252 \]

The possible stock prices at the end of one year are:

\[ 50u = 50 \times 1.46228 = 73.1142 \]
\[ 50d = 50 \times 0.80252 = 40.1259 \]

The payoffs for the call option in 1 year are:

- Up state: $\max[0, 73.1142 - 48] = 25.1142$
- Down state: $\max[0, 40.1259 - 48] = 0.0000$

The stock price tree and the call price tree are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>73.1142</td>
<td>25.1142</td>
</tr>
<tr>
<td>50.0000</td>
<td>V</td>
</tr>
<tr>
<td>40.1259</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.08 - 0.00)(1) - 0.80252}}{1.46228 - 0.80252} = 0.42556 \]

The value of the call option is:

\[ V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.08(1)} \left[ 0.42556(25.1142) + (1 - 0.42556)(0.0000) \right] = 9.8659 \]

Solution 8

D Chapter 11, Expected Return

The values of $u$ and $d$ are:

\[ u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.08 - 0.00)(1) + 0.30\sqrt{\frac{1}{2}}} = 1.46228 \]
\[ d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.08 - 0.00)(1) - 0.30\sqrt{\frac{1}{2}}} = 0.80252 \]

The stock price tree and the call price tree are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>73.1142</td>
<td>25.1142</td>
</tr>
<tr>
<td>50.0000</td>
<td>V</td>
</tr>
<tr>
<td>40.1259</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.08 - 0.00)1} - 0.80252}{1.46228 - 0.80252} = 0.42556 \]

The value of the call option is:

\[ V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.08} \left[ 0.42556(25.1142) + (1 - 0.42556)(0.0000) \right] = 9.8659 \]

Since we know the price of the option and the distribution of its payoffs, we can now find the expected return of the option:

\[ V = e^{-rh} \left[ (p)V_u + (1 - p)V_d \right] \]

\[ 9.8659 = e^{-\gamma} \left[ 0.46 \times 25.1142 + (1 - 0.46) \times 0.0000 \right] \]

\[ \gamma = -\ln \left( \frac{9.8659}{0.46 \times 25.1142} \right) = 0.15783 \]

Solution 9

C Chapter 11, Expected Return

The return on a put option is generally less than the risk-free rate of return. For a put option, \( \Delta \) is negative and \( \beta \) is positive. This means that buying a put option is equivalent to selling stock and lending at the risk-free rate. The expected return is less than the risk-free rate since the stock's expected return is greater than the risk-free rate.

The values of \( u \) and \( d \) are:

\[ u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.08 - 0.00)1 + 0.30 \sqrt{1}} = 1.46228 \]

\[ d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.08 - 0.00)1 - 0.30 \sqrt{1}} = 0.80252 \]

The stock price tree and the put price tree are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>73.1142</td>
<td>V 0.0000</td>
</tr>
<tr>
<td>50.0000</td>
<td>V 7.8741</td>
</tr>
<tr>
<td>40.1259</td>
<td>V 7.8741</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.08 - 0.00)1} - 0.80252}{1.46228 - 0.80252} = 0.42556 \]

The value of the put option is:

\[ V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.08} \left[ 0.42556(0.0000) + (1 - 0.42556)(7.8741) \right] = 4.1754 \]
Since we know the price of the option and the distribution of its payoffs, we can now found the expected return of the option:

\[
V = e^{-\gamma h} \left[ (p) V_u + (1-p) V_d \right]
\]

\[
4.1754 = e^{-\gamma} \left[ 0.46 \times 0.0000 + (1-0.46) \times 7.8741 \right]
\]

\[
\gamma = -\ln \left( \frac{4.1754}{(1-0.46) \times 7.8741} \right) = 0.01817
\]

**Solution 10**

**A Chapter 11, Expected Return**

The values of \(u\) and \(d\) are:

\[
u = e^{(r-\delta)h + \sigma \sqrt{h}} = e^{0.08-0.04(1) + 0.24 \sqrt{1}} = 1.32313
\]

\[
d = e^{(r-\delta)h - \sigma \sqrt{h}} = e^{0.08-0.04(1) - 0.24 \sqrt{1}} = 0.81873
\]

The stock price tree and the put price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>82.0340</td>
<td>0.0000</td>
</tr>
<tr>
<td>62.0000</td>
<td>(V)</td>
</tr>
<tr>
<td>50.7613</td>
<td>13.2387</td>
</tr>
</tbody>
</table>

The realistic probability of an upward movement is:

\[
p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{0.12-0.04(1) - 0.81873}}{1.32313 - 0.81873} = 0.52450
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.08-0.04(1) - 0.81873}}{1.32313 - 0.81873} = 0.44029
\]

The value of the put option can be calculated using the risk-neutral probability:

\[
V = e^{-\gamma h} \left[ (p^*) V_u + (1-p^*) V_d \right] = e^{-0.08(1)} \left[ 0.44029(0.0000) + (1-0.44029)(13.2387) \right] = 6.8402
\]

Since we know the price of the option and the distribution of its payoffs, we can now found the expected return of the option:

\[
V = e^{-\gamma h} \left[ (p) V_u + (1-p) V_d \right]
\]

\[
6.8402 = e^{-\gamma} \left[ 0.52450 \times 0.0000 + (1-0.52450) \times 13.2387 \right]
\]

\[
\gamma = -\ln \left( \frac{6.8402}{(1-0.52450) \times 13.2387} \right) = -0.08305
\]
Solution 11

B Chapter 11, Realistic Probability

The values of $u$ and $d$ are:

\[
\begin{align*}
  u &= e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.07 - 0.02)(1) + 0.27\sqrt{1}} = 1.37713 \\
  d &= e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.07 - 0.02)(1) - 0.27\sqrt{1}} = 0.80252
\end{align*}
\]

The stock price tree and the call price tree are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>52.3309</td>
<td>12.3309</td>
</tr>
<tr>
<td>38.0000</td>
<td>V</td>
</tr>
<tr>
<td>30.4957</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.07 - 0.02)(1) - 0.80252}}{1.37713 - 0.80252} = 0.43291
\]

The value of the call option is:

\[
V = e^{-rh} \left[(p^*)V_u + (1 - p^*)V_d\right] = e^{-0.07(1)} \left[0.43291(12.3309) + (1 - 0.43291)(0.0000)\right] = 4.9772
\]

Since we know the price of the option, its possible payoffs, and its expected rate of return, we can find the realistic probability of an upward movement:

\[
V = e^{-rh} \left[(p)V_u + (1 - p)V_d\right] \\
4.9772 = e^{-0.34836(1)} \left[p \times 12.3309 + (1 - p) \times 0.0000\right] \\
p = \frac{4.9772 \times e^{0.34836}}{12.3309} = 0.57185
\]

The realistic probability that the stock price moves down is:

\[1 - 0.57185 = 0.42815\]

Solution 12

C Chapter 11, Expected Return

The values of $u$ and $d$ are:

\[
\begin{align*}
  u &= e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.07 - 0.02)(1) + 0.27\sqrt{1}} = 1.37713 \\
  d &= e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.07 - 0.02)(1) - 0.27\sqrt{1}} = 0.80252
\end{align*}
\]
The stock price tree and the call price tree are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>38.0000</td>
<td>V</td>
</tr>
<tr>
<td>30.4957</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{0.07 - 0.02}(1) - 0.80252}{1.37713 - 0.80252} = 0.43291 \]

The value of the call option is:

\[ V = e^{-rh} [(p^*)V_u + (1 - p^*)V_d] = e^{-0.07(1)} [0.43291(12.3309) + (1 - 0.43291)(0.0000)] = 4.9772 \]

Since we know the price of the option, its possible payoffs, and its expected rate of return, we can find the realistic probability of an upward movement:

\[ V = e^{-rh} [(p)V_u + (1 - p)V_d] = 4.9772 = e^{-0.34836(1)} [p \times 12.3309 + (1 - p) \times 0.0000] \]

\[ p = \frac{4.9772 \times e^{0.34836}}{12.3309} = 0.57185 \]

We can now use the realistic probability to find the expected return on the stock:

\[ p = \frac{e^{(\alpha - \delta)h} - d}{u - d} = \frac{e^{(\alpha - 0.02) \times 1} - 0.80252}{1.37713 - 0.80252} \]

\[ \alpha = 0.1432 \]

**Solution 13**

**B Chapter 11, Expected Return**

The realistic probability of an up jump is constant throughout the tree. The expected return on the stock is also constant throughout the tree. The return on the option, however, can change from one node to the next.

The values of \( u \) and \( d \) are:

\[ u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.06 - 0.05)(0.5) + 0.30 \sqrt{0.5}} = 1.24251 \]

\[ d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.06 - 0.05)(0.5) - 0.30 \sqrt{0.5}} = 0.81291 \]
The risk-neutral probability of an upward movement is:
\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.06-0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.44716
\]

The stock price tree and the call price tree are shown below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>European Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>77.1913</td>
<td>30.1913</td>
</tr>
<tr>
<td>62.1254</td>
<td>14.9806</td>
</tr>
<tr>
<td>50.0000</td>
<td>50.5025</td>
</tr>
<tr>
<td>40.6456</td>
<td>7.3162</td>
</tr>
<tr>
<td>33.0413</td>
<td>1.5199</td>
</tr>
</tbody>
</table>

Although the entire tree is shown above, it is not necessary to calculate the values along the lowest path in order to answer this problem.

The realistic probability of an up jump is:
\[
p = \frac{e^{(\alpha-\delta)h} - d}{u - d} = \frac{e^{(0.10-0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.49442
\]

If the stock price increases during the first 6 months, then the expected rate of return during the second 6 months must satisfy the following equation:
\[
\gamma = \ln\left(\frac{0.49442 \times 30.1913 + (1 - 0.49442) \times 3.5025}{14.9806}\right) \times \frac{1}{0.5} = 0.21708
\]

Solution 14

C Chapter 11, Expected Return

The values of \( u \) and \( d \) are:
\[
\begin{align*}
  u &= e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.06-0.05)(0.5)+0.30\sqrt{0.5}} = 1.24251 \\
  d &= e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.06-0.05)(0.5)-0.30\sqrt{0.5}} = 0.81291
\end{align*}
\]

The risk-neutral probability of an upward movement is:
\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.06-0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.44716
\]
The stock price tree and the call price tree are shown below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>American Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>77.1913</td>
<td>0.0000</td>
</tr>
<tr>
<td>62.1254</td>
<td>0.0000</td>
</tr>
<tr>
<td>50.0000</td>
<td>4.0177</td>
</tr>
<tr>
<td>40.6456</td>
<td>7.4888</td>
</tr>
<tr>
<td>33.0413</td>
<td>13.9587</td>
</tr>
</tbody>
</table>

Although the entire tree is shown above, it is not necessary to calculate the values along the uppermost path in order to answer this problem.

Early exercise of the American put option is never optimal.

The realistic probability of an up-move is:

\[
p = \frac{e^{(\alpha - \delta)h} - d}{u - d} = \frac{e^{(0.10 - 0.05)(0.5)} - 0.81291}{1.24251 - 0.81291} = 0.49442
\]

If the stock price decreases during the first 6 months, then the expected rate of return during the second 6 months must satisfy the following equation:

\[
V = e^{-\gamma h} \left[ (p)V_u + (1 - p)V_d \right]
\]

\[
7.4888 = e^{-0.5\gamma} \left[ 0.49442 \times 0.0000 + (1 - 0.49442) \times 13.9587 \right]
\]

\[
\gamma = \ln \left( \frac{(1 - 0.49442) \times 13.9587}{7.4888} \right) \times \frac{1}{0.5} = -0.1187
\]

**Solution 15**

E Chapter 11, Expected Return

The values of \(u\) and \(d\) are:

\[
u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.11 - 0.10)(1.0) + 0.40\sqrt{1}} = 1.50682
\]

\[
d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.11 - 0.10)(1.0) - 0.40\sqrt{1}} = 0.67706
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.11 - 0.10)(1.0) - 0.67706}}{1.50682 - 0.67706} = 0.40131
\]
The stock price tree and the call price tree are shown below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>113.5250</th>
<th>75.3409</th>
<th>50.0000</th>
<th>33.8528</th>
<th>22.9203</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Call</td>
<td>73.5250</td>
<td>35.3409</td>
<td>14.8283</td>
<td>3.9582</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

If the stock moves up during the first year, then it is optimal to exercise the American option early. This is indicated above by the bold type for the option price of $35.3409.

The realistic probability of an up-move is:

$$p = \frac{e^{(\alpha - \delta)h - d}}{u - d} = \frac{e^{(0.24 - 0.10)(1.0) - 0.67706}}{1.50682 - 0.67706} = 0.57031$$

The expected return on the call option can be calculated using the realistic probability:

$$V = e^{-\gamma h} \left[ (p)V_u + (1 - p)V_d \right]$$

$$14.8283 = e^{-\gamma(1)} \left[ 0.57031 \times 35.3409 + (1 - 0.57031) \times 3.9582 \right]$$

$$\gamma = \ln \left( \frac{0.57031 \times 35.3409 + (1 - 0.57031) \times 3.9582}{14.8283} \right) = 0.3879$$

**Solution 16**

**D** Chapter 11, Expected Return

The magnitude of $\gamma$ decreases as the stock price increases, so we only need to check the top nodes to answer this question.

The risk-neutral probability of an upward movement is constant throughout the tree for the standard binomial model. Therefore we can calculate it using any node. Below we use the first node:

$$p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.08 - 0.00)(1/3) - 54.4126}}{63.000 - 54.4126} = 0.45681$$

Working from right to left, we create the tree of prices for the American call option:

<table>
<thead>
<tr>
<th>Stock</th>
<th>114.7493</th>
<th>93.9609</th>
<th>76.9385</th>
<th>63.0000</th>
<th>54.4126</th>
<th>46.9958</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Call</td>
<td>55.7493</td>
<td>36.5134</td>
<td>21.4521</td>
<td>9.8535</td>
<td>4.3827</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

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Since the stock does not pay dividends, it is never optimal to exercise the call early.

The realistic probability is also constant throughout the tree, so as with the risk-neutral probability, we can use the values at the first node to calculate it:

\[ p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.15-0.00)(1/3)} - \frac{54.126}{63.000}}{63.000 - 63.000} = 0.52462 \]

Since \( \gamma \) decreases as the stock price increases, we need only check the uppermost branch of the tree of prices to find the lowest value of \( \gamma \).

When the stock price is $63.00, we have:

\[
e^{\gamma h} = \frac{[(p)V_u + (1-p)V_d]}{V} = \frac{(0.52462)21.4521 + (1-0.52462)(4.3827)}{11.8596} = 21.4521
\]

\[
\gamma = \ln\left(\frac{(0.52462)21.4521 + (1-0.52462)(4.3827)}{11.8596}\right) \times 3 = 0.352
\]

When the stock price is $76.9385, we have:

\[
e^{\gamma h} = \frac{[(p)V_u + (1-p)V_d]}{V} = \frac{(0.52462)36.5134 + (1-0.52462)(9.8535)}{21.4521} = 21.4521
\]

\[
\gamma = \ln\left(\frac{(0.52462)36.5134 + (1-0.52462)(9.8535)}{21.4521}\right) \times 3 = 0.317
\]

When the stock price is $93.9609, we have:

\[
e^{\gamma h} = \frac{[(p)V_u + (1-p)V_d]}{V} = \frac{(0.52462)55.7493 + (1-0.52462)(22.1533)}{36.5134} = 36.5134
\]

\[
\gamma = \ln\left(\frac{(0.52462)55.7493 + (1-0.52462)(22.1533)}{36.5134}\right) \times 3 = 0.257
\]

The lowest value of \( \gamma \) occurs when the stock price is $93.9609.
Solution 17

D Chapter 11, Expected Return

The risk-neutral probability of an upward movement is constant throughout the tree for the standard binomial model. Therefore we can calculate it using any node. Below we use the first node:

\[ p^* = \frac{e^{(r-\delta)h - d}}{u-d} = \frac{e^{(0.08-0.00)(1/3)} - 54.4126}{76.9385 - 54.4126} = \frac{63.000}{54.4126} = 0.45681 \]

Working from right to left, we create the tree of prices for the European put option:

<table>
<thead>
<tr>
<th>Stock</th>
<th>114.7493</th>
<th>European Put</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>93.9609</td>
<td>114.7493</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>76.9385</td>
<td>81.1533</td>
<td>3.3235</td>
<td>0.8497</td>
</tr>
<tr>
<td>63.0000</td>
<td>54.4126</td>
<td>5.9058</td>
<td>1.6066</td>
</tr>
<tr>
<td>46.9958</td>
<td>40.5899</td>
<td>10.4517</td>
<td>18.4101</td>
</tr>
</tbody>
</table>

The realistic probability is also constant throughout the tree, so as with the risk-neutral probability, we can use the values at the first node to calculate it:

\[ p = \frac{e^{(\alpha-\delta)h - d}}{u-d} = \frac{e^{(0.15-0.00)(1/3)} - 54.4126}{76.9385 - 54.4126} = \frac{63.000}{54.4126} = 0.52462 \]

Since \( \gamma \) decreases as the stock price increases, we need only check the uppermost branch of the tree of prices to find the lowest value of \( \gamma \).

When the stock price is $63.00, we have:

\[ e^{\gamma h} = \frac{[(p)V_u + (1-p)V_d]}{V} \]

\[ e^{\gamma(1/3)} = \frac{(0.52462)(0.4494 + (1-0.52462)(5.9058))}{3.3235} \]

\[ \gamma = \ln\left(\frac{(0.52462)(0.4494 + (1-0.52462)(5.9058))}{3.3235}\right) \times 3 = -0.264 \]
When the stock price is $76.9385, we have:
\[
e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d \right]}{V} \\
e^{\gamma (1/3)} = \frac{(0.52462)(0.0000) + (1 - 0.52462)(0.8497)}{0.4494} \\
\gamma = \ln\left(\frac{(0.52462)(0.0000) + (1 - 0.52462)(0.8497)}{0.4494}\right) \times 3 = -0.320
\]

When the stock price is $54.4126, we have:
\[
e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d \right]}{V} \\
e^{\gamma (1/3)} = \frac{(0.52462)(0.8497) + (1 - 0.52462)(10.4517)}{5.9058} \\
\gamma = \ln\left(\frac{(0.52462)(0.8497) + (1 - 0.52462)(10.4517)}{5.9058}\right) \times 3 = -0.261
\]

When the stock price is $66.4512 we have:
\[
e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d \right]}{V} \\
e^{\gamma (1/3)} = \frac{(0.52462)(0.0000) + (1 - 0.52462)(1.6066)}{0.8497} \\
\gamma = \ln\left(\frac{(0.52462)(0.0000) + (1 - 0.52462)(1.6066)}{0.8497}\right) \times 3 = -0.320
\]

When the stock price is $46.9958 we have:
\[
e^{\gamma h} = \frac{\left[(p)V_u + (1-p)V_d \right]}{V} \\
e^{\gamma (1/3)} = \frac{(0.52462)(1.6066) + (1 - 0.52462)(18.4101)}{10.4517} \\
\gamma = \ln\left(\frac{(0.52462)(1.6066) + (1 - 0.52462)(18.4101)}{10.4517}\right) \times 3 = -0.257
\]

The tree of values for $\gamma$ is:

|       | N/A | -0.320 | -0.264 | -0.261 | -0.257 |

The lowest value is -32.0%.
It might seem surprising that $\gamma$ is lowest at the upper nodes. After all, the put option can be viewed as being more risky as it becomes more out-of-the-money. So as the stock price increases, shouldn’t the expected return increase as well? The answer is no, because the put option is a hedging instrument. Therefore, its "risk" is inversely related to the risk of the stock price declining. That’s why put options earn less than the risk-free rate of return. Armed with this knowledge, we see that the lowest value of $\gamma$ is sure to be in one of the upper nodes. Therefore, we could have solved this problem by calculating the value of $\gamma$ at only the three uppermost nodes.

The general rule for both calls and puts is that the lowest value of $\gamma$ for each column in a tree is at the highest node.

Solution 18

C Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of $u$ and $d$ are:

$$u = e^{\sigma \sqrt{h}} = e^{0.30 \sqrt{1}} = 1.34986$$
$$d = e^{-\sigma \sqrt{h}} = e^{-0.30 \sqrt{1}} = 0.74082$$

The possible stock prices at the end of one year are:

$$38u = 38 \times 1.34986 = 51.2946$$
$$38d = 38 \times 0.74082 = 28.1511$$

The payoffs for the call option in 1 year are:

Up state: $\text{Max}[0, 51.2946 - 40] = 11.2946$
Down state: $\text{Max}[0, 28.1511 - 40] = 0.0000$

The stock price tree and the call price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>51.2946</td>
<td>11.2946</td>
</tr>
<tr>
<td>38.0000</td>
<td></td>
</tr>
<tr>
<td>28.1511</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.07-0.00)(1)} - 0.74082}{1.34986 - 0.74082} = 0.54461$$

The value of the call option is:

$$V = e^{-rh} [p^*V_u + (1 - p^*)V_d] = e^{-0.07(1)} [0.54461(11.2946) + (1 - 0.54461)(0.0000)]$$
$$= 5.735$$
Solution 19
A  Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of $u$ and $d$ are:

$$u = e^{\sigma \sqrt{h}} = e^{0.30 \sqrt{1}} = 1.34986$$
$$d = e^{-\sigma \sqrt{h}} = e^{-0.30 \sqrt{1}} = 0.74082$$

We can now solve for the expected return on the stock:

$$p = \frac{e^{(\alpha - \delta)h} - d}{u - d}$$
$$0.661 = \frac{e^{(\alpha - 0) \times 1} - 0.74082}{1.34986 - 0.74082}$$
$$\alpha = 0.1340$$

Solution 20
B  Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of $u$ and $d$ are:

$$u = e^{\sigma \sqrt{h}} = e^{0.30 \sqrt{1}} = 1.34986$$
$$d = e^{-\sigma \sqrt{h}} = e^{-0.30 \sqrt{1}} = 0.74082$$

The possible stock prices at the end of one year are:

$$38u = 38 \times 1.34986 = 51.2946$$
$$38d = 38 \times 0.74082 = 28.1511$$

The payoffs for the put option in 1 year are:

Up state: $Max[0, 40 - 51.2946] = 0.0000$
Down state: $Max[0, 40 - 28.1511] = 11.8489$

The stock price tree and the put price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>51.2946</td>
<td>0.0000</td>
</tr>
<tr>
<td>38.0000</td>
<td>V</td>
</tr>
<tr>
<td>28.1511</td>
<td>11.8489</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.07-0.00) \times 1} - 0.74082}{1.34986 - 0.74082} = 0.54461$$
The value of the put option is:

\[
V = e^{-rh} \left[ (p^*) V_u + (1 - p^*) V_d \right] = e^{-0.07(1)} \left[ 0.54461(0.0000) + (1 - 0.54461)(11.8489) \right] = 5.031
\]

**Solution 21**

**A**  Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of \(u\) and \(d\) are:

\[
u = e^{\sigma \sqrt{h}} = e^{0.30 \sqrt{0.25}} = 1.16183
\]

\[
d = e^{-\sigma \sqrt{h}} = e^{-0.30 \sqrt{0.25}} = 0.86071
\]

The stock price tree and the call price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>116.1834</td>
<td>21.1834</td>
</tr>
<tr>
<td>100.0000</td>
<td>10.1239</td>
</tr>
<tr>
<td>86.0708</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.08 - 0.05)(0.25)} - 0.86071}{1.16183 - 0.86071} = 0.48757
\]

The value of the call option is:

\[
V = e^{-rh} \left[ (p^*) V_u + (1 - p^*) V_d \right] = e^{-0.08(0.25)} \left[ 0.48757(21.1834) + (1 - 0.48757)(0.00) \right] = 10.124
\]

**Solution 22**

**B**  Chapter 11, Cox-Ross-Rubinstein Binomial Tree

The values of \(u\) and \(d\) are:

\[
u = e^{\sigma \sqrt{h}} = e^{0.30 \sqrt{0.25}} = 1.16183
\]

\[
d = e^{-\sigma \sqrt{h}} = e^{-0.30 \sqrt{0.25}} = 0.86071
\]

The stock price tree and the call price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>116.1834</td>
<td>21.1834</td>
</tr>
<tr>
<td>100.0000</td>
<td>10.1239</td>
</tr>
<tr>
<td>86.0708</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.08-0.05)(0.25)} - 0.86071}{1.16183 - 0.86071} = 0.48757 \]

The value of the call option is:

\[ V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.08(0.25)} \left[ 0.48757(21.1834) + (1 - 0.48757)(0.00) \right] \]
\[ p = 10.1239 \]

The realistic probability can now be determined:

\[ V = e^{-rh} \left[ (p)V_u + (1 - p)V_d \right] \]
\[ 10.1239 = e^{-0.2153(0.25)} \left[ (p)(21.1834) + (1 - p)(0.0000) \right] \]
\[ p = \frac{10.1239e^{0.2153(0.25)}}{21.1834} \]
\[ p = 0.50434 \]

Now that we have the realistic probability, we can find the expected return for the stock:

\[ S = e^{-\alpha h} \left[ (p)Sue^{\delta h} + (1 - p)Sde^{\delta h} \right] \]
\[ 100 = e^{-\alpha(0.25)} \left[ (0.50434)(116.1834)e^{0.05(0.25)} + (1 - 0.50434)(86.0708)e^{0.05(0.25)} \right] \]
\[ \alpha = \ln \left( \frac{(0.50434)(116.1834)e^{0.05(0.25)} + (1 - 0.50434)(86.0708)e^{0.05(0.25)}}{100} \right) \times \frac{1}{0.25} \]
\[ = 0.100 \]

Solution 23

D Chapter 11, Jarrow and Rudd Binomial Tree

The values of \( u \) and \( d \) are:

\[ u = e^{(r - \delta - 0.5\sigma^2)h + \sigma \sqrt{h}} = e^{(0.07-0.00-0.5\times0.3^2)(1)+0.3\sqrt{1}} = 1.38403 \]
\[ d = e^{(r - \delta - 0.5\sigma^2)h - \sigma \sqrt{h}} = e^{(0.07-0.00-0.5\times0.3^2)(1)-0.3\sqrt{1}} = 0.75957 \]

The possible stock prices at the end of one year are:

\[ 38u = 38 \times 1.38403 = 52.5932 \]
\[ 38d = 38 \times 0.75957 = 28.8637 \]

The payoffs for the call option in 1 year are:

Up state: \( Max[0, 52.5932 - 40] = 12.5932 \)
Down state: \( Max[0, 28.8637 - 40] = 0.0000 \)
The stock price tree and the call price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>52.5932</td>
<td>12.5932</td>
</tr>
<tr>
<td>38.0000</td>
<td>5.8842</td>
</tr>
<tr>
<td>28.8637</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.07 - 0.00)(1) - 0.75957}}{1.38403 - 0.75957} = 0.50113 \]

The value of the call option is:

\[ V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.07(1)} [0.50113(12.5932) + (1 - 0.50113)(0.0000)] \]
\[ = 5.884 \]

**Solution 24**

C Chapter 11, Jarrow and Rudd Binomial Tree

The values of \( u \) and \( d \) are:

\[ u = e^{(r - \delta - 0.5\sigma^2)h + \sigma \sqrt{h}} = e^{0.07 - 0.00 - 0.5 \times 0.30^2(1) + 0.3 \sqrt{1}} = 1.38403 \]
\[ d = e^{(r - \delta - 0.5\sigma^2)h - \sigma \sqrt{h}} = e^{0.07 - 0.00 - 0.5 \times 0.30^2(1) - 0.3 \sqrt{1}} = 0.75957 \]

The possible stock prices at the end of one year are:

\[ 38u = 38 \times 1.38403 = 52.5932 \]
\[ 38d = 38 \times 0.75957 = 28.8637 \]

The stock price tree and the put price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>52.5932</td>
<td>0.0000</td>
</tr>
<tr>
<td>38.0000</td>
<td>5.1799</td>
</tr>
<tr>
<td>28.8637</td>
<td>11.1363</td>
</tr>
</tbody>
</table>

The risk-neutral probability of an upward movement is:

\[ p^* = \frac{e^{(r - \delta)h - d}}{u - d} = \frac{e^{(0.07 - 0.00)(1) - 0.75957}}{1.38403 - 0.75957} = 0.50113 \]

The value of the put option is:

\[ V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.07(1)} [0.50113(0.0000) + (1 - 0.50113)(11.1363)] \]
\[ = 5.180 \]
Solution 25

C Chapter 11, Jarrow and Rudd Binomial Tree

The values of \( u \) and \( d \) are:

\[
\begin{align*}
u &= e^{(r - \delta - 0.5\sigma^2)h + \sigma\sqrt{h}} = e^{(0.08 - 0.05 - 0.5 \times 30^2)(0.25) + 0.3 \sqrt{0.25}} = 1.15749 \\
d &= e^{(r - \delta - 0.5\sigma^2)h - \sigma\sqrt{h}} = e^{(0.08 - 0.05 - 0.5 \times 30^2)(0.25) - 0.3 \sqrt{0.25}} = 0.85749
\end{align*}
\]

The stock price tree and the call price tree are below:

\[
\begin{array}{c|c}
\text{Stock} & \text{Call} \\
115.7486 & 20.7486 \\
100.0000 & 10.1717 \\
85.7486 & 0.0000
\end{array}
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.08 - 0.05)(0.25)} - 0.85749}{1.15749 - 0.85749} = 0.50014
\]

The value of the call option is:

\[
V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.08(0.25)} \left[ 0.50014(20.7486) + (1 - 0.50014)(0.0000) \right] = 10.172
\]

Solution 26

D Chapter 11, Jarrow and Rudd Binomial Tree

The values of \( u \) and \( d \) are:

\[
\begin{align*}
u &= e^{(r - \delta - 0.5\sigma^2)h + \sigma\sqrt{h}} = e^{(0.08 - 0.05 - 0.5 \times 30^2)(0.25) + 0.3 \sqrt{0.25}} = 1.15749 \\
d &= e^{(r - \delta - 0.5\sigma^2)h - \sigma\sqrt{h}} = e^{(0.08 - 0.05 - 0.5 \times 30^2)(0.25) - 0.3 \sqrt{0.25}} = 0.85749
\end{align*}
\]

The stock price tree and the call price tree are below:

\[
\begin{array}{c|c}
\text{Stock} & \text{Call} \\
115.7486 & 20.7486 \\
100.0000 & 10.1717 \\
85.7486 & 0.0000
\end{array}
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.08 - 0.05)(0.25)} - 0.85749}{1.15749 - 0.85749} = 0.50014
\]

The value of the call option is:

\[
V = e^{-rh} \left[ (p^*)V_u + (1 - p^*)V_d \right] = e^{-0.07(1)} \left[ 0.50014(20.7486) + (1 - 0.50014)(0.0000) \right] = 10.1717
\]
The realistic probability can now be determined:

\[ V = e^{-\gamma h} [(p)V_u + (1-p)V_d] \]

\[ 10.1717 = e^{-(0.5281)(0.25)} [(p)(20.7486) + (1-p)(0.0000)] \]

\[ p = \frac{10.1717 e^{(0.5281)(0.25)}}{20.7486} \]

\[ p = 0.55943 \]

Now that we have the realistic probability, we can find the expected return for the stock:

\[ p = \frac{e^{(\alpha - \delta)h} - d}{u - d} \]

\[ 0.55943 = \frac{e^{(0.05 - 0.25) \times 0.25} - 0.85749}{1.15749 - 0.85749} \]

\[ \alpha = 0.150 \]

Solution 27

A Chapter 11, Jarrow and Rudd Binomial Tree

The values of \( u \) and \( d \) are:

\[ u = e^{(r - \delta - 0.5\sigma^2)h + \sigma\sqrt{h}} = e^{(0.12 - 0.045 - 0.5 \times 0.27^2)(0.5) + 0.27\sqrt{0.5}} = 1.23392 \]

\[ d = e^{(r - \delta - 0.5\sigma^2)h - \sigma\sqrt{h}} = e^{(0.12 - 0.045 - 0.5 \times 0.27^2)(0.5) - 0.27\sqrt{0.5}} = 0.84228 \]

The possible stock prices at the end of one year are:

\[ 38u = 53 \times 1.23392 = 65.3976 \]

\[ 38d = 53 \times 0.84228 = 44.6408 \]

The stock price tree and the call price tree are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>65.3976</td>
<td>5.3976</td>
</tr>
<tr>
<td>53.0000</td>
<td>2.5431</td>
</tr>
<tr>
<td>44.6408</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The value of delta is:

\[ \Delta = e^{-\delta h} \frac{V_u - V_d}{S(u - d)} = e^{-0.045(0.5)} \frac{5.3976 - 0.0000}{65.3976 - 44.6408} = 0.254 \]
Solution 28

A  Chapter 11, Jarrow and Rudd Binomial Tree

The values of \(u\) and \(d\) are:

\[
u = e^{(r - \delta - 0.5\sigma^2)h + \sigma \sqrt{h}} = e^{(0.05 - 0.01 - 0.5 \times 0.35^2)(0.25) + 0.35 \sqrt{0.25}} = 1.18493
\]

\[
d = e^{(r - \delta - 0.5\sigma^2)h - \sigma \sqrt{h}} = e^{(0.05 - 0.01 - 0.5 \times 0.35^2)(0.25) - 0.35 \sqrt{0.25}} = 0.83501
\]

The stock price tree, the put payoffs and the risk-neutral probability of each payoff are below:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Put Payoff</th>
<th># of Paths</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.5706</td>
<td>0.0000</td>
<td>1</td>
<td>0.06261</td>
</tr>
<tr>
<td>83.1865</td>
<td>69.4615</td>
<td>4</td>
<td>0.25022</td>
</tr>
<tr>
<td>70.2035</td>
<td>58.6206</td>
<td>4</td>
<td>0.25022</td>
</tr>
<tr>
<td>59.2467</td>
<td>48.9487</td>
<td>4</td>
<td>0.24978</td>
</tr>
<tr>
<td>50.0000</td>
<td>41.3092</td>
<td>4</td>
<td>0.24978</td>
</tr>
<tr>
<td>41.7505</td>
<td>34.4936</td>
<td>1</td>
<td>0.06239</td>
</tr>
<tr>
<td>34.8620</td>
<td>29.1101</td>
<td>1</td>
<td>0.06239</td>
</tr>
<tr>
<td>29.1101</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Although we show the entire table above, there is no need to calculate the stock prices higher than $34.4936 to answer this question. At those prices, the option is clearly out-of-the-money.

Only one of the final stock prices is below the strike price of $34. The payoff if the lowest stock price is reached is:

\[
34 - 24.3072 = 9.6928
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.05 - 0.01)(0.25)} - 0.83501}{1.18493 - 0.83501} = 0.50022
\]

The probability of 4 downward movements is:

\[
(1 - p^*)^4 = (1 - 0.50022)^4 = 0.06239
\]

Since the option is a European option, we can find its value directly as the present value of its expected value:

\[
V(S_0, K, 0) = e^{-r(hn)} \sum_{i=0}^{n} \left( \binom{n}{i} p^*(n-i) (1-p^*)^i V(S_0 u^{n-i} d^i, K, hn) \right)
\]

\[
= e^{-0.05} \times 0.06239 \times 9.6928 = 0.5752
\]
Solution 29

D  Chapter 11, J-R and CRR Binomial Trees

In the J-R model, the values of $u$ and $d$ are:

$$u = e^{(r-\delta - 0.5\sigma^2)h + \sigma\sqrt{h}} = e^{(0.07-0.025-0.5\times0.30^2)(0.25)+0.30\sqrt{0.25}} = e^{0+0.30\sqrt{0.25}}$$
$$d = e^{(r-\delta - 0.5\sigma^2)h - \sigma\sqrt{h}} = e^{(0.07-0.025-0.5\times0.30^2)(0.25)-0.30\sqrt{0.25}} = e^{0-0.30\sqrt{0.25}}$$

In the CRR model, the values of $u$ and $d$ are:

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{0.25}}$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{0.25}}$$

Since this J-R model’s up and down factors are the same as this CRR model’s up and down factors, Ann and Betsy are using the same binomial model. Therefore, the prices they produce satisfy put-call parity:

$$C_{Eur} + Ke^{-rT} = S_0e^{-\delta T} + P_{Eur}$$
$$A + 50e^{-0.07\times1} = 50e^{-0.025\times1} + B$$
$$A - B = 2.1458$$

Solution 30

B  Chapter 11, Alternative Binomial Trees

If the option has a positive payoff in both the up and down states, then:

$$\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u-d)} = e^{-0.11\times0.25} \frac{(38 - S_u) - (38 - S_d)}{S_u - S_d} = 0.9729 \frac{S_d - S_u}{S_u - S_d} = -0.9729$$

But $\Delta = -0.262$, so it must be the case that $V_u = 0$.

Since we are given the values of $\Delta$ and $B$, we can solve for $u$ and $d$:

$$-0.262 = \Delta = e^{-\delta h} \frac{V_u - V_d}{S(u-d)} = e^{-0.11\times0.25} \frac{0 - (38 - 40d)}{40u - 40d} = 0.9729 \frac{40d - 38}{40(u-d)}$$
$$11.76 = B = e^{-rh} \frac{uV_d - dV_u}{u-d} = e^{-0.15\times0.25} \frac{u(38 - 40d) - d \times 0}{u-d} = 0.9632 \frac{u(38 - 40d)}{u-d}$$

Dividing the first equation by the second allows us to solve for $u$:

$$\frac{-0.262}{11.76} = \frac{0.9729 \frac{40d - 38}{40(u-d)}}{0.9632 \frac{u(38 - 40d)}{u-d}}$$

$$-0.02228 = 0.02525 \times \frac{1}{u}$$
$$u = 1.1334$$
Now we can find the value of $d$:

$$-0.262 = 0.9729 \frac{40d - 38}{40(u - d)}$$

$$-0.262 = 0.9729 \frac{40d - 38}{40(1.1334 - d)}$$

$$-12.2094 + 10.7722d = 40d - 38$$

$$d = 0.8824$$

Both the Cox-Ross-Rubinstein model and the Jarrow-Rudd model have the same ratio of $u$ to $d$:

$$\frac{u}{d} = e^{2\sigma \sqrt{t}}$$

$$\frac{1.1334}{0.8824} = e^{2\sigma \sqrt{0.25}}$$

$$\sigma = 0.2503$$

**Solution 31**

**C** Chapter 11, Utility Values and State Prices

The payoffs of the call option are:

$$V_u = Max(0, 200 - 130) = 70$$

$$V_d = Max(0, 50 - 130) = 0$$

The price of the call option is:

$$V = Q_u V_u + Q_d V_d = p U_u \times 70 + (1 - p) U_d \times 0 = 0.55(0.9) \times 70 + 0.45(1.03) \times 0 = 34.65$$

**Solution 32**

**E** Chapter 11, Utility Values and State Prices

The payoffs of the call option are:

$$V_u = Max(0, 200 - 130) = 70$$

$$V_d = Max(0, 50 - 130) = 0$$

The price of the call option is:

$$V = Q_u V_u + Q_d V_d = p U_u \times 70 + (1 - p) U_d \times 0 = 0.55(0.9) \times 70 + 0.45(1.03) \times 0 = 34.65$$

The expected return on the call option can now be calculated:

$$\left(1 + \gamma_{Call}\right)^{0.5} = \frac{p V_u + (1 - p) V_d}{V}$$

$$\left(1 + \gamma_{Call}\right)^{0.5} = \frac{0.55 \times 70 + 0.45 \times 0}{34.65}$$

$$\gamma_{Call} = 0.2346$$
The payoffs of the put option are:

\[ V_u = \text{Max}(0, 100 - 200) = 0 \]
\[ V_d = \text{Max}(0, 100 - 50) = 50 \]

The price of the put option is:

\[ V = Q_u V_u + Q_d V_d = p U_u \times 0 + (1 - p) U_d \times 50 = 0.55(0.9) \times 0 + 0.45(1.03) \times 50 \]
\[ = 23.175 \]

The expected return on the put option can now be calculated:

\[ (1 + \gamma_{Put})^{0.5} = \frac{pV_u + (1 - p)V_d}{V} \]
\[ (1 + \gamma_{Put})^{0.5} = \frac{0.55 \times 0 + 0.45 \times 50}{23.175} \]
\[ \gamma_{Put} = -0.0574 \]

The difference in the expected returns is:

\[ \gamma_{Call} - \gamma_{Put} = 0.2346 - (-0.0574) = 0.2920 \]

Solution 33

E Chapter 11, Utility Values and State Prices

The price of the stock is:

\[ S = Q_u S_u e^{\delta h} + Q_d S_d e^{\delta h} = p U_u S_u e^{\delta h} + (1 - p) U_d S_d e^{\delta h} \]
\[ = 0.55 \times 0.87 \times 200 e^{0.06 \times 1} + 0.45 \times 0.98 \times 50 e^{0.06 \times 1} = 125.0313 \]

The expected return can now be calculated:

\[ (1 + \alpha)^h = \frac{p S_u e^{\delta h} + (1 - p) S_d e^{\delta h}}{S} \]
\[ (1 + \alpha)^1 = \frac{0.55 \times 200 e^{0.06 \times 1} + 0.45 \times 50 e^{0.06 \times 1}}{125.0313} \]
\[ \alpha = 0.1253 \]

Solution 34

D Chapter 11, Utility Values and State Prices

Although we called the two states "up" and "down" in the Review Notes, any consistent labeling system is valid. In the formulas, we can replace references to the up state with references to Scenario 1, and we can replace references to the down state with references to Scenario 2.
The price of the stock is:

\[ S = e^{-\alpha h} \left[ pS_1e^{\delta h} + (1 - p)S_2e^{\delta h} \right] = e^{-0.07 \times 1} \left[ 0.3 \times 50e^{0 \times 1} + 0.7 \times 100e^{0 \times 1} \right] \]

\[ = 79.2535 \]

We can use Stock X and the risk-free asset to obtain two equations and two unknowns:

\[ 79.2535 = 50Q_1 + 100Q_2 \]
\[ 0.92 = Q_1 + Q_2 \]

Multiplying the bottom equation by 50 and subtracting it from the top equation, we have:

\[ 79.2535 - 50 \times 0.92 = 50Q_1 - 50Q_1 + 100Q_2 - 50Q_2 \]
\[ 33.2535 = 50Q_2 \]
\[ Q_2 = 0.6651 \]

Solution 35

A Chapter 11, Utility Values and State Prices

Although the textbook presented state prices with 2 states, it is not difficult to extend the analysis to cover 3 states.

We can find the utility value for the third state:

\[ S = Q_1S_1e^{\delta h} + Q_2S_2e^{\delta h} + Q_3S_3e^{\delta h} \]
\[ S = p_1U_1S_1e^{\delta h} + p_2U_2S_2e^{\delta h} + p_3U_3S_3e^{\delta h} \]
\[ 100 = 0.46 \times 0.8554 \times 200 + 0.06 \times U_2 \times 120 + 0.48 \times 0.9819 \times 35 \]
\[ U_2 = 0.6677 \]

The values of the call option on Stock B at time T are:

\[ V_1 = Max(0, 70 - 60) = 10 \]
\[ V_2 = Max(0, 90 - 60) = 30 \]
\[ V_3 = Max(0, 40 - 60) = 0 \]

The value of the option is:

\[ V = Q_1V_1 + Q_2V_2 + Q_3V_3 \]
\[ V = p_1U_1V_1 + p_2U_2V_2 + p_3U_3V_3 \]
\[ V = 0.46 \times 0.8554 \times 10 + 0.06 \times 0.6677 \times 30 + 0.48 \times 0.9819 \times 0 = 5.137 \]
Solution 36

D Chapter 11, Utility Values and State Prices

This stock has a higher value in the low state than it does in the high state. Although this situation does not affect our solution to the question, it suggests that the stock could be used as a hedge for whatever asset (which might, for example, be the entire market) is used to define high state and low state.

The price of the risk-free asset is:

\[ 110e^{-r_h} = 110(Q_u + Q_d) = 110(pU_u + (1-p)U_d) = 110(0.4 \times 0.85 + 0.6 \times 0.95) = 100.10 \]

The price of the stock is:

\[ S = Q_uS_u e^{\delta_h} + Q_dS_d e^{\delta_h} = pU_uS_u e^{\delta_h} + (1-p)U_dS_d e^{\delta_h} \]
\[ = 0.40 \times 0.85 \times 75e^{0.05 \times 1} + 0.60 \times 0.95 \times 125e^{0.05 \times 1} = 101.71 \]

The stock price exceeds the price of the risk-free bond by:

\[ 101.71 - 100.10 = 1.61 \]

Solution 37

A Chapter 11, Utility Values and State Prices

Since the probability of the high state is 0.65, the probability of the low state is:

\[ 1 - p = 1 - 0.65 = 0.35 \]

The stock’s cash flow in the low state can now be determined:

\[ S = Q_uS_u e^{\delta_h} + Q_dS_d e^{\delta_h} \]
\[ S = pU_uS_u e^{\delta_h} + (1-p)U_dS_d e^{\delta_h} \]
\[ 36.53 = 0.65 \times 0.90 \times 50e^{0 \times 2} + 0.35 \times 1.04 \times C_L e^{0 \times 2} \]
\[ C_L = 20 \]

The payoffs of the derivative are:

\[ V_u = \ln(50^3) = 11.7361 \]
\[ V_d = \ln(20^3) = 8.9872 \]

The value of the derivative is:

\[ V = Q_uV_u + Q_dV_d = pU_u \times V_u + (1-p)U_d \times V_d \]
\[ = 0.65(0.9) \times 11.7361 + 0.35(1.04) \times 8.9872 = 10.1369 \]
Solution 38

C  Chapter 11, Utility Values and State Prices

Since the probability of the high state is 0.65, the probability of the low state is:

\[ 1 - p = 1 - 0.65 = 0.35 \]

The stock’s cash flow in the low state can now be determined:

\[ S = Q_u S_u e^{\delta h} + Q_d S_d e^{\delta h} \]
\[ S = p U_u S_u e^{\delta h} + (1 - p) U_d S_d e^{\delta h} \]
\[ 36.53 = 0.65 \times 0.90 \times 50 e^{0 \times 2} + 0.35 \times 1.04 \times C_L e^{0 \times 2} \]
\[ C_L = 20 \]

The risk-free rate can be found with the probabilities and utility values:

\[ e^{-r h} = Q_u + Q_d \]
\[ e^{-r h} = p U_u + (1 - p) U_d \]
\[ e^{-r x^2} = 0.65 \times 0.90 + 0.35 \times 1.04 \]
\[ r = 0.0262 \]

The payoffs of the derivative at time 3 are:

\[ V_u(3) = \ln(50^3) = 11.7361 \]
\[ V_d(3) = \ln(20^3) = 8.9872 \]

Although the derivative does not pay until time 3, its payoffs are known with certainty at time 2. Therefore, we can discount the payoffs back to time 2 at the risk-free interest rate.

\[ V_u(2) = V_u(3)e^{-r x^1} = 11.7361 e^{-0.0262} = 11.4329 \]
\[ V_d(2) = V_d(3)e^{-r x^1} = 8.9872 e^{-0.0262} = 8.7550 \]

The value of the derivative is:

\[ V(0) = Q_u V_u(2) + Q_d V_d(2) = p U_u V_u(2) + (1 - p) U_d V_d(2) \]
\[ = 0.65 \times 0.9 \times 11.4329 + 0.35 \times 1.04 \times 8.7550 = 9.8751 \]

Solution 39

B  Chapter 11, Utility Values and State Prices

The expected return on the call options can be found with the formula below:

\[ 1 + \gamma_{Call} = \frac{p V_u + (1 - p) V_d}{V} = \frac{p V_u + (1 - p) V_d}{p U_u V_u + (1 - p) U_d V_d} \]
The expected returns for the first 2 choices are:

A: \[ \gamma_{\text{Call}} = \frac{pV_u + (1-p)V_d}{pU_u V_u + (1-p)U_d V_d} \]
\[ = \frac{0.6 \times 60 + 0.4 \times 20}{0.6 \times 0.92 \times 60 + 0.4 \times 1.05 \times 20} = 5.97\% \]

B: \[ \gamma_{\text{Call}} = \frac{pV_u + (1-p)V_d}{pU_u V_u + (1-p)U_d V_d} \]
\[ = \frac{0.6 \times 35 + 0.4 \times 0}{0.6 \times 0.92 \times 35 + 0.4 \times 1.05 \times 0} = 8.70\% \]

For Choice C, the strike price is greater than either of the possible stock prices, so the value of the call option is zero. Consequently, the expected return is undefined for the option described in Choice C.

The expected returns for the fourth and fifth choices are:

D: \[ \gamma_{\text{Put}} = \frac{pV_u + (1-p)V_d}{pU_u V_u + (1-p)U_d V_d} \]
\[ = \frac{0.6 \times 0 + 0.4 \times 15}{0.6 \times 0.92 \times 0 + 0.4 \times 1.05 \times 15} = -4.76\% \]

E: \[ \gamma_{\text{Put}} = \frac{pV_u + (1-p)V_d}{pU_u V_u + (1-p)U_d V_d} \]
\[ = \frac{0.6 \times 20 + 0.4 \times 60}{0.6 \times 0.92 \times 20 + 0.4 \times 1.05 \times 60} = -0.66\% \]

The option described in Choice B has the highest expected return.

Solution 40

B Chapter 11, Arbitrage

The Cox-Ross-Rubinstein model is:

\[ u = e^{\sigma\sqrt{h}} \]
\[ d = e^{-\sigma\sqrt{h}} \]

The Jarrow-Rudd model is:

\[ u = e^{(r-\delta -0.5\sigma^2)h + \sigma\sqrt{h}} \]
\[ d = e^{(r-\delta -0.5\sigma^2)h - \sigma\sqrt{h}} \]

From Chapter 10, we know that to avoid arbitrage, a model must satisfy the following:

\[ d < e^{(r-\delta)h} < u \]

The values of \( d, \ e^{(r-\delta)h}, \) and \( u \) are calculated for each model below:

<table>
<thead>
<tr>
<th>Model</th>
<th>( r )</th>
<th>( \delta )</th>
<th>( \sigma )</th>
<th>( h )</th>
<th>( d )</th>
<th>( e^{(r-\delta)h} )</th>
<th>( u )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.25</td>
<td>0.01</td>
<td>0.30</td>
<td>0.25</td>
<td>0.8607</td>
<td>1.0618</td>
<td>1.1618</td>
<td>CRR</td>
</tr>
<tr>
<td>B</td>
<td>0.21</td>
<td>0.00</td>
<td>0.20</td>
<td>1.00</td>
<td>0.8187</td>
<td>1.2337</td>
<td>1.2214</td>
<td>CRR</td>
</tr>
<tr>
<td>C</td>
<td>0.14</td>
<td>0.03</td>
<td>0.10</td>
<td>0.50</td>
<td>0.9317</td>
<td>1.0565</td>
<td>1.0733</td>
<td>CRR</td>
</tr>
<tr>
<td>D</td>
<td>0.14</td>
<td>0.05</td>
<td>0.50</td>
<td>0.50</td>
<td>0.6900</td>
<td>1.0460</td>
<td>1.3994</td>
<td>J-R</td>
</tr>
<tr>
<td>E</td>
<td>0.07</td>
<td>0.00</td>
<td>0.10</td>
<td>1.00</td>
<td>0.9656</td>
<td>1.0725</td>
<td>1.1794</td>
<td>J-R</td>
</tr>
</tbody>
</table>

Model B violates the restriction because it has the risk-free asset earning more than the risky asset after an upward move:

\[ 1.2337 > 1.2214 \]
Solution 41

**B  Chapter 11, Realistic Probability**

We can solve for \( p \), the true probability of the stock price going up, using the following formula:

\[
p = \frac{e^{(\alpha - \delta)h} - d}{u - d} = \frac{e^{(0.10 - 0) \times 1 - 0.763}}{1.477 - 0.763} = 0.47923
\]

Solution 42

**D  Chapter 11, Greeks in the Jarrow-Rudd Binomial Model**

*How do we know that gamma in the question refers to \( \Gamma \) and not \( \gamma \)? Because we would need to know the realistic probability of an upward movement in order to determine the expected return on the option, \( \gamma \), but there is no way of knowing the realistic probability of an upward movement.*

The values of \( u \) and \( d \) are:

\[
u = e^{(r - \delta - 0.5\sigma^2)h + \sigma \sqrt{h}} = e^{(0.11 - 0.04 - 0.5 \times 0.32^2)(1) + 0.32 \sqrt{1}} = 1.40326
\]

\[
d = e^{(r - \delta - 0.5\sigma^2)h - \sigma \sqrt{h}} = e^{(0.11 - 0.04 - 0.5 \times 0.32^2)(1) - 0.32 \sqrt{1}} = 0.73993
\]

The risk-neutral probability of an upward movement is:

\[
p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.11 - 0.04)(1) - 0.73993}}{1.40326 - 0.73993} = 0.50137
\]

The stock price tree and its corresponding tree of option prices are:

<table>
<thead>
<tr>
<th>Stock</th>
<th>American Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>78.7658</td>
<td>0.0000</td>
</tr>
<tr>
<td>56.1305</td>
<td>0.2088</td>
</tr>
<tr>
<td>41.5326</td>
<td>5.6339</td>
</tr>
<tr>
<td>21.8998</td>
<td>20.1002</td>
</tr>
<tr>
<td>12.4028</td>
<td></td>
</tr>
</tbody>
</table>

If the stock price initially moves down, then the resulting put price is $12.4028. This price is in bold type above to indicate that it is optimal to exercise early at this node:

\[42 - 29.5972 = 12.4028\]

This exercise value of 12.4028 is greater than the value of holding the option, which is:

\[e^{-0.11(1)} [(0.50137)(0.4674) + (1 - 0.50137)(20.1002)] = 9.1884\]

There is no need to calculate the current value of the American put to answer this question, but it is provided in the tree above for completeness.
We need to calculate the two possible values of delta at time 1:

\[
\Delta(Su, h) = e^{-\delta h} \frac{V_{uu} - V_{ud}}{Su^2 - Sud} = e^{-0.04 \times 1} \frac{0.0000 - 0.4674}{78.7658 - 41.5326} = -0.0121
\]

\[
\Delta(Sd, h) = e^{-\delta h} \frac{V_{ud} - V_{dd}}{Sud - Sd^2} = e^{-0.04 \times 1} \frac{0.4674 - 20.1002}{41.5326 - 21.8998} = -0.9608
\]

We can now calculate gamma:

\[
\Gamma(S, 0) = \Gamma(S_h, h) = \frac{\Delta(Su, h) - \Delta(Sd, h)}{Su - Sd} = \frac{-0.0121 - (-0.9608)}{56.1305 - 29.5972} = 0.0358
\]

Solution 43

**D Chapter 11, State Prices**

Since the strike price is $10, the put option pays \text{Max}[0, 10 - 15] = 0$ if the up state occurs. If the down state occurs, the call option pays \text{Max}[0, 10 - 7] = 3. We can use the stock price and the option price to solve for the original state prices:

\[
\begin{align*}
10 &= 15Q_u + 7Q_d \\
1.69 &= 0Q_u + 3Q_d
\end{align*}
\]

\[
\Rightarrow Q_u = 0.4038 \quad Q_d = 0.5633
\]

The discount factor for one year is:

\[
e^{-\delta} = Q_u + Q_d = 0.4038 + 0.5633 = 0.9671
\]

After the correction is made, the new state prices still result in the same risk-free rate of return:

\[
e^{-\hat{\delta}} = \hat{Q}_u + \hat{Q}_d = 0.9671
\]

We can use the stock price with the corrected value after a down-move and the expression for the risk-free discount factor to obtain another system of 2 equations and 2 unknowns:

\[
\begin{align*}
10 &= 15\hat{Q}_u + 5\hat{Q}_d \\
0.9671 &= \hat{Q}_u + \hat{Q}_d
\end{align*}
\]

\[
\Rightarrow \hat{Q}_d = 0.4507
\]

The new state price can now be used to find the value of the option. Since the payoff of the put option is \text{Max}[0, 10 - 5] = 5 when we use the correct lower stock price of 5, we have the following price for the put option:

\[
0\hat{Q}_u + 5\hat{Q}_d = 5 \times 0.4507 = 2.2533
\]

**Alternate Solution**

*We don’t have to use state prices to answer this question.*
We can use the stock price and the option price to solve for the risk-neutral probability of an upward movement and the discount factor:

\[
10 = e^{-r}[15p^* + 7(1 - p^*)] \quad \Rightarrow \quad e^{-r} = 0.9671
\]

\[
1.69 = e^{-r}[0p^* + 3(1 - p^*)] \quad \Rightarrow \quad p^* = 0.4175
\]

After the correction is made, we still have the same discount factor of 0.9671, but we have a new risk-neutral probability of an upward movement. We can use the stock price with the corrected value after a down-move to solve for the new, corrected risk-neutral probability of an up move:

\[
10 = 0.9671[15\hat{p}^* + 5(1 - \hat{p}^*)]
\]

\[
\frac{10}{0.9671} = 10\hat{p}^* + 5
\]

\[
\hat{p}^* = 0.5340
\]

The payoff of the put option is $5 when we use the correct lower stock price of $5. We have the following price for the put option:

\[
5(1 - \hat{p}^*)e^{-r} = 5(1 - 0.5340)(0.9671) = 2.2533
\]

**Solution 44**

**E** Chapter 11, Utility Values and State Prices

The payoffs of the call option are:

\[V_u = Max(0, 100 - K_C) = 100 - K_C\]
\[V_d = Max(0, 60 - K_C) = 0\]

The price of the call option is:

\[V = Q_uV_u + Q_dV_d = pU_u \times (100 - K_C) + (1 - p)U_d \times 0 = 0.6(0.92)(100 - K_C)\]

The expected return on the call option can now be calculated:

\[1 + \gamma_{Call} = \frac{pV_u + (1 - p)V_d}{V}\]
\[1 + \gamma_{Call} = \frac{0.6(100 - K_C)}{0.6(0.92)(100 - K_C)}\]
\[1 + \gamma_{Call} = \frac{1}{0.92}\]
\[\gamma_{Call} = 0.0870\]

The payoffs of the put option are:

\[V_u = Max(0, K_P - 100) = 0\]
\[V_d = Max(0, K_P - 60) = K_P - 60\]
The price of the put option is:

\[ V = Q_u V_u + Q_d V_d = p U_u \times 0 + (1 - p) U_d \times (K_P - 60) = 0.4(1.05)(K_P - 60) \]

The expected return on the put option can now be calculated:

\[ 1 + \gamma_{Put} = \frac{p V_u + (1 - p)V_d}{V} \]

\[ 1 + \gamma_{Put} = \frac{0.4(K_P - 60)}{0.4(1.05)(K_P - 60)} \]

\[ 1 + \gamma_{Put} = \frac{1}{1.05} \]

\[ \gamma_{Put} = -0.0476 \]

The difference in the expected returns is:

\[ \gamma_{Call} - \gamma_{Put} = 0.0870 - (-0.0476) = 0.1346 \]

Solution 45

B Chapter 11, Utility Values and State Prices

Notice that the expected return on the call option is expressed as an annual effective rate, not a continuously compounded rate.

Since the call option consists of 20 of the up-state Arrow-Debreu securities, the expected return of each up-state Arrow-Debreu security is equal to the expected return of the call option:

\[ \gamma_u = \gamma_{Call} = 0.23 \]

The utility value is the present value of $1, calculated using the expected return of the corresponding Arrow-Debreu security. Therefore, the up-state utility value is:

\[ U_u = \frac{1}{1 + \gamma_u} = \frac{1}{1.23} \]

The call option pays off $20 in the up state and $0 in the down state. We can use the call option to obtain the state price for the up state:

\[ 7.37 = 20Q_u + 0Q_d \]

\[ Q_u = \frac{7.37}{20} = 0.3685 \]

We can use the state price of the up-state to obtain the realistic probability that the up state occurs:

\[ Q_u = p \times U_u \]

\[ 0.3685 = p \times \frac{1}{1.23} \]

\[ p = 0.4533 \]
We can now find the down-state utility value:

\[ S = Q_u S_u + Q_d S_d \]
\[ S = U_u p S_u + U_d (1 - p) S_d \]

\[ 76 = \frac{1}{1.23} \times 0.4533 \times 100 + U_d (1 - 0.4533)70 \]
\[ U_d = 1.0229 \]

The sum of the up-state and the down-state utility values is:

\[ U_u + U_d = \frac{1}{1.23} + 1.0229 = 1.8359 \]
Chapter 12 – Solutions

Solution 1

Chapter 12, Black-Scholes Call Price

The question doesn’t specifically tell us to use the Black-Scholes formula, but we can’t use the binomial model since we are not told the number of binomial periods. Therefore, we must use the Black-Scholes formula.

The first step is to calculate $d_1$ and $d_2$:

$$d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(46 / 45) + (0.08 - 0.02 + 0.5 \times 0.35^2) \times 0.25}{0.35\sqrt{0.25}} = 0.29881$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.29881 - 0.35\sqrt{0.25} = 0.12381$$

In the Formula document for the MFE/3F exam, the SOA states that students should use 5 decimal places for both the inputs and the outputs of the online normal distribution calculator, so we will follow that convention in our Solutions.

We have:

$$N(d_1) = N(0.29881) = 0.61746$$

$$N(d_2) = N(0.12381) = 0.54927$$

The value of the European call option is:

$$C_{Eur} = S e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

$$= 46 e^{-0.02(0.25)} \times 0.61746 - 45 e^{-0.08(0.25)} \times 0.54927 = 4.03378$$

Solution 2

Chapter 12, Black-Scholes Put Price

The first step is to calculate $d_1$ and $d_2$:

$$d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(46 / 45) + (0.08 - 0.02 + 0.5 \times 0.35^2) \times 0.25}{0.35\sqrt{0.25}} = 0.29881$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.29881 - 0.35\sqrt{0.25} = 0.12381$$

We have:

$$N(d_1) = N(0.29981) = 0.61746$$

$$N(d_2) = N(0.12381) = 0.54927$$
The value of the European put option is:

\[
P_{\text{Eur}} = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)
\]

\[
= 45e^{-0.08(0.25)} \times (1 - 0.54927) - 46e^{-0.02(0.25)} \times (1 - 0.61746)
\]

\[
= 2.37215
\]

**Solution 3**

**A  Chapter 12, Black-Scholes Formula Using Prepaid Forward Prices**

The prepaid forward price of the stock is:

\[
F_{0,T}^P(S) = S_0 - PV_{0,T}(\text{Div}) = 46 - 5e^{(-0.08 \times 1)/12} = 41.0332
\]

The volatility of the prepaid forward price is:

\[
\sigma_{PF} = 0.3924
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma_{PF}^2T}{\sigma_{PF}\sqrt{T}} = \frac{\ln \left( \frac{41.0332}{45e^{-0.08(0.25)}} \right) + 0.5 \times 0.3924^2 \times 0.25}{0.3924\sqrt{0.25}}
\]

\[
= -0.27030
\]

\[
d_2 = d_1 - \sigma_{PF}\sqrt{T} = -0.27030 - 0.3924\sqrt{0.25} = -0.46650
\]

We have:

\[
N(d_1) = N(-0.27030) = 0.39346
\]

\[
N(d_2) = N(-0.46650) = 0.32043
\]

The value of the European call option is:

\[
C_{\text{Eur}} = F_{0,T}^P(S)N(d_1) - F_{0,T}^P(K)N(d_2) = 41.0332 \times 0.39346 - 44.1089 \times 0.32043
\]

\[
= 2.0111
\]

**Solution 4**

**B  Chapter 12, Options on Currencies**

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln(x_0 / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1.25/1.3) + (0.06 - 0.035 + 0.5 \times 0.11^2)(0.5)}{0.11\sqrt{0.5}}
\]

\[
= -0.30464
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = -0.30464 - 0.11\sqrt{0.5} = -0.38243
\]
We have:

\[
N(d_1) = N(-0.30464) = 0.38032 \\
N(d_2) = N(-0.38243) = 0.35107
\]

The value of the call option is:

\[
C_{Eur} = x_0 e^{-rT} N(d_1) - Ke^{-rT} N(d_2) \\
= 1.25e^{-0.035(0.5)} \times 0.38032 - 1.30e^{-0.06(0.5)} \times 0.35107 = 0.024245
\]

Solution 5

Chapter 12, Options on Currencies

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(1.25/1.3) + (0.06 - 0.035 + 0.5 \times 0.11^2)(0.5)}{0.11 \sqrt{0.5}} \\
= -0.30464
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = -0.30464 - 0.11 \sqrt{0.5} = -0.38243
\]

We have:

\[
N(d_1) = N(-0.30464) = 0.38032 \\
N(d_2) = N(-0.38243) = 0.35107
\]

The put price can be determined directly:

\[
P_{Eur} = Ke^{-rT} N(-d_2) - x_0 e^{-rT} N(-d_1) \\
= 1.30e^{-0.06(0.5)} \times (1 - 0.35107) - 1.25e^{-0.035(0.5)} \times (1 - 0.38032) = 0.05751
\]

Alternatively, we can find the value of the otherwise equivalent call option and then use put-call parity to find the value of the put option. The value of the call option is:

\[
C_{Eur} = x_0 e^{-rT} N(d_1) - Ke^{-rT} N(d_2) \\
= 1.25e^{-0.035(0.5)} \times 0.38032 - 1.30e^{-0.06(0.5)} \times 0.35107 = 0.024245
\]

We can now use put-call parity to find the value of the put option:

\[
P_{Eur} = C_{Eur} + Ke^{-rT} - x_0 e^{-rT} \\
= 0.024245 + 1.30e^{-0.06(0.5)} - 1.25e^{-0.035(0.5)} \\
= 0.05751
\]
Solution 6

Chapter 12, Options on Currencies

Since the put option is euro-denominated, we use the euro as the base currency. This means that the current exchange rate is:

\[ x_0 = \frac{1}{1.25} = 0.80 \]

Since the option is at-the-money, the strike price is also 0.80 €/$:

\[ K = 0.80 \]

We also have:

\[ r = 0.05 \]
\[ r_f = 0.08 \]

The values of \( d_1 \) and \( d_2 \) are:

\[ d_1 = \frac{\ln(x_0 / K) + (r-r_f + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(0.80/0.80) + (0.05-0.08 + 0.5 \times 0.10^2) \times 1}{0.10 \sqrt{1}} \]
\[ = -0.25000 \]
\[ d_2 = d_1 - \sigma \sqrt{T} = -0.25000 - 0.10 \sqrt{1} = -0.35000 \]

The values of \( N(d_1) \) and \( N(d_2) \) are:

\[ N(d_1) = N(-0.25000) = 0.40129 \]
\[ N(d_2) = N(-0.35000) = 0.36317 \]

The put price can be determined directly:

\[ P_{Eur} = Ke^{-rT} N(-d_2) - x_0 e^{-r_f T} N(-d_1) \]
\[ = 0.80 e^{-0.05(1)} \times (1 - 0.36317) - 0.80 e^{-0.08(1)} \times (1 - 0.40129) = 0.04247 \]

Alternatively, we can find the value of the otherwise equivalent call option and then use put-call parity to find the value of the put option. The value of the call option is:

\[ C_{Eur} = x_0 e^{-r_f T} N(d_1) - Ke^{-rT} N(d_2) \]
\[ = 0.80 e^{-0.08(1)} \times 0.40129 - 0.80 e^{-0.05(1)} \times 0.36317 = 0.01998 \]

We can now use put-call parity to find the value of the put option:

\[ P_{Eur} = C_{Eur} + Ke^{-rT} - x_0 e^{-r_f T} \]
\[ = 0.01998 + 0.80 e^{-0.05(1)} - 0.80 e^{-0.08(1)} \]
\[ = 0.04247 \]
Solution 7

Chapter 12, Options on Futures

The 1-year futures price is $60:

\[ F_{0,1} = 60 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln \left( \frac{F_{0,T_F}}{K} \right) + 0.5 \sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln (60/60) + 0.5(0.25)^2(1)}{0.25 \sqrt{1}} = 0.12500
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.12500 - 0.25 \sqrt{1} = -0.12500
\]

We have:

\[
N(d_1) = N(0.12500) = 0.54974
\]

\[
N(d_2) = N(-0.12500) = 0.45026
\]

The value of the call option is:

\[
C_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) = F_{0,T_F} e^{-rT} N(d_1) - Ke^{-rT} N(d_2)
\]

\[
= 60e^{-0.08(1)} \times 0.54974 - 60e^{-0.08(1)} \times 0.45026
\]

\[
= 5.5099
\]

Solution 8

Chapter 12, Options on Futures

The 1-year futures price is $60:

\[ F_{0,1} = 60 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln \left( \frac{F_{0,T_F}}{K} \right) + 0.5 \sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln (60/60) + 0.5(0.25)^2(1)}{0.25 \sqrt{1}} = 0.12500
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.12500 - 0.25 \sqrt{1} = -0.12500
\]

We have:

\[
N(d_1) = N(0.12500) = 0.54974
\]

\[
N(d_2) = N(-0.12500) = 0.45026
\]
The put price is:

\[
P_{\text{Eur}}(F_{0,T_F}, K, \sigma, r, T, r) = Ke^{-rT}N(-d_2) - F_{0,T_F}e^{-rT}N(-d_1)
\]
\[
= 60e^{-0.08(1)} \times (1 - 0.45026) - 60e^{-0.08(1)} \times (1 - 0.54974)
\]
\[
= 5.5099
\]

Alternatively, we can find the value of the otherwise equivalent call option and then use put-call parity to find the value of the put option. The value of the call option is:

\[
C_{\text{Eur}}(F_{0,T_F}, K, \sigma, r, T, r) = F_{0,T_F}e^{-rT}N(d_1) - Ke^{-rT}N(d_2)
\]
\[
= 60e^{-0.08(1)} \times 0.54974 - 60e^{-0.08(1)} \times 0.45026
\]
\[
= 5.5099
\]

We now can use put-call parity to find the value of the put option:

\[
P_{\text{Eur}}(F_{0,T_F}, K, \sigma, r, T, r) = C_{\text{Eur}}(F_{0,T_F}, K, \sigma, r, T, r) + Ke^{-rT} - F_{0,T_F}e^{-rT}
\]
\[
= 5.5099 + 60e^{-0.08(1)} - 60e^{0.08(1)} = 5.5099
\]

*When the strike price is equal to the futures price, the call price and the put price are the same.*

**Solution 9**

**B Chapter 12, Options on Futures**

The 6-month futures price is:

\[
F_{t,T} = S_t e^{(r-\delta)(T-t)}
\]
\[
F_{0,0.5} = 70e^{(0.09-0.05)(0.5)} = 71.4141
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln(F_{0,T_F} / K) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln(71.4141/65) + 0.5(0.30)^2(0.5)}{0.30\sqrt{0.5}} = 0.54970
\]
\[
d_2 = d_1 - \sigma\sqrt{T} = 0.54970 - 0.30\sqrt{0.5} = 0.33756
\]

We have:

\[
N(d_1) = N(0.54970) = 0.70874
\]
\[
N(d_2) = N(0.33756) = 0.63215
\]

The value of the call option is:

\[
C_{\text{Eur}}(F_{0,T_F}, K, \sigma, r, T, r) = F_{0,T_F}e^{-rT}N(d_1) - Ke^{-rT}N(d_2)
\]
\[
= 71.4141e^{-0.09(0.5)} \times 0.70874 - 65e^{-0.09(0.5)} \times 0.63215
\]
\[
= 9.10518
\]
Solution 10

A Chapter 12, Options on Futures

In this case, the option expires before the underlying futures contract:

\[ T = 0.5 \]
\[ T_F = 1.0 \]

The 1-year futures price is:

\[ F_{T,T} = S_te^{(r-\delta)(T-t)} \]
\[ F_{0,1} = 70e^{(0.09-0.05)(1)} = 72.8568 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln\left(\frac{F_{0,T_F}}{K}\right) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln(72.8568/65) + 0.5(0.30)^2(0.5)}{0.30\sqrt{0.5}} = 0.64398
\]
\[
d_2 = d_1 - \sigma\sqrt{T} = 0.64398 - 0.30\sqrt{0.5} = 0.43184
\]

We have:

\[ N(-d_1) = N(-0.64398) = 0.25979 \]
\[ N(-d_2) = N(-0.43184) = 0.33293 \]

The value of the put option is:

\[
P_{Eur}(F_{0,T_F},K,r,T,r) = Ke^{-rT}N(-d_2) - F_{0,T_F}e^{-rT}N(-d_1)
\]
\[
= 65e^{-0.09(0.5)} \times 0.33293 - 72.8568^{-0.09(0.5)}0.25979
\]
\[
= 2.5936
\]

Solution 11

B Chapter 12, Greek Measures for Portfolios

A call bull spread consists of a position in which a low-strike call is purchased and a high-strike call is sold. The low-strike call has the higher premium, so Option 1 is the one that is purchased and Option 2 is the one that is sold.

The quantity of Option 1 in the portfolio is \( q_1 \), and the quantity of Option 2 in the portfolio is \( q_2 \):

\[
q_1 = 1
\]
\[
q_2 = -1
\]
The value of vega for the bull spread is:

\[ Vega_{Port} = \sum_{i=1}^{n} q_i Vega_i = 1 \times 0.1329 - 1 \times 0.1646 = -0.0317 \]

**Solution 12**

**E** Chapter 12, Delta

The value of \( d_1 \) is:

\[
d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(86/90) + (0.095 - 0.03 + 0.5 \times 0.35^2) \times 0.75}{0.35 \sqrt{0.75}} \]

\[ = 0.16240 \]

We have:

\[ N(d_1) = N(0.16240) = 0.56450 \]

The delta of the put option is:

\[ \Delta_{Put} = -e^{-\delta T} N(-d_1) = -e^{-0.03(0.75)} \times (1 - 0.56450) = -0.42581 \]

**Solution 13**

**C** Chapter 12, Delta

The value of \( d_1 \) is:

\[
d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(86/90) + (0.095 - 0.03 + 0.5 \times 0.35^2) \times 0.75}{0.35 \sqrt{0.75}} \]

\[ = 0.16240 \]

We have:

\[ N(d_1) = N(0.16240) = 0.56450 \]

The delta of the call option is:

\[ \Delta_{Call} = e^{-\delta T} N(d_1) = e^{-0.03(0.75)} \times 0.56450 = 0.55194 \]

The delta of the put option is:

\[ \Delta_{Put} = \Delta_{Call} - e^{-\delta T} = 0.55194 - e^{-0.03(0.75)} = -0.42581 \]

The delta of the position is:

\[ \Delta_{Port} = \sum_{i=1}^{n} q_i \Delta_i = 100 \times 0.55194 + 50 \times (-0.42581) = 33.9035 \]
Solution 14
B Chapter 12, Elasticity

The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(86/90) + (0.095 - 0.03 + 0.5 \times 0.35^2) \times 0.75}{0.35\sqrt{0.75}}$$

$$d_1 = 0.16240$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.16240 - 0.35\sqrt{0.75} = -0.14071$$

We have:

$$N(-d_1) = N(-0.16240) = 0.43550$$

$$N(-d_2) = N(0.14071) = 0.55595$$

The value of the European put option is:

$$P_{Eur} = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

$$= 90e^{-0.095(0.75)} \times 0.55595 - 86e^{-0.03(0.75)} \times 0.43550 = 9.9749$$

The delta of the put option is:

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{-0.03(0.75)} \times 0.43550 = -0.42581$$

The elasticity of the put option is:

$$\Omega = \frac{S\Delta}{V} = \frac{86 \times (-0.42581)}{9.9749} = -3.6712$$

Solution 15
E Chapter 12, Greek Measures for Portfolios

The quantity of each option purchased is determined below:

$$q_1 \times 10 = 40\% \times 50 \quad \Rightarrow \quad q_1 = \frac{20}{10} = 2$$

$$q_2 \times 20 = 40\% \times 50 \quad \Rightarrow \quad q_2 = \frac{20}{20} = 1$$

$$q_3 \times 5 = 20\% \times 50 \quad \Rightarrow \quad q_3 = \frac{10}{5} = 2$$

The value of rho for the portfolio is:

$$\rho_{Port} = \sum_{i=1}^{n} q_i \rho_i = 2 \times 0.2754 + 1 \times 0.3641 + 2 \times (-0.2087) = 0.4975$$
Solution 16

E  Chapter 12, Greek Measures for Portfolios

The graph in the question is comparable to Figure 12.8 of the textbook.

Let’s consider each of the choices.

The value of delta for a call option is always positive, so Choice A cannot be correct.
The value of vega for a put option is always positive, so Choice B cannot be correct.
The value of rho is always negative for a put option, so Choice C cannot be correct.
The value of psi is always negative for a call option, so Choice D cannot be correct.
The value of theta is usually negative, but it can be positive for a deep-in-the-money put option. This is the pattern in the graph, so Choice E is the correct answer.

Solution 17

B  Chapter 12, Elasticity

The risk premium of the option is:

\[ \gamma - r = \Omega \times (\alpha - r) = 4.5 \times (0.15 - 0.07) = 0.36 \]

Solution 18

E  Chapter 12, Elasticity

The value of the option is:

\[ V = \Delta S + B \]

The percentage of the value of the option that is invested in the stock is the elasticity of the stock:

\[ \frac{\Delta S}{V} = \Omega = -5.35 \]

The amount invested in the stock is:

\[ \Delta S = \Omega V = -5.35 \times 4.17 = -22.3095 \]

We can now solve for the amount that is lent at the risk-free rate of return:

\[ V = \Delta S + B \]
\[ 4.17 = -22.3095 + B \]
\[ B = 26.4795 \]
Solution 19

A  Chapter 12, Sharpe Ratio

The Sharpe ratio for any asset is the ratio of the asset’s risk premium to its volatility:

\[
\text{Sharpe ratio for call option} = \frac{\gamma_{\text{Call}} - r}{\sigma_{\text{Call}}} = \frac{\Omega \times (\alpha - r)}{\sigma} = \frac{4.377(0.15 - 0.06)}{1.40} = 0.28138
\]

Solution 20

C  Chapter 12, Elasticity

If someone writes an option, that person receives the price of the option in exchange for agreeing to provide the payoff if the option is exercised. Writing an option is essentially the same as selling the option.

The investor purchases 1 each of Option A and Option B, and the investor sells 1 of Option C:

\[
q_A = 1 \quad q_B = 1 \quad q_C = -1
\]

The value of the portfolio is:

\[
Port = 1 \times 9.986 + 1 \times 7.985 - 1 \times 6.307 = 11.664
\]

The elasticity of the portfolio is:

\[
\Omega_{\text{Port}} = \sum_{i=1}^{n} q_i \Omega_i = \frac{9.986}{11.664} \times 4.367 + \frac{7.985}{11.664} \times 4.794 - \frac{6.307}{11.664} \times 5.227 = 4.1943
\]
Solution 21

**D Chapter 12, Options on Currencies**

The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(1.40 / 1.50) + (0.05 - 0.025 + 0.5 \times 0.10^2)}{0.10 \sqrt{1}} = -0.38993$$

$$d_2 = d_1 - \sigma \sqrt{T} = 0.38993 - 0.10 \sqrt{1} = -0.48993$$

From the standard normal table, we have:

$$N(-d_1) = N(0.38993) = 0.65171$$

$$N(-d_2) = N(0.48993) = 0.68791$$

The value of one of the put options is:

$$P_{Eur} = Ke^{-rT}N(-d_2) - x_0e^{-r_f T}N(-d_1)$$

$$=1.50e^{-0.05(1)} \times 0.68791 - 1.40e^{-0.025(1)} \times 0.65171 = 0.09167$$

The value of 1,000 of the options is:

$$1,000 \times 0.09167 = 91.67$$

Solution 22

**B Chapter 12, Option Elasticity**

The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(25 / 26) + (0.06 - 0.01 + 0.5 \times 0.20^2)}{0.20 \sqrt{2}} = 0.35631$$

$$d_2 = d_1 - \sigma \sqrt{T} = 0.35631 - 0.20 \sqrt{2} = 0.07347$$

From the standard normal table, we have:

$$N(-d_1) = N(-0.35631) = 0.36080$$

$$N(-d_2) = N(-0.07347) = 0.47072$$

The value of the put option is:

$$P_{Eur} = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

$$= 26e^{-0.06(2)} \times 0.47072 - 25e^{-0.01(2)} \times 0.36080 = 2.01338$$

The delta of the put option is:

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{-0.01(2)}(0.36080) = -0.35366$$
The elasticity of the put option is:
\[ \Omega = \frac{S\Delta}{V} = \frac{25 \times (-0.35366)}{2.01338} = -4.3913 \]

**Solution 23**

D  Chapter 12, Elasticity of a Portfolio

Using put-call parity, we can determine the cost to establish the portfolio:

\[
C_{Eur} + Ke^{-rT} = S_0e^{-\delta T} + P_{Eur} \\
C_{Eur} - P_{Eur} = S_0e^{-\delta T} - Ke^{-rT} \\
C_{Eur} - P_{Eur} = 60e^{-0.05(0.5)} - 62e^{-0.13(0.5)} \\
C_{Eur} - P_{Eur} = 0.42041
\]

The value of the portfolio is therefore:
\[ V_{Port} = C_{Eur} - P_{Eur} = 0.42041 \]

The delta of the portfolio is the delta of the call minus the delta of the put:
\[ \Delta_{Port} = \sum_{i=1}^{2} q_i \Delta_i = 1 \times \Delta_{Call} - 1 \times \Delta_{Put} = \Delta_{Call} - (\Delta_{Call} - e^{-\delta T}) = e^{-0.05(0.5)} = 0.97531 \]

The elasticity of the portfolio is:
\[ \Omega_{Port} = \frac{S\Delta_{Port}}{V_{Port}} = \frac{60 \times 0.97531}{0.42041} = 139.2 \]

**Solution 24**

B  Chapter 12, Risk Premium of a Portfolio

We can use put-call parity to determine the value of the put option:

\[
C_{Eur} + Ke^{-rT} = S_0e^{-\delta T} + P_{Eur} \\
5.10 + 60e^{-0.06(0.5)} = 60e^{-0.06(0.5)} + P_{Eur} \\
P_{Eur} = 5.10
\]

The value of the portfolio is:
\[ V_{Port} = 2 \times C_{Eur} + 1 \times P_{Eur} = 2 \times 5.10 + 1 \times 5.10 = 15.30 \]

The delta of the put option is:
\[ \Delta_{Put} = \Delta_{Call} - e^{-\delta T} = 0.5277 - e^{-0.06(0.5)} = -0.4427 \]
The delta of the portfolio is:

\[ \Delta_{\text{Port}} = \sum_{i=1}^{2} q_i \Delta_i = 2 \times \Delta_{\text{Call}} + 1 \times \Delta_{\text{Put}} = 2 \times 0.5277 + 1 \times (-0.4427) = 0.6127 \]

The elasticity of the portfolio is:

\[ \Omega_{\text{Port}} = \frac{S\Delta_{\text{Port}}}{V_{\text{Port}}} = \frac{60 \times 0.6127}{15.30} = 2.4026 \]

The expected return of the portfolio is:

\[ \gamma_{\text{Port}} = (\alpha - r)\Omega_{\text{Port}} + r = (0.22 - 0.06)(2.4026) + 0.06 = 0.4444 \]

**Solution 25**

**E**  Chapter 12, Sharpe Ratio

The Sharpe ratio of a call option is the same as the Sharpe ratio of the underlying stock:

\[ \text{Sharpe ratio for call option} = \frac{\gamma_{\text{Call}} - r}{\sigma_{\text{Call}}} = \frac{\alpha - r}{\sigma} \]

The Sharpe ratio for the underlying stock is:

\[ \frac{\alpha - r}{\sigma} = \frac{0.08 - 0.04}{0.25} = 0.16 \]

Therefore, the Sharpe ratio for the call option is 0.16.

**Solution 26**

**A**  Chapter 12, General

_The page references below refer to the Second Edition of Derivatives Markets._

Choice A is false. From the third sentence from the bottom of page 391: the elasticity decreases as the strike price decreases. This is because the elasticity is highest for out-of-the-money options. As the strike price decreases, a call option becomes more in-the-money.

Choice B is true. On page 394, the last paragraph of the risk premium section: the expected return of a call option goes down as the price of the underlying stock goes up.

Choice C is true. On page 394, the last sentence of the risk premium section: the expected return of a put option is less than that of the underlying stock.

Choice D is true. Refer to the second to last sentence of the first paragraph of the section titled “Using Implied Volatility” on page 402.

Choice E is true. On page 394, the last full sentence of the page: the Sharpe ratio for a call equals the Sharpe ratio for the underlying stock.
Solution 27

A calendar spread consists of buying and selling options that have different expiration dates. Since the investor believes that the stock price will not change, the investor focuses on the theta measure when deciding which option to buy and which one to sell. Both options decline in value over time, but Option 1 declines more quickly than Option 2, since Option 1’s theta is more negative than Option 2’s theta. Therefore, the investor sells Option 1 and buys Option 2.

The quantity of Option 1 in the portfolio is \( q_1 \), and the quantity of Option 2 in the portfolio is \( q_2 \):

\[
q_1 = -1 \\
q_2 = 1
\]

The value of psi for the calendar spread position is:

\[
\psi_{Port} = \sum_{i=1}^{n} q_i \psi_i = -1 \times (-0.0184) + 1 \times (-0.2704) = -0.2520
\]

Solution 28

The holding period profit is the current value of the position minus the cost of the position, including interest.

At the end of 6 months, the 1-year put option becomes a 6-month put option, and its value is shown in the 6-month column of the table. Since the final stock price is $38, the value of this option from the table is $3.50.

The 6-month option expires at the end of 6 months, and since the final stock price is $38 and the strike price is $40, the 6-month option has a payoff of $2.

The cost of establishing the position is the cost of the 12-month option minus the proceeds from writing (i.e., selling) the 6-month option, accumulated for interest:

\[
(3.25 - 2.67)e^{0.095(0.5)} = 0.6082
\]

The holding period profit is the value of the 12-month option minus the value of the 6-month option minus the cost of establishing the position, all evaluated at the end of 6 months:

\[
3.50 - 2.00 - 0.6082 = 0.8918
\]
Solution 29

D  Chapter 12, Black-Scholes Call Price

As the number of steps in the binomial model becomes large, the resulting price for an option approaches the Black-Scholes price. For an example of this, see Table 12.1 of the textbook. 1,000 is a very large number of steps, so the binomial model price, rounded to two decimal places, is the same as the price obtained with the Black-Scholes formula.

The first step is to calculate \( d_1 \) and \( d_2 \):

\[
d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40/35) + (0.10 - 0.02 + 0.5 \times 0.30^2) \times 1}{0.30\sqrt{1}} = 0.86177
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = 0.86177 - 0.30\sqrt{1} = 0.56177
\]

We have:

\[
N(d_1) = N(0.86177) = 0.80559
\]

\[
N(d_2) = N(0.56177) = 0.71286
\]

The value of the European call option is:

\[
C_{Eur} = S e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) = 40 e^{-0.02(1)} \times 0.80559 - 35 e^{-0.10(1)} \times 0.71286 = 9.00975
\]

Solution 30

B  Chapter 12, Black-Scholes Put Price

The dividend of $9 at the end of 9 months occurs after the option expires, so it does not affect the calculation of the price of the 6-month option.

The prepaid forward price of the stock is:

\[
P_{0,T}^P(S) = S_0 - PV_{0,T}(Div) = 77 - 7e^{-0.10(0.25)} = 70.1728
\]

The volatility of the prepaid forward price is:

\[
\sigma_{PF} = 0.2743
\]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln\left(\frac{P_{0,T}^P(S)}{P_{0,T}^P(K)}\right) + 0.5\sigma_{PF}^2T}{\sigma_{PF}\sqrt{T}} = \frac{\ln\left(\frac{70.1728}{73e^{-0.10(0.5)}}\right) + 0.5 \times 0.2743^2 \times 0.5}{0.2743\sqrt{0.5}} = 0.15112
\]

\[
d_2 = d_1 - \sigma_{PF}\sqrt{T} = 0.15112 - 0.2743\sqrt{0.5} = -0.04284
\]
We have:
\[ N(-d_1) = N(-0.15112) = 0.43974 \]
\[ N(-d_2) = N(0.04284) = 0.51709 \]

The value of the put option is:
\[ P_{Eur} = Ke^{-rT}N(-d_2) - [S_0 - PV_{0,T}(D_i)]N(-d_1) \]
\[ = 73e^{-0.10(0.5)} \times 0.51709 - 70.1728 \times 0.43994 = 5.0348 \]

**Solution 31**

C Chapter 12, Greek Measures for Portfolios

A put bear spread consists of a position in which a high-strike put is purchased and a low-strike put is sold. The high-strike put has the higher price, so Option 2 is the high-strike put, and Option 1 is the low-strike put. Therefore, Option 2 is purchased, and Option 1 is sold.

The quantity of Option 1 in the portfolio is \( q_1 \), and the quantity of Option 2 in the portfolio is \( q_2 \):\n
\[ q_1 = -1 \]
\[ q_2 = 1 \]

The value of gamma for the put bear spread position is:
\[ \Gamma_{Port} = \sum_{i=1}^{n} q_i \Gamma_i = -1 \times (0.0181) + 1 \times (0.0225) = 0.0044 \]

**Solution 32**

E Chapter 12, Black-Scholes Formula Using Prepaid Forward Prices

The key to this question is recognizing that the ratio of the prepaid forward price of the stock to the present value of the strike price is the same for both options.

For the option on Stock X:
\[ \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} = \frac{80e^{-0.03(0.75)}}{70e^{-r(0.75)}} = 1.11743e^{r(0.75)} \]

For the option on Stock Y:
\[ \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} = \frac{118.48e^{-0.013(0.75)}}{105e^{-r(0.75)}} = 1.11743e^{r(0.75)} \]
The values of $d_1$ and $d_2$ are based on that ratio, $\sigma$, and $T$:

$$d_1 = \ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma^2T \div \sigma\sqrt{T}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Therefore, both options have the same values of $d_1$ and $d_2$, and they also have the same values of $N(d_1)$ and $N(d_2)$.

The formula for the price of a European call option is:

$$C_{Eur} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) = F_{0,T}^P(S)N(d_1) - F_{0,T}^P(K)N(d_2)$$

Therefore, for the option on Stock X, we have:

$$16.05 = 80e^{-0.03(0.75)}N(d_1) - 70e^{-r(0.75)}N(d_2)$$

$$16.05 = 78.220N(d_1) - 70e^{-r(0.75)}N(d_2)$$

For the option on Stock Y, we have:

$$C_{Eur} = 118.48e^{-0.013(0.75)}N(d_1) - 105e^{-r(0.75)}N(d_2)$$

$$C_{Eur} = 117.330N(d_1) - 105e^{-r(0.75)}N(d_2)$$

Notice that the strike price of the option on Stock Y is 150% of the Strike price of the option on Stock X:

$$\frac{105}{70} = 1.50000$$

Furthermore, the prepaid forward price of Stock Y is 150% of the prepaid forward price of Stock X:

$$\frac{117.330}{78.220} = 1.50000$$

If we multiply both sides of the Black-Scholes formula for the option on Stock X by 1.5, we have the value of the option on Stock Y:

$$1.5 \times 16.05 = 1.5 \times \left[ 78.220N(d_1) - 70e^{-r(0.75)}N(d_2) \right]$$

$$24.075 = 117.330N(d_1) - 105e^{-r(0.75)}N(d_2)$$

The right side of the equation above is the Black-Scholes formula for the value of the call option on Stock Y. Therefore, the value of the call option on Stock Y is $24.075.$
Solution 33

Chapter 12, Implied Volatility

D

If the volatility is 30%, then \( N(-d_1) \) and \( N(-d_2) \) are calculated as shown below:

\[
N(-d_1) = N(-0.21670) = 0.41422
\]

\[
N(-d_2) = N(0.08330) = 0.53319
\]

If the volatility is 30%, then the price of the put option is:

\[
P_{Eur}(S,K,\sigma,r,T,\delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)
\]

\[
= 35e^{-0.08(1)} \times 0.53319 - 35e^{-0.06(1)} \times 0.41422 = 3.57345
\]

Since the calculated price of $3.57 matches the observed price of the option, the implied volatility is 30%.

The prices for the rest of the volatilities are shown in the table below:

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( N(-d_1) )</th>
<th>( N(-d_2) )</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.20833</td>
<td>0.05830</td>
<td>0.41750</td>
<td>0.47675</td>
<td>1.64</td>
</tr>
<tr>
<td>0.20</td>
<td>0.20000</td>
<td>0.00000</td>
<td>0.42074</td>
<td>0.50000</td>
<td>2.29</td>
</tr>
<tr>
<td>0.25</td>
<td>0.20500</td>
<td>-0.04500</td>
<td>0.41879</td>
<td>0.51795</td>
<td>2.93</td>
</tr>
<tr>
<td>0.30</td>
<td>0.21670</td>
<td>-0.08330</td>
<td>0.41422</td>
<td>0.53319</td>
<td>3.57</td>
</tr>
<tr>
<td>0.35</td>
<td>0.23210</td>
<td>-0.11790</td>
<td>0.40823</td>
<td>0.54693</td>
<td>4.21</td>
</tr>
</tbody>
</table>

To answer the question, it is not necessary to calculate more than 3 of the prices of the table. We can begin by calculating the price that corresponds to the middle volatility of 25%. Since that price is $2.93, which is too low, we choose a higher volatility next. That way, we don’t need to calculate the prices that correspond to the 15% and 20% volatilities.

Solution 34

Chapter 12, Calendar Spread

D

A calendar spread position involves buying and selling options with different expiration dates. Since Option 1 and Option 2 have the same expiration dates, Choices A and B cannot be correct. Since Option 3 and Option 4 have the same expiration dates, Choice E cannot be correct either.

This leaves us with Choice C and Choice D as possible correct answers. Theta is the time decay measure, which means that the lower the value of theta (i.e., the more negative), the more the option’s price decreases over time, assuming the stock price doesn’t change. Since the investor does not expect the stock price to change, the investor purchases an option with a high theta and sells an option with a low theta. Since the values of theta are negative, this means buying the option with the less negative value of theta and selling the option with the more negative value of theta. Therefore, the investor buys Option 4 and sells Option 2. The correct answer is Choice D.
Solution 35
C Chapter 12, Options on Currencies

Since the put option is yen-denominated, we use the yen as the base currency. This means that the current exchange rate is:

\[ x_0 = 132 \]

We also have:

\[ r = 0.01 \]
\[ r_f = 0.04 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(132/125) + (0.01 - 0.04 + 0.5 \times 0.12^2) \times 0.5}{0.12\sqrt{0.5}}
\]

\[ = 0.50780 \]
\[ d_2 = d_1 - \sigma\sqrt{T} = 0.50780 - 0.12\sqrt{0.5} = 0.42295 \]

We have:

\[ N(d_1) = N(0.50780) = 0.69420 \]
\[ N(d_2) = N(0.42295) = 0.66383 \]

The value of the put option is:

\[
P_{Eur} = Ke^{-rT}N(-d_2) - x_0e^{-r_fT}N(-d_1)
\]

\[ = 125e^{-0.01(0.5)} \times (1 - 0.66383) - 132e^{-0.04(0.5)} \times (1 - 0.69420) = 2.2454 \]

Alternatively, we can find the value of the otherwise equivalent call option and then use put-call parity to find the value of the put option. The value of the call option is:

\[
C_{Eur} = x_0e^{-r_fT}N(d_1) - Ke^{-rT}N(d_2)
\]

\[ = 132e^{-0.04(0.5)} \times 0.69420 - 125e^{-0.01(0.5)} \times 0.66383 = 7.2550 \]

We can now use put-call parity to find the value of the put option:

\[
P_{Eur} = C_{Eur} + Ke^{-rT} - x_0e^{-r_fT}
\]

\[ = 7.2550 + 125e^{-0.01(0.5)} - 132e^{-0.04(0.5)} = 2.2454 \]

Solution 36
A Chapter 12, Options on Currencies

Since the call option is euro-denominated, we use the euro as the base currency. This means that the current exchange rate is:

\[ x_0 = \frac{1}{132} \]
We also have:

\[ r = 0.04 \]
\[ r_f = 0.01 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln\left(\frac{1/132}{0.008}\right) + (0.04 - 0.01 + 0.5 \times 0.12^2) \times 0.5}{0.12\sqrt{0.5}}
\]
\[ = -0.42295 \]
\[
d_2 = d_1 - \sigma \sqrt{T} = -0.42295 - 0.12\sqrt{0.5} = -0.50780
\]

We have:

\[ N(d_1) = N(-0.42295) = 0.33617 \]
\[ N(d_2) = N(-0.50780) = 0.30580 \]

The value of the call option is:

\[
C_{Eur} = x_0 e^{-rT} N(d_1) - Ke^{-rT} N(d_2)
\]
\[ = \frac{1}{132} e^{-0.01(0.5)} \times 0.33617 - 0.008 e^{-0.04(0.5)} \times 0.30580 = 0.00014
\]

**Solution 37**

**D** Chapter 12, Options on Futures Contracts

The 1-year futures price is:

\[
F_{t,T} = S_t e^{(r-\delta)(T-t)}
\]
\[ F_{0,1} = 100e^{(0.07-0.00)(1)} = 107.2508 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln\left(F_{0,T_F} / K\right) + 0.5\sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln(107.2508/105) + 0.5(0.35)^2(1)}{0.35\sqrt{1}} = 0.23560
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = 0.23560 - 0.35\sqrt{1} = -0.11440
\]

We have:

\[ N(d_1) = N(0.23560) = 0.59313 \]
\[ N(d_2) = N(-0.11440) = 0.45446 \]
The value of the call option is:

\[
C_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) = F_{0,T_F}e^{-rT}N(d_1) - Ke^{-rT}N(d_2)
\]

\[
= 107.2508e^{-0.07(1)} \times 0.59313 - 105e^{-0.07(1)} \times 0.45446
\]

\[
= 14.8208
\]

**Solution 38**

**C  Chapter 12, Black-Scholes Put Price**

The first step is to calculate \(d_1\) and \(d_2\):

\[
d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}}
\]

\[
= \frac{\ln(100/103) + (0.06 - 0.025 + 0.5 \times 0.5^2) \times 0.5}{0.5 \sqrt{0.5}} = 0.14267
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.14267 - 0.5 \sqrt{0.5} = -0.21088
\]

We have:

\[
N(-d_1) = N(-0.14267) = 0.44328
\]

\[
N(-d_2) = N(0.21088) = 0.58351
\]

The value of the European put option is:

\[
P_{Eur} = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)
\]

\[
= 103e^{-0.06(0.5)} \times 0.58351 - 100e^{-0.025(0.5)} \times 0.44328
\]

\[
= 14.5479
\]

**Solution 39**

**A  Chapter 12, Black-Scholes and Delta**

*Since the stock does not pay dividends, the price of the American call option is equal to the price of the otherwise equivalent European call option.*

Since the stock does not pay dividends, delta is equal to \(N(d_1)\):

\[
N(d_1) = 0.48405
\]

\[
d_1 = -0.03999
\]
The formula for $d_1$ can be solved for the risk-free rate of return:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$-0.03999 = \frac{\ln(40/42) + (r - 0 + 0.5 \times 0.36^2)0.25}{0.36\sqrt{0.25}}$$

$$r = 0.10157$$

The value of $d_2$ is:

$$d_2 = d_1 - \sigma\sqrt{T} = -0.03999 - 0.36\sqrt{0.25} = -0.21999$$

The value of the call option is:

$$C_{Eur}(S, K, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

$$= 40(0.48405) - 42e^{-0.10157 \times 0.25}N(-0.21999)$$

$$= 19.36 - 40.9470[1 - N(0.21999)]$$

$$= -21.5850 + 40.9470N(0.21999)$$

$$= 40.9470 \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.21999} e^{-0.5x^2} dx - 21.5850$$

$$= 16.3355 \int_{-\infty}^{0.22} e^{-0.5x^2} dx - 21.5850$$

The correct answer is A.

**Solution 40**

**A** Chapter 12, Black-Scholes Call Price

*It is never optimal to exercise the American call option early because the stock does not pay dividends. Therefore, its value is equal to an otherwise equivalent European call option, and we can use the Black-Scholes formula to find its price.*

The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(78/82) + (0.10 - 0.00 + 0.5 \times 0.27^2) \times 0.25}{0.27\sqrt{0.25}}$$

$$= -0.11776$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.11776 - 0.27\sqrt{0.25} = -0.25276$$

We have:

$$N(d_1) = N(-0.11776) = 0.45313$$

$$N(d_2) = N(-0.25276) = 0.40023$$
The value of the call option is:

\[ C_{Eur} = S e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \]
\[ = 78e^{-0.00(0.25)} \times 0.45313 - 82e^{-0.10(0.25)} \times 0.40023 = 3.3356 \]

**Solution 41**

C Chapter 12, Volatility of an Option

The first step is to calculate \( d_1 \) and \( d_2 \):

\[ d_1 = \frac{\ln(S/K) + (r - \delta + 0.5 \sigma^2)T}{\sigma \sqrt{T}} \]
\[ = \frac{\ln(90/95) + (0.065 - 0.00 + 0.5 \times 0.45^2)1}{0.45\sqrt{1}} = 0.24930 \]
\[ d_2 = d_1 - \sigma \sqrt{T} = 0.24930 - 0.45\sqrt{1} = -0.20070 \]

We have:

\[ N(d_1) = N(0.24930) = 0.59844 \]
\[ N(d_2) = N(-0.20070) = 0.42047 \]

The value of the call option is:

\[ C_{Eur}(S, K, \sigma, r, T, \delta) = S e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \]
\[ = 90e^{-0(1)} \times 0.59844 - 95e^{-0.065(1)} \times 0.42047 = 16.4288 \]

The value of delta is:

\[ \Delta_{Call} = e^{-\delta T} N(d_1) = e^{-0.0(1)} \times 0.59844 = 0.59844 \]

The elasticity of the option is:

\[ \Omega = \frac{S \Delta}{V} = \frac{90 \times 0.59844}{16.4288} = 3.2784 \]

The volatility of the call option is:

\[ \sigma_{Option} = \sigma_{Stock} \times |\Omega_{Option}| = 0.45 \times 3.2784 = 1.4753 \]

**Solution 42**

C Chapter 12, Black-Scholes Formula

The annual volatility is:

\[ \sigma = \sqrt{\frac{\text{Var} [\ln S(t)]}{t}} = \sqrt{\frac{0.5t}{t}} = \sqrt{0.5} \]

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We can use the version of the Black-Scholes formula that is based on prepaid forward prices to find the value of the call option:

\[
\begin{align*}
    d_1 &= \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{50}{50} \right) + 0.5 \times 0.5 \times 8}{\sqrt{0.5 \times 8}} = 1.00000 \\
    d_2 &= d_1 - \sigma \sqrt{T} = 1.00000 - \sqrt{0.5 \times 8} = -1.00000 \\
    N(d_1) &= N(1.00000) = 0.84134 \\
    N(d_2) &= N(-1.00000) = 0.15866
\end{align*}
\]

The price of the call option is:

\[
C_{Eur} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) = F_{0,T}^P(S)N(d_1) - F_{0,T}^P(K)N(d_2) = 50 \times 0.84134 - 50 \times 0.15866 = 34.134
\]

**Solution 43**

Chapter 12, Black-Scholes Formula Using Prepaid Forward Prices

The prepaid forward prices of the stock and the strike price are:

\[
F_{0,T}^P(S) = 100 - 4.00e^{-0.07(4/12)} = 96.0923 \\
F_{0,T}^P(K) = 105e^{-0.07(0.5)} = 101.3886
\]

The volatility of the prepaid forward price is:

\[
\sigma_{PF} = \sqrt{\frac{\text{Var}[\ln F_{t,0.5}]}{t}} = \frac{0.2602\sqrt{t}}{\sqrt{t}} = 0.2602
\]

We use the prepaid forward volatility in the Black-Scholes Formula:

\[
\begin{align*}
    d_1 &= \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma_{PF}^2 T}{\sigma_{PF} \sqrt{T}} = \frac{\ln \left( \frac{96.0923}{101.3886} \right) + 0.5 \times 0.2602^2 \times 0.5}{0.2602\sqrt{0.5}} = -0.19961 \\
    d_2 &= d_1 - \sigma_{PF} \sqrt{T} = -0.19961 - 0.2602\sqrt{0.5} = -0.38360
\end{align*}
\]

We have:

\[
N(-d_1) = N(0.19961) = 0.57911 \\
N(-d_2) = N(0.38360) = 0.64936
\]
The value of the European put option is:

$$P_{Eur} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) = Ke^{-rT} N(-d_2) - \left[ S_0 - PV_{0,T}(Div) \right] N(-d_1)$$

$$= 101.3886 \times 0.64936 - 96.0923 \times 0.57911$$

$$= 10.1897$$

**Solution 44**

**D ** Chapter 12, Black-Scholes Formula Using Prepaid Forward Prices

The prepaid forward price of the stock is:

$$F_{0,T}^P(S) = S_0 - PV_{0,T}(Div) = 82 - 4e^{-0.10\times1} = 78.3807$$

The volatility of the prepaid forward price is:

$$\sigma_{PF} = 0.2615$$

The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma_{PF}^2 T}{\sigma_{PF}\sqrt{T}} = \frac{\ln \left( \frac{78.3807}{87e^{-0.10\times2}} \right) + 0.5 \times 0.2615^2 \times 2}{0.2615\sqrt{2}}$$

$$= 0.44360$$

$$d_2 = d_1 - \sigma_{PF}\sqrt{T} = 0.4436 - 0.2615\sqrt{2} = 0.07378$$

We have:

$$N(-d_1) = N(-0.44360) = 0.32867$$

$$N(-d_2) = N(-0.07378) = 0.47059$$

The value of the European put option is:

$$P_{Eur} = F_{0,T}^P(K)N(-d_2) - F_{0,T}^P(S)N(-d_1) = 87e^{-0.10\times2} \times 0.47059 - 78.3807 \times 0.32867$$

$$= 71.2296 \times 0.47059 - 78.3807 \times 0.32867 = 7.7586$$
Solution 45

E  Chapter 12, Currency Options and Black-Scholes

This question is similar to Question #7 of the SOA MFE/3F Sample Exam.

Statement (vii) contains a lot of information:

- Let’s define $x(t)$ to be the dollar per ruble exchange rate. Then (vii) tells us that:

$$\ln \left( \frac{1}{x(t)} \right) = \alpha dt + \sigma dZ(t) \quad \Rightarrow \quad \ln [x(t)] = -\alpha dt - \sigma dZ(t)$$

Since $-Z(t)$ is a standard Brownian motion, the logarithm of the dollar per ruble exchange rate can be said to have the same volatility as the ruble per dollar exchange rate. For both $\ln \left( \frac{1}{x(t)} \right)$ and $\ln [x(t)]$, the standard deviation is $\sigma \sqrt{t}$.

Therefore:

$$0.366397\% = \sigma \sqrt{\frac{1}{365}} \quad \Rightarrow \quad \sigma = 0.07$$

- Since the logarithm of the dollar per ruble exchange rate is an arithmetic Brownian motion, the Black-Scholes framework applies. This means that put and call options on the ruble can be priced using the Black-Scholes formula.

The currency option is a put option with the ruble as its underlying asset. The domestic currency is dollars, and the current value of the one ruble is:

$$x_0 = \frac{1}{26} \text{ dollars}$$

Since the option is at-the-money, the strike price is equal to the value of one ruble:

$$K = \frac{1}{26} \text{ dollars}$$

The domestic interest rate is 5%, and the foreign interest rate is 8%:

$$r = 5\%$$

$$r_f = 8\%$$

The volatility of the ruble per dollar exchange rate is equal to the volatility of the dollar per ruble exchange rate. We convert the daily volatility into annual volatility:

$$\sigma = \frac{\sigma_h}{\sqrt{h}} = \frac{0.00366397}{\sqrt{\frac{1}{365}}} = 0.00366397 \times \sqrt{365} = 0.07000$$
The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1) + (0.05 - 0.08 + 0.5 \times 0.07^2)0.25}{0.07\sqrt{0.25}}$$

$$d_1 = -0.19679$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.19679 - 0.07\sqrt{0.25} = -0.23179$$

We round $d_1$ and $d_2$ when using the normal distribution table:

$$N(-d_1) = N(0.19679) = 0.57800$$

$$N(-d_2) = N(0.23179) = 0.59165$$

The value of a put option on one ruble is:

$$P_{Eur}(S, K, \sigma, r, T, \delta) = Ke^{-rT}N(-d_2) - Se^{-r_f T}N(-d_1)$$

$$= \frac{1}{26} e^{-0.05(0.25)}(0.59165) - \frac{1}{26} e^{-0.08(0.25)}(0.57800)$$

$$= 0.000682522$$

Since the option is for 250,000,000 rubles the value of the put option in dollars is:

$$250,000,000 \times 0.000682522 = 170,630.50$$

Solution 46

C Chapter 12, Portfolio Delta and Elasticity

The elasticity of portfolio A is the weighted average of the elasticities of its components:

$$\Omega_{Port} = \sum_{i=1}^{n} \omega_i \Omega_i$$

$$2.82 = \frac{2 \times 10}{2 \times 10 + 5} \Omega_{Call} + \frac{5}{2 \times 10 + 5} \Omega_{Put}$$

$$70.5 = 20\Omega_{Call} + 5\Omega_{Put}$$

We can express the delta of an option in terms of the elasticity of the option:

$$\Omega = \frac{S\Delta}{V} \quad \Rightarrow \quad \Delta = \Omega \frac{V}{S}$$
The delta of portfolio B is the sum of the deltas of its components:

\[ \Delta_{Port} = \sum_{i=1}^{n} q_i \Delta_i \]

\[ 3.50 = 4 \Delta_{Call} - 5 \Delta_{Put} \]

\[ 3.50 = 4 \Delta_{Call} \frac{10}{86} - 5 \Delta_{Put} \frac{5}{86} \]

\[ 301 = 40 \Delta_{Call} - 25 \Delta_{Put} \]

We now have a system of 2 equations with 2 unknowns:

\[ 70.5 = 20 \Delta_{Call} + 5 \Delta_{Put} \]

\[ 301 = 40 \Delta_{Call} - 25 \Delta_{Put} \]

Let's subtract the second equation from twice the first equation:

\[ 2 \times 70.5 - 301 = 2 \times 20 \Delta_{Call} - 40 \Delta_{Call} + 2 \times 5 \Delta_{Put} - (-25 \Delta_{Put}) \]

\[ -160 = 35 \Delta_{Put} \]

\[ \Delta_{Put} = -4.5714 \]

**Solution 47**

D Chapter 12, Delta

The delta of a bear spread is the same regardless of whether it is constructed of calls or puts. Let's assume that the bear spread consists of calls.

Since the bear spread consists of purchasing the higher-strike call and selling the lower-strike call, the delta of the bear spread is:

\[ \Delta_{50} - \Delta_{45} \]

We can find the values of delta using:

\[ \Delta_{Call} = e^{-\delta T} N(d_1) \quad \text{and} \quad d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} \]

With \( T = 0.25 \) and \( K = 50 \), we have:

\[ d_1 = \frac{\ln(51/50) + \left(0.06 - 0.02 + 0.5(0.25)^2\right)(0.25)}{0.25 \sqrt{0.25}} = 0.30092 \]

\[ N(0.30092) = 0.61826 \]

\[ \Delta_{50} = e^{-0.02 \times 0.25} \times 0.61826 = 0.61518 \]
With \( T = 0.25 \) and \( K = 45 \), we have:

\[
d_1 = \frac{\ln(51/45) + \left(0.06 - 0.02 + 0.5(0.25)^2\right)(0.25)}{0.25\sqrt{0.25}} = 1.14381
\]

\[
N(1.14381) = 0.87365
\]

\[
\Delta_{45} = e^{-0.02\times0.25} \times 0.87365 = 0.86929
\]

Therefore, the delta of the bear spread is initially:

\[
\Delta_{50} - \Delta_{45} = 0.61518 - 0.86929 = -0.25412
\]

With \( T = \frac{2}{12} \) and \( K = 50 \), we have:

\[
d_1 = \frac{\ln(51/50) + \left(0.06 - 0.02 + 0.5(0.25)^2\right)\frac{2}{12}}{0.25\sqrt{\frac{2}{12}}} = 0.31038
\]

\[
N(0.31038) = 0.62186
\]

\[
\Delta_{50} = e^{-0.02\times2/12} \times 0.62186 = 0.61979
\]

With \( T = \frac{2}{12} \) and \( K = 45 \), we have:

\[
d_1 = \frac{\ln(51/45) + \left(0.06 - 0.02 + 0.5(0.25)^2\right)\frac{2}{12}}{0.25\sqrt{\frac{2}{12}}} = 1.34269
\]

\[
N(1.34269) = 0.91031
\]

\[
\Delta_{45} = e^{-0.02\times2/12} \times 0.91031 = 0.90728
\]

Therefore, the delta of the bear spread after 1 month is:

\[
\Delta_{50} - \Delta_{45} = 0.61979 - 0.90728 = -0.28749
\]

The change in the delta of the bear spread is:

\[-0.28749 - (-0.25412) = -0.03337\]

**Solution 48**

**B**  Chapter 12, Options on Currencies

Sam wants to give up $125,000 and receive €100,000 at the end of 1 year. This can be accomplished with 2 different purchases:

- 100,000 dollar-denominated call options on euros, each with a strike price of $1.25.
- 125,000 euro-denominated put options on dollars, each with a strike price of €0.80.

Therefore, the correct answer must be either Choice B or Choice E.

Let’s find the price of the dollar-denominated call options.
The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1.25/1.25) + (0.08 - 0.05 + 0.5 \times 0.10^2)(1)}{0.10\sqrt{1}} = 0.35000$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.35000 - 0.10\sqrt{1} = 0.25000$$

We have:

$$N(d_1) = N(0.35000) = 0.63683$$

$$N(d_2) = N(0.25000) = 0.59871$$

The value of the call option is:

$$C_{Eur} = x_0 e^{-r_f T} N(d_1) - Ke^{-r T} N(d_2)$$

$$= 1.25e^{-0.05(1)} \times 0.63683 - 1.25e^{-0.08(1)} \times 0.59871 = 0.06637$$

The value of 100,000 of the dollar-denominated call options is:

$$0.066337 \times 100,000 = $6,637$$

*If we had begun by finding the value of the 125,000 euro-denominated put options, we would have obtained a value of €5,309, implying that Choice E is not correct.*

**Solution 49**

**E** Chapter 12, Options on Currencies

Sam wants to give up $125,000 and receive €100,000 at the end of 1 year. This can be accomplished with 2 different purchases:

- 100,000 dollar-denominated call options on euros, each with a strike price of $1.25.
- 125,000 euro-denominated put options on dollars, each with a strike price of €0.80.

Therefore, the correct answer must be either Choice B or Choice E.

Let's find the price of the dollar-denominated call options.

The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1.25/1.25) + (0.08 - 0.05 + 0.5 \times 0.10^2)(1)}{0.10\sqrt{1}} = 0.35000$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.35000 - 0.10\sqrt{1} = 0.25000$$
We have:
\[ N(d_1) = N(0.35000) = 0.63683 \]
\[ N(d_2) = N(0.25000) = 0.59871 \]

The value of the call option is:
\[ C_{Eur} = x_0 e^{-rT} N(d_1) - Ke^{-rT} N(d_2) \]
\[ = 1.25e^{-0.05(1)} \times 0.63683 - 1.25e^{-0.08(1)} \times 0.59871 = 0.06637 \]

The value of 100,000 of the dollar-denominated call options is:
\[ $0.06637 \times 100,000 = $6,637 \]

Therefore, Choice B is not correct.

The 100,000 dollar-denominated calls with a strike price of $1.25 (described in Choice B) allow their owner the right to give up $125,000 to obtain €100,000. Likewise, 125,000 euro-denominated puts with a strike price of €0.80 (described in Choice E) allow their owner the right to give up $125,000 to obtain €100,000. Since both option positions have the same payoff, they must have the same current cost:
\[ $6,637 / €1.25 = €5,309 \]

Therefore, Choice E is correct.

Solution 50

D  Chapter 12, Elasticity

The contingent claim can be replicated with a portfolio consisting of the present value of $85 and a short position in a European put option with a strike price of $85.

To find the value of this contingent claim, we must find the value of the put option:

The first step is to calculate \( d_1 \) and \( d_2 \):
\[ d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(90 / 85) + (0.08 - 0.05 + 0.5 \times 0.35^2) \times 0.5}{0.35 \sqrt{0.5}} \]
\[ d_1 = 0.41531 \]
\[ d_2 = d_1 - \sigma \sqrt{T} = 0.41531 - 0.35 \sqrt{0.5} = 0.16782 \]

We have:
\[ N(-d_1) = N(-0.41531) = 0.33896 \]
\[ N(-d_2) = N(-0.16782) = 0.43336 \]

The value of the European put option is:
\[ P(85) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1) \]
\[ = 85e^{-0.08 \times 0.5} \times 0.43336 - 90e^{-0.05 \times 0.5} \times 0.33896 = 5.6381 \]

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The current value of the contingent claim is the present value of $85 minus the value of the put:

\[ V = 85e^{-0.08 \times 0.5} - 5.6381 = 76.0290 \]

The delta of the put option is:

\[ \Delta_{put} = -e^{-\delta T}N(-d_1) = -e^{-0.05 \times 0.5} \times 0.33896 = -0.33059 \]

The delta of the contingent claim is the delta of the present value of $85 (i.e., zero) minus the delta of the put:

\[ \Delta = 0 - (-0.33059) = 0.33059 \]

The elasticity of the contingent claim is:

\[ \Omega = \frac{SV \Delta}{V} = \frac{90 \times 0.33059}{76.0290} = 0.3913 \]

**Solution 51**

C Chapter 12, Elasticity

The contingent claim can be replicated with a portfolio consisting of the present value of $85 and a short position in a European call option with a strike price of $85.

To find the value of this contingent claim, we must find the value of the call option:

The first step is to calculate \( d_1 \) and \( d_2 \):

\[ d_1 = \frac{\ln(S/K) + (r - \delta + 0.5 \sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(90/85) + (0.08 - 0.05 + 0.5 \times 0.35^2) \times 0.5}{0.35 \sqrt{0.5}} = 0.41531 \]

\[ d_2 = d_1 - \sigma \sqrt{T} = 0.41531 - 0.35 \sqrt{0.5} = 0.16782 \]

We have:

\[ N(d_1) = N(0.41531) = 0.66104 \]
\[ N(d_2) = N(0.16782) = 0.56664 \]

The value of the European call option is:

\[ C(85) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = 90e^{-0.05 \times 0.5} \times 0.66104 - 85e^{-0.08 \times 0.5} \times 0.56664 = 11.74885 \]
The current value of the contingent claim is the present value of $85 minus the value of the call:

\[ V = 85e^{-0.08 \times 0.5} - 11.74885 = 69.91825 \]

The delta of the call option is:

\[ \Delta_{\text{Call}} = e^{-\delta T} N(d_1) = e^{-0.05 \times 0.5} \times 0.66104 = 0.64472 \]

The delta of the contingent claim is the delta of the present value of $85 (i.e., zero) minus the delta of the call:

\[ \Delta = 0 - 0.64472 = -0.64472 \]

The elasticity of the contingent claim is:

\[ \Omega = \frac{S\Delta}{V} = \frac{90 \times (-0.64472)}{69.91825} = -0.8299 \]

Solution 52

Chapter 12, Options on Futures

At time 0, the values of \( d_1 \) and \( d_2 \) are:

\[
\begin{align*}
  d_1 &= \frac{\ln(F_0 T_f / K) + 0.5\sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln(30/30) + 0.5\sigma^2 T}{\sigma \sqrt{T}} = 0.5\sigma \sqrt{T} \\
  d_2 &= d_1 - \sigma \sqrt{T} = 0.5\sigma \sqrt{T} - \sigma \sqrt{T} = -0.5\sigma \sqrt{T}
\end{align*}
\]

We can find \( N(-d_1) \) in terms of \( N(-d_2) \):

\[
\begin{align*}
  N(-d_1) &= N(-0.5\sigma \sqrt{T}) \\
  N(-d_2) &= N(0.5\sigma \sqrt{T}) \\
  N(-d_1) &= 1 - N(-d_2)
\end{align*}
\]

We can substitute this value of \( N(-d_1) \) into the time 0 formula for the price in order to find \( d_2 \):

\[
\begin{align*}
  P_{\text{Eur}}(F_0 T_f, K, \sigma, r, T, r) &= Ke^{-r T} N(-d_2) - F_0 T_f e^{-r T} N(-d_1) \\
  3.40 &= 30e^{-0.08 \times 0.75} N(-d_2) - 30e^{-0.08 \times 0.75} N(-d_1) \\
  3.40 &= 30e^{-0.06} [N(-d_2) - (1 - N(-d_2))] \\
  \frac{3.40}{30} e^{0.06} &= 2N(-d_2) - 1 \\
  N(-d_2) &= 0.56017 \\
  -d_2 &\approx 0.15140 \\
  d_2 &\approx -0.15140
\end{align*}
\]
We can now find $\sigma$:

\[
d_2 = -0.5\sigma\sqrt{T} \\
-0.15140 = -0.5\sigma\sqrt{0.75} \\
\sigma = 0.34964
\]

After 3 months, the new values of $d_1$ and $d_2$ are:

\[
d_1 = \frac{\ln \left( \frac{F_{0,T_F}}{K} \right) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln (27/30) + 0.5 \times 0.34964^2 \times 0.5}{0.34964\sqrt{0.5}} = -0.30254 \\
d_2 = d_1 - \sigma\sqrt{T} = -0.30254 - 0.34964\sqrt{0.5} = -0.54975
\]

We can look up the values of $N(-d_1)$ and $N(-d_2)$ from the normal distribution table:

\[
N(-d_1) = N(0.30254) = 0.61888 \\
N(-d_2) = N(0.54975) = 0.70876
\]

After 3 months, the value of the put option is:

\[
P_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) = Ke^{-rT}N(-d_2) - F_{0,T_F}e^{-rT}N(-d_1) \\
= 30e^{-0.08\times0.5} \times 0.70876 - 27e^{-0.08\times0.5} \times 0.61888 \\
= 4.3745
\]

**Solution 53**

**D** Chapter 12, Options on Futures

Put-call parity implies that the put and call options must be at-the-money:

\[
C_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) + Ke^{-rT} = F_{0,T_F}e^{-rT} + P_{Eur}(F_{0,T_F}, K, \sigma, r, T, r) \\
5.61 + Ke^{-0.08\times0.75} = 50e^{-0.08\times0.75} + 5.61 \\
K = 50
\]

At time 0, the values of $d_1$ and $d_2$ are:

\[
d_1 = \frac{\ln \left( \frac{F_{0,T_F}}{K} \right) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln (50/50) + 0.5\sigma^2T}{\sigma\sqrt{T}} = 0.5\sigma\sqrt{T} \\
d_2 = d_1 - \sigma\sqrt{T} = 0.5\sigma\sqrt{T} - \sigma\sqrt{T} = -0.5\sigma\sqrt{T}
\]

We can find $N(d_1)$ in terms of $N(d_2)$:

\[
N(d_1) = N \left( 0.5\sigma\sqrt{T} \right) \\
N(d_2) = N \left( -0.5\sigma\sqrt{T} \right) \\
N(d_1) = 1 - N(d_2)
\]
We can substitute this value of $N(d_1)$ into the time 0 formula for the price of the call option in order to find $d_2$:

$$C_{Eur}(F_{0,T_f}, K, \sigma, r, T, r) = F_{0,T_f} e^{-rT} N(d_1) - Ke^{-rT} N(d_2)$$

5.61 = 50e^{-0.08 \times 0.75} N(d_1) - 50e^{-0.08 \times 0.75} N(d_2)

5.61 = 50e^{-0.06}[N(d_1) - N(d_2)]

5.61 = 50e^{-0.06}[1 - N(d_2) - N(d_2)]

$$5.61 \times e^{-0.06} = 1 - 2N(d_2)$$

$N(d_2) = 0.44043$

$d_2 = -0.14988$

We can now find $\sigma$:

$$d_2 = -0.5\sigma\sqrt{T}$$

$-0.14988 = -0.5\sigma\sqrt{0.75}$

$\sigma = 0.34613$

After 3 months, the new values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln\left(F_{0,T_f} / K\right) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln(52/50) + 0.5 \times 0.34613^2 \times 0.5}{0.34613\sqrt{0.5}} = 0.28262$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.28262 - 0.34613\sqrt{0.5} = 0.03787$$

We can look up the values of $N(d_1)$ and $N(d_2)$ from the normal distribution table:

$N(d_1) = N(0.28262) = 0.61127$

$N(d_2) = N(0.03787) \approx N(0.04) = 0.51510$

After 3 months, the value of the call option is:

$$C_{Eur}(F_{0,T_f}, K, \sigma, r, T, r) = F_{0,T_f} e^{-rT} N(d_1) - Ke^{-rT} N(d_2)$$

$$= 52e^{-0.08 \times 0.5} \times 0.61127 - 50e^{-0.08 \times 0.5} \times 0.51510$$

$$= 5.7946$$
Solution 54

A Chapter 12, Holding Period Profit

We begin by finding the value of the put option today. The first step is to calculate $d_1$ and $d_2$:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5 \sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(80/90) + (0.06 - 0 + 0.5 \times 0.28^2) \times 4/12}{0.28 \times \sqrt{4/12}}$$

$$= -0.52405$$

$$d_2 = d_1 - \sigma \sqrt{T} = -0.52405 - 0.28 \times \sqrt{4/12} = -0.68571$$

We have:

$$N(-d_1) = N(0.52405) = 0.69988$$

$$N(-d_2) = N(0.68571) = 0.75355$$

The value of the European put option today is:

$$P_{Eur} = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

$$= 90e^{-0.06(4/12)} \times 0.75355 - 80e^{0.00(4/12)} \times 0.69988 = 10.4862$$

The holding period profit is current value of the position minus the cost of the position, including interest:

$$10.4862 - 7e^{0.06(8/12)} = 3.2005$$

Solution 55

A Chapter 12, Call Option Delta

The value of delta can be used to determine $d_1$:

$$\Delta_{Call} = e^{-\delta T}N(d_1)$$

$$0.6 = e^{-0.02 \times 1}N(d_1)$$

$$e^{0.02} = N(d_1)$$

$$N(d_1) = 0.61212$$

$$d_1 = 0.28485$$
The formula for $d_1$ is used to find a quadratic equation in terms of $\sigma$:

$$d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$0.28485 = \frac{\ln(S / S) + (0.0494 - 0.02 + 0.5\sigma^2) \times 1}{\sigma\sqrt{1}}$$

$$0.28485 = \frac{(0.0294 + 0.5\sigma^2)}{\sigma}$$

$$0.28485\sigma = 0.0294 + 0.5\sigma^2$$

$$0.5\sigma^2 - 0.28485\sigma + 0.0294 = 0$$

$$\sigma^2 - 0.5697\sigma + 0.0588 = 0$$

The two solutions to the quadratic equation are found below:

$$\sigma = \frac{0.5697 \pm \sqrt{(-0.5697)^2 - 4(1)(0.0588)}}{2}$$

$$\sigma = 0.13539 \quad \text{or} \quad \sigma = 0.43431$$

**Shortcut:** The higher the value of $\sigma$, the higher the call price. Since (i) tells us that the call price is less than 10% of the stock price, we can guess that the volatility will be the lower of the two values, so $\sigma = 13.539\%$. Below, we show that this guess is correct.

The value of $N(d_2)$ depends on the value of $\sigma$:

$$\sigma = 0.13539 \quad \Rightarrow \quad d_2 = d_1 - \sigma\sqrt{T} = 0.28485 - 0.13539 = 0.14946$$

$$\Rightarrow \quad N(d_2) = 0.55940$$

$$\sigma = 0.43431 \quad \Rightarrow \quad d_2 = d_1 - \sigma\sqrt{T} = 0.28485 - 0.43431 = -0.14946$$

$$\Rightarrow \quad N(d_2) = 1 - 0.55940 = 0.44060$$

For an at-the-money option, $K = S$. We are given that the call price is less than 10% of the stock price:

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

$$\frac{C}{S} = e^{-0.02}N(d_1) - e^{-0.0494}N(d_2)$$

$$e^{-0.02}N(d_1) - e^{-0.0494}N(d_2) < 0.10$$

$$0.6 - e^{-0.0494}N(d_2) < 0.10$$

$$0.50 < e^{-0.0494}N(d_2)$$

$$0.52532 < N(d_2)$$
For the inequality above to be satisfied, it must be the case that:
\[
\sigma = 0.13539
\]

**Solution 56**

**B Chapter 12, Black-Scholes Formula**

The prepaid forward volatility is:
\[
\sigma_{PF} = \sqrt{\frac{\text{Var} \left[ \ln \left( F_{t,0.5}^P(S) \right) \right]}{t}} = \sqrt{\frac{0.04}{t}} = 0.20
\]

The prepaid forward prices of the stock and the strike price are:
\[
F_{0,T}^P(S) = 70 - 4.00e^{-0.07(4/12)} = 66.0923
\]
\[
F_{0,T}^P(K) = 75e^{-0.07(0.5)} = 72.4204
\]

We use the prepaid forward volatility in the Black-Scholes Formula:
\[
d_1 = \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma_{PF}^2T}{\sigma_{PF}\sqrt{T}} = \frac{\ln \left( \frac{66.0923}{72.4204} \right) + 0.5 \times 0.20^2 \times 0.5}{0.20\sqrt{0.5}} = -0.57584
\]
\[
d_2 = d_1 - \sigma_{PF}\sqrt{T} = -0.57584 - 0.20\sqrt{0.5} = -0.71726
\]

We have:
\[
N(d_1) = N(-0.57584) = 0.28236
\]
\[
N(d_2) = N(-0.71726) = 0.23661
\]

The value of the European call option is:
\[
C_{\text{Eur}} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) = \left[ S_0 - PV_{0,T}(\text{Div}) \right] N(d_1) - Ke^{-rT}N(d_2)
\]
\[
= 66.0923 \times 0.28236 - 72.4204 \times 0.23661
\]
\[
= 1.5264
\]
Solution 57

D Chapter 12, Black-Scholes Formula

The prepaid forward price can be written in terms of the forward price:

\[ F_{t,T}^P(S) = e^{-r(T-t)} F_{t,T} \quad \Rightarrow \quad F_{t,T} = e^{r(T-t)} F_{t,T}^P(S) \]

The variance of the prepaid forward price is equal to the variance of the forward price:

\[
\begin{align*}
\text{Var} \left[ \ln \left( F_{t,0.75}^P(S) \right) \right] &= \text{Var} \left[ \ln \left( e^{r(0.75-t)} F_{t,0.75}(S) \right) \right] \\
&= \text{Var} \left[ \ln(e^{r(0.75-t)}) + \ln(F_{t,0.75}(S)) \right] \\
&= 0 + \text{Var} \left[ \ln(F_{t,0.75}(S)) \right] = 0.04 \times t
\end{align*}
\]

The prepaid forward volatility is:

\[
\sigma_{PF} = \sqrt{\frac{\text{Var} \left[ \ln \left( F_{t,0.75}^P(S) \right) \right]}{t}} = \sqrt{\frac{0.04t}{t}} = 0.20
\]

The prepaid forward prices of the stock and the strike price are:

\[
\begin{align*}
F_{0,T}^P(S) &= 70 - 5.00 e^{-0.08(6/12)} = 65.1961 \\
F_{0,T}^P(K) &= 75 e^{-0.08(0.75)} = 70.6323
\end{align*}
\]

We use the prepaid forward volatility in the Black-Scholes Formula:

\[
\begin{align*}
d_1 &= \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5 \sigma_{PF}^2 T}{\sigma_{PF} \sqrt{T}} = \frac{\ln \left( \frac{65.1961}{70.6323} \right) + 0.5 \times 0.20^2 \times 0.75}{0.20 \sqrt{0.75}} \\
&= -0.37579 \\
d_2 &= d_1 - \sigma_{PF} \sqrt{T} = -0.37579 - 0.20 \sqrt{0.75} = -0.54900
\end{align*}
\]

We have:

\[
\begin{align*}
N(-d_1) &= N(0.37579) = 0.64646 \\
N(-d_2) &= N(0.54900) = 0.70850
\end{align*}
\]

The value of the European put option is:

\[
\begin{align*}
P_{\text{Eur}} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) &= Ke^{-rT} N(-d_2) - \left[ S_0 - PV_{0,T}(\text{Div}) \right] N(-d_1) \\
&= 70.6323 \times 0.70850 - 65.1961 \times 0.64646 \\
&= 7.8964
\end{align*}
\]
Solution 58

D  Chapter 12, Theta

We can take the derivative of both sides of the put-call parity equation and solve for the theta of the put option:

\[
C_{Eur} + Ke^{-r(T-t)} = S_te^{-\delta(T-t)} + P_{Eur}
\]

\[
\theta_{Call} + \frac{d}{dt}\left(Ke^{-r(T-t)}\right) = \frac{d}{dt}\left(S_te^{-\delta(T-t)}\right) + \theta_{Put}
\]

\[
\theta_{Call} + rKe^{-r(T-t)} = \delta S_te^{-\delta(T-t)} + \theta_{Put}
\]

\[
-7.16 + 0.10 \times 105e^{-0.10(1-0)} = 0.05 \times 100e^{-0.05(1-0)} + \theta_{Put}
\]

\[
\theta_{Put} = -2.4154
\]

A straddle consists of a put and a call. Therefore, the theta of the straddle is the sum of the theta of the put plus the theta of the call:

\[
\theta_{Put} + \theta_{Call} = -2.4154 - 7.16 = -9.5754
\]

Solution 59

A  Chapter 12, Option Volatility

The delta of the put option can be used to solve for \(N(-d_1)\):

\[
\Delta_{Put} = -e^{-\delta T}N(-d_1)
\]

\[
-0.1514 = -e^{-0.07 \times 9}N(-d_1)
\]

\[
N(-d_1) = 0.28427
\]

We can use the standard normal table to find the value of \(d_1\):

\[
N(d_1) = 1 - 0.28427 = 0.71573
\]

\[
d_1 = 0.57020
\]

The formula for \(d_1\) can be used to find \(\sigma\):

\[
d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}
\]

\[
0.57020 = \frac{\ln(9/2) + (0.07 - 0.07 + 0.5\sigma^2)9}{\sigma\sqrt{9}}
\]

\[
0.57020 = 1.5\sigma
\]

\[
\sigma = 0.380133
\]

We can now find the value of \(d_2\):

\[
d_2 = d_1 - \sigma\sqrt{T} = 0.57020 - 0.380133\sqrt{9} = -0.57020
\]
Next, we find the elasticity of the option:
\[
\Omega = \frac{S\Delta}{V} = \frac{S \times e^{-\delta T} N(-d_1)}{Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)} = \frac{e^{-0.07 \times 9} N(-d_1)}{e^{-0.07 \times 9} N(-d_2) - e^{-0.07 \times 9} N(-d_1)}
\]

\[
= \frac{-0.1514}{e^{-0.63} \times 0.71573 - 0.1514} = -0.65886
\]

The volatility of the option is:
\[
\sigma_{Option} = \sigma_{Stock} \times |\Omega_{Option}| = 0.380133 \times |-0.65886| = 0.25045
\]

Solution 60

E Chapter 12, Option Volatility

Since \(\sigma_{Put}\) is defined as a standard deviation, it must be positive. Therefore, we use the following formula from Chapter 12 of the textbook:
\[
\sigma_{Put} = \sigma_{Stock} \times |\Omega_{Put}|
\]

The expected return on the stock is equal to the rate of price appreciation plus the dividend yield:
\[\alpha = 0.08 + 0.07 = 0.15\]

Since the Sharpe ratio of the call option is equal to the call option of the stock, we can find \(\sigma_{Stock}\):
\[\frac{\alpha - r}{\sigma} = 0.24\]
\[
\frac{0.15 - 0.09}{\sigma} = 0.24
\]
\[\sigma = 0.25\]
\[\sigma_{Stock} = 0.25\]

The values of \(d_1\) and \(d_2\) are:
\[
d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(105/100) + (0.09 - 0.07 + 0.5 \times 0.25^2)1}{0.25 \sqrt{1}} = 0.40016
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = 0.40016 - 0.25 \sqrt{1} = 0.15016
\]

We use the cumulative normal distribution calculator to obtain:
\[N(d_1) = N(0.40016) = 0.65548\]
\[N(d_1) = N(0.15016) = 0.55968\]
The value of the call option is:
\[ C_{\text{Eur}} = S e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) = 105 e^{-0.07} \times 0.65548 - 100 e^{-0.09} \times 0.55968 \]
\[ = 13.02148 \]

The value of the put option is:
\[ P_{\text{Eur}} = K e^{-rT} N(-d_2) - S e^{-\delta T} N(-d_1) = K e^{-rT} [1 - N(d_2)] - S e^{-\delta T} [1 - N(d_1)] \]
\[ = 100e^{-0.99} \times (1 - 0.55968) - 105e^{-0.07} \times (1 - 0.65548) \]
\[ = 6.51324 \]

The deltas of the call and put options are:
\[ \Delta_{\text{Call}} = e^{-\delta T} N(d_1) = e^{-0.07} \times 0.65548 = 0.61117 \]
\[ \Delta_{\text{Put}} = \Delta_{\text{Call}} - e^{-\delta T} = 0.61117 - e^{-0.07} = -0.32123 \]

The standard deviations are:
\[ \sigma_{\text{Call}} = \sigma_{\text{Stock}} \times |\Omega_{\text{Call}}| = 0.25 \times \left| \frac{S \Delta_{\text{Call}}}{C_{\text{Eur}}} \right| = 0.25 \times \left| \frac{105 \times 0.61117}{13.02148} \right| = 1.23205 \]
\[ \sigma_{\text{Put}} = \sigma_{\text{Stock}} \times |\Omega_{\text{Put}}| = 0.25 \times \left| \frac{S \Delta_{\text{Put}}}{P_{\text{Eur}}} \right| = 0.25 \times \left| \frac{105 \times (-0.32123)}{6.51324} \right| = 1.29463 \]

The ranked standard deviations are:
\[ 1.29463 > 1.23205 > 0.25 \]
\[ \sigma_{\text{Put}} > \sigma_{\text{Call}} > \sigma_{\text{Stock}} \]
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Chapter 13 – Solutions

Solution 1
A  Chapter 13, Delta-Hedging

The required investment can consist of the market-maker’s capital, or it can be borrowed. The delta of the position resulting from the sale of the 100 call options is:

\[
\text{Delta of a short position in 100 calls} = -100 \times 0.4998 = -49.98
\]

To delta-hedge the position, the market-maker purchases 49.98 shares of stock. The cost of this position is:

\[
49.98 \times 60 = 2,998.80
\]

The investment required is the cost of the stock minus the proceeds received from selling the options:

\[
2,998.80 - 100 \times 3.76 = 2,622.80
\]

Solution 2
C  Chapter 13, Market-Maker Profit

The market-maker sells 100 of the call options. From the market-maker’s perspective, the value of this short position is:

\[
-100 \times 3.90 = -390.00
\]

The delta of the position, from the perspective of the market-maker, is:

\[
\text{Delta of a short position in 100 calls} = -100 \times 0.5743 = -57.43
\]

To delta-hedge the position, the market-maker purchases 57.43 shares of stock. The value of this position is:

\[
57.43 \times 50 = 2,871.50
\]

Since the market-maker received only $390.00 from the sale of the calls, the differential must be borrowed. The value of the loan, from the perspective of the market-maker, is:

\[
390.00 - 2,871.50 = -2,481.50
\]

The initial position, from the perspective of the market-maker, is:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>-390.0000</td>
</tr>
<tr>
<td>Shares</td>
<td>2,871.5000</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>-2,481.5000</td>
</tr>
<tr>
<td>Net</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
After 1 day, the value of the options has changed by:
\[-100 \times (4.32 - 3.90) = -42.00\]

After 1 day, the value of the shares of stock has changed by:
\[57.43 \times (50.75 - 50.00) = 43.0725\]

After 1 day, the value of the funds that were borrowed at the risk-free rate has changed by:
\[-2,481.50 \left( e^{0.07/365} - 1 \right) = -0.4759\]

The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>-42.0000</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>43.0725</td>
</tr>
<tr>
<td>Interest</td>
<td>-0.4759</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>0.5966</td>
</tr>
</tbody>
</table>

The change in the value of the position is $0.60, and this is the overnight profit.

**Solution 3**

The market-maker sells 100 of the put options. From the perspective of the market-maker, the value of this position is:
\[-100 \times 3.04 = -304.00\]

The delta of the position, from the perspective of the market-maker, is:

\[\text{Delta of a short position in 100 puts} = -100 \times (-0.4257) = 42.57\]

To delta-hedge the position, the market-maker sells 42.57 shares of stock. From the perspective of the market-maker, the value of this position is:
\[-42.57 \times 50 = -2,128.50\]

These proceeds plus the $304.00 received from the sale of the puts are lent at the risk-free rate:
\[2,128.50 + 304.00 = 2,432.50\]

The initial position, from the perspective of the market-maker, is:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>-304.0000</td>
</tr>
<tr>
<td>Shares</td>
<td>-2,128.5000</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>2,432.5000</td>
</tr>
<tr>
<td>Net</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
After 1 day, the value of the options has changed by:
\[-100 \times (2.71 - 3.04) = 33.00\]

After 1 day, the value of the shares of stock has changed by:
\[-42.57 \times (50.75 - 50.00) = -31.9275\]

After 1 day, the value of the funds that were lent at the risk-free rate has changed by:
\[2,432.50 \left( e^{0.07/365} - 1 \right) = 0.4666\]

The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>33.000</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>-31.93</td>
</tr>
<tr>
<td>Interest</td>
<td>0.4666</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>1.5391</td>
</tr>
</tbody>
</table>

The overnight profit is $1.54.

**Solution 4**

**A** Chapter 13, Delta-Hedging

A ratio spread consists of a long position in one option and a short position in another option. The only difference between the two options is that they have different strike prices.

The delta of the position to be hedged is:
\[100 \times 0.4043 - 300 \times 0.5854 = -135.19\]

To delta-hedge the position, the market-maker purchases 135.19 shares of stock. The cost of this position is:
\[135.19 \times 60 = 8,111.40\]

The investment required is the cost of the stock minus the proceeds received from setting up the call ratio spread:
\[8,111.40 - (300 \times 4.83 - 100 \times 2.75) = 6,937.40\]

**Solution 5**

**B** Chapter 13, Market-Maker Profit

From the market-maker’s perspective, the initial value of the position to be hedged is:
\[-300 \times 4.83 + 100 \times 2.75 = -1,174.00\]

The delta of the position to be hedged, from the perspective of the market-maker, is:
\[-300 \times 0.5854 + 100 \times 0.4043 = -135.19\]
To delta-hedge the position, the market-maker purchases 135.19 shares of stock. The value of this position is:

$$135.19 \times 60 = 8,111.40$$

Since the market-maker received only $1,174.00 for entering into the call ratio spread, the differential must be borrowed. The value of the loan, from the perspective of the market-maker, is:

$$1,174.00 - 8,111.40 = -6,937.40$$

The initial position, from the perspective of the market-maker, is:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>-1,174.00</td>
</tr>
<tr>
<td>Shares</td>
<td>8,111.40</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>-6,937.40</td>
</tr>
<tr>
<td>Net</td>
<td>0.00</td>
</tr>
</tbody>
</table>

After 1 day, the value of the options has changed by:

$$-300 \times (6.04 - 4.83) + 100 \times (3.61 - 2.75) = -277.00$$

After 1 day, the value of the shares of stock has changed by:

$$135.19 \times (62.00 - 60.00) = 270.38$$

After 1 day, the value of the funds that were borrowed at the risk-free rate has changed by:

$$-6,937.40 \left( e^{0.09/365} - 1 \right) = -1.7108$$

The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>-277.00</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>270.38</td>
</tr>
<tr>
<td>Interest</td>
<td>-1.7108</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>-8.3308</td>
</tr>
</tbody>
</table>

The position experienced an overnight loss of $8.33.

**Solution 6**

C Chapter 13, Market-Maker Profit

From the market-maker’s perspective, the initial value of the position to be hedged is:

$$100 \times 1.60 - 200 \times 3.50 = -540.00$$

The delta of the position to be hedged, from the perspective of the market-maker, is:

$$100 \times (-0.2377) - 200 \times (-0.4146) = 59.15$$
To delta-hedge the position, the market-maker sells 59.15 shares of stock. The value of this position is:

\[-59.15 \times 60 = -3,549.00\]

The market-maker receives $540.00 for entering into the put ratio spread. The market-maker also receives $3,549 for the stock that it sells. The sum of the proceeds is lent at the risk-free rate of return:

\[540 + 3,549 = 4,089\]

The initial position, from the perspective of the market-maker, is:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>-540.0000</td>
</tr>
<tr>
<td>Shares</td>
<td>-3,549.0000</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>4,089.0000</td>
</tr>
<tr>
<td>Net</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

After 1 day, the value of the options has changed by:

\[100 \times (1.17 - 1.60) - 200 \times (2.73 - 3.50) = 111.00\]

After 1 day, the value of the shares of stock has changed by:

\[-59.15 \times (62.00 - 60.00) = -118.30\]

After 1 day, the value of the funds that were lent at the risk-free rate has changed by:

\[4,089.00 \left( e^{0.09/365} - 1 \right) = 1.0084\]

The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>111.0000</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>-118.3000</td>
</tr>
<tr>
<td>Interest</td>
<td>1.0084</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>-6.2916</td>
</tr>
</tbody>
</table>

The position experienced an overnight loss of $6.29.

**Solution 7**

**E** Chapter 13, Market-Maker Profit

The market-maker purchases 100 of the call options. From the market-maker’s perspective, the value of this position is:

\[100 \times 4.69 = 469.00\]

The delta of the position, from the perspective of the market-maker, is:

\[\text{Delta of long position in 100 calls} = 100 \times (0.5218) = 52.18\]
To delta-hedge the position, the market-maker sells 52.18 shares of stock. The value of this position is:

\[-52.18 \times 70 = -3,652.60\]

The market-maker receives $3,652.60 from the sale of the stock and has an outlay of $469 to purchase the calls. Therefore, the amount available to lend is:

\[3,652.60 - 469.00 = 3,183.60\]

The initial position, from the perspective of the market-maker, is:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>469.0000</td>
</tr>
<tr>
<td>Shares</td>
<td>-3,652.6000</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>3,183.6000</td>
</tr>
<tr>
<td>Net</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

After 1 day, the value of the options has changed by:

\[100 \times (3.68 - 4.69) = -101.00\]

After 1 day, the value of the shares of stock has changed by:

\[-52.18 \times (68.00 - 70.00) = 104.36\]

After 1 day, the value of the funds that were lent at the risk-free rate has changed by:

\[3,183.60 \left( e^{0.09/365} - 1 \right) = 0.7851\]

The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>-101.0000</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>104.3600</td>
</tr>
<tr>
<td>Interest</td>
<td>0.7851</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>4.1451</td>
</tr>
</tbody>
</table>

The overnight profit is $4.15.

Solution 8

D Chapter 13, Delta

* A call bull spread is described in Section 9.3 of the Chapter 9 Review Note.

A call bull spread consists of buying the low-strike call and selling the high-strike call. Therefore, the delta of the speculator’s position is:

\[1 \times 0.5218 - 1 \times 0.3994 = 0.1224\]
Solution 9

A Chapter 13, Market-Maker Profit

There is no need to calculate the first day’s profit to answer this question. It might help to envision the market-maker liquidating the position at the end of the first day and then immediately re-selling the 100 calls and delta-hedging them.

At the end of the first day, the market-maker is short 100 call options. From the market-maker’s perspective, the value of this position is:

\[-100 \times 3.68 = -368.00\]

The delta of the position, from the perspective of the market-maker, is:

\[
\text{Delta of short position in 100 calls} = -100 \times 0.4545 = -45.45
\]

To delta-hedge the position, the market-maker holds 45.45 shares of stock. The value of this position is:

\[45.45 \times 68.00 = 3,090.60\]

Rebalancing is equivalent to liquidating the position, re-selling the call options, and delta-hedging the position. Since the cost of the stock is greater than the proceeds from selling the call options, the market-maker must borrow (or equivalently, invest its own funds). The value of the loan, from the market-maker’s perspective, is:

\[368.00 - 3,090.60 = -2,722.60\]

After rebalancing, the position at the end of Day 1 is shown in the rightmost column below:

<table>
<thead>
<tr>
<th>Component</th>
<th>Day 0</th>
<th>Day 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>-469.0000</td>
<td>-368.0000</td>
</tr>
<tr>
<td>Shares</td>
<td>3,652.6000</td>
<td>3,090.6000</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>-3,183.6000</td>
<td>-2,722.6000</td>
</tr>
<tr>
<td>Net</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

We’ve shown the Day 0 values for information only. They do not need to be calculated to answer this question. For illustrative purposes, the Day 0 values are calculated below:

\[-469.00 = -100 \times 4.69\]

\[3,652.60 = 100 \times 0.5218 \times 70\]

\[-3,183.60 = 469.00 - 3,652.60\]

During the second day, the value of the options has changed by:

\[-100 \times (4.62 - 3.68) = -94.00\]

During the second day, the value of the shares of stock has changed by:

\[45.45 \times (70.00 - 68.00) = 90.90\]
During the second day, the value of the funds that were borrowed at the risk-free rate has changed by:

\[-2,722.60\left(e^{0.09/365} - 1\right) = -0.6714\]

The sum of these changes is the overnight profit shown in the Day 2 column below:

<table>
<thead>
<tr>
<th>Component</th>
<th>Day 1</th>
<th>Day 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>101.00</td>
<td>-94.00</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>-104.36</td>
<td>90.90</td>
</tr>
<tr>
<td>Interest</td>
<td>-0.7851</td>
<td>-0.6714</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>-4.1451</td>
<td>-3.7714</td>
</tr>
</tbody>
</table>

We’ve shown the Day 1 values for information only. They do not need to be calculated to answer this question. For illustrative purposes, the Day 1 Changes are determined below:

\[101.00 = -368.00 - (-469.00)\]
\[-104.36 = 0.5218 \times 100 \times 68 - 3,652.60\]
\[-0.7851 = -3 \times 183.60 \left(e^{0.09/365} - 1\right)\]

The market-maker experienced a loss of $3.77 during the second day.

**Solution 10**

**D** Chapter 13, Delta Approximation

The delta approximation is:

\[V(t + h) \approx V(t) + \varepsilon \Delta_t = 6.13 + 2 \times 0.5910 = 7.312\]

**Solution 11**

**D** Chapter 13, Delta-Gamma Approximation

The delta-gamma approximation is:

\[V(t + h) \approx V(t) + \varepsilon \Delta_t + \frac{\varepsilon^2}{2} \Gamma_t = 6.13 + 2 \times 0.5910 + \frac{2^2}{2} \times 0.0296 = 7.371\]

**Solution 12**

**C** Chapter 13, Delta-Gamma-Theta Approximation

The delta-gamma-theta approximation is:

\[V(t + h) \approx V(t) + \varepsilon \Delta_t + \frac{\varepsilon^2}{2} \Gamma_t + h\theta_t = 6.13 + 2 \times 0.5910 + \frac{2^2}{2} \times 0.0296 + \frac{1}{365} \times (-14.0317) = 7.333\]
Solution 13

B Chapter 13, Market-Maker Profit

The market-maker’s profit is:

\[
\text{Market-maker's profit} = -\frac{\epsilon^2}{2} \Gamma_t - h\theta_t - rh \left[ S_t \Delta_t - V(t) \right]
\]

\[
= -\frac{2^2}{2} (0.0296) - \frac{(-14.0317)}{365} - \frac{0.10}{365} (75 \times 0.5910 - 6.13)
\]

\[=-0.0312\]

Solution 14

B Chapter 13, Market-Maker Profit

This subject is discussed in Section 13.4 of the Chapter 13 Review Note.

The market-maker’s profit is:

\[
\text{Market-maker's profit} = -\frac{\epsilon^2}{2} \Gamma_t - h\theta_t - rh \left[ S_t \Delta_t - V(t) \right]
\]

We can use the formula above to analyze each of the statements.

A high value of gamma leads to a lower (i.e., more negative) value of \(-\frac{\epsilon^2}{2} \Gamma_t\), so Choice A is false.

A small stock price movement leads to a higher (i.e., less negative) value of \(-\epsilon^2 \Gamma_t\), so Choice B is true.

A high value of theta leads to a lower value of \(-h\theta_t\), so Choice C is false. For example, if theta is \(-12\), then the market-maker’s profit is less than it would be if theta were \(-13\). Therefore, the relatively high value of \(-12\) leads to a lower profit than the relatively low value of \(-13\).

A high interest rate makes the term \(-rh \left[ S_t \Delta_t - V(t) \right]\) smaller, since for a call option, \(\left[ S_t \Delta_t - V(t) \right]\) is positive. Therefore, Choice D is false.

The direction of the stock price change does not affect the market-maker’s profit. It is the magnitude of the change that affects the market-maker’s profit. Therefore Choice E is false.
Solution 15

D  Chapter 13, Market-Maker Profit

The market-maker’s profit is:

\[ \text{Market-maker's profit} = -\frac{\sigma^2}{2} \Gamma_t - h\theta_t - rh [S_t \Delta_t - V(t)] \]

We can use the formula above to analyze each of the statements.

A large stock price movement leads to a lower value of \(-\frac{\sigma^2}{2} \Gamma_t\), so Choice A is false.

A high value of gamma leads to a lower value of \(-\frac{\sigma^2}{2} \Gamma_t\), so Choice B is false.

A high value of theta leads to a lower value of \(-h\theta_t\), so Choice C is false. For example, if theta is –12, then the market-maker’s profit is less than it would be if theta were –13. Therefore, the relatively high value of –12 leads to a lower profit than the relatively low value of –13.

A high interest rate makes the term \(-rh [S_t \Delta_t - V(t)]\) larger, since the put option’s negative delta implies that \([S_t \Delta_t - V(t)]\) is negative. Therefore, Choice D is true.

The direction of the stock price change does not affect the market-maker’s profit. It is the magnitude of the change that affects the market-maker’s profit. Therefore Choice E is false.

Solution 16

B  Chapter 13, Black-Scholes Equation

The Black-Scholes equation can be used to find the price of the option:

\[ \frac{\sigma^2 S_t^2}{2} \Gamma_t + rS_t \Delta_t + \theta_t = rV(t) \]

\[ 0.3^2 \cdot \frac{30^2}{2} - 0.0866 + (0.08)(30)(0.4118) - 4.3974 = 0.08V(t) \]

\[ 0.09822 = 0.08V(t) \]

\[ V(t) = 1.22775 \]

The Black-Scholes equation does not require the time until maturity of the option to determine the price of the option.
Solution 17

E Chapter 13, Black-Scholes Equation

*The Black-Scholes equation works for put options as well as call options.*

The Black-Scholes equation can be used to find the price of the option:

\[
\frac{\sigma^2 S_t^2}{2} \Gamma_t + r S_t \Delta t + \theta_t = rV(t)
\]

\[
\frac{0.3^2 30^2}{2} - 0.0866 + (0.08)(30)(-0.5882) - 1.8880 = 0.08V(t)
\]

\[
0.20762 = 0.08V(t)
\]

\[
V(t) = 2.59525
\]

Solution 18

B Chapter 13, Black-Scholes Equation

The Black-Scholes equation can be used to find the price of the call option:

\[
\frac{\sigma^2 S_t^2}{2} \Gamma_t + r S_t \Delta t + \theta_t = rC(t)
\]

\[
\frac{0.32^2 50^2}{2} 0.0243 + (0.08)(50)(0.5901) - 4.9231 = 0.08C(t)
\]

\[
0.5477 = 0.08C(t)
\]

\[
C(t) = 6.84625
\]

Put-call parity can be used to find the value of the put option:

\[
C(t) + Ke^{-rT} = S_0 + P(t)
\]

\[
6.84525 + 53e^{-0.08(1)} = 50 + P(t)
\]

\[
P(t) = 5.77
\]

Solution 19

B Chapter 13, Frequency of Re-Hedging

The variance of the hourly returns of the delta-hedged position for one call option is:

\[
Var[R_h] = \frac{1}{2} \left( S^2 \sigma^2 \Gamma h \right)^2
\]

\[
Var \left[ R_{\frac{1}{365}} \right] = \frac{1}{2} \left( 50^2 \times 0.30^2 \times 0.0521 \times \frac{1}{365 \times 24} \right)^2
\]

\[
= 0.00000089537
\]
The standard deviation of the hourly returns for one call option is:

\[
\sqrt{\text{Var} \left[ R_{\frac{1}{365 \times 24}} \right]} = \sqrt{0.00000089537} = 0.00094624
\]

The standard deviation of the hourly returns for 100 of the call options is just 100 times the standard deviation for one call option:

\[100 \times 0.00094624 = 0.094624\]

**Solution 20**

D Chapter 13, Frequency of Re-Hedging

The variance of the daily returns of the delta-hedged position is the square of the standard deviation:

\[
\text{Var} \left[ R_{\frac{1}{365}} \right] = (2.271)^2 = 5.157441
\]

If the market-maker delta-hedges hourly, then the daily variance of the returns is:

\[
\frac{\text{Var}[R_h]}{n} = \frac{\text{Var} \left[ R_{\frac{1}{365}} \right]}{24} = \frac{1}{24} \times 5.157441 = 0.214893
\]

Therefore, with hourly re-hedging, the daily standard deviation is:

\[\sqrt{0.214893} = 0.46357\]

**Solution 21**

D Chapter 13, Frequency of Re-Hedging

Since the market-maker delta-hedges hourly, the daily variance of the returns for one call option is:

\[
\frac{\text{Var}[R_h]}{n} = \frac{1}{n} \times \frac{1}{2} \left(S^2 \sigma^2 \Gamma h \right) \]

\[
\text{Var} \left[ R_{\frac{1}{365}} \right] = \frac{1}{24} \times \frac{1}{2} \left(60^2 \times 0.30^2 \times 0.0434 \times \frac{1}{365} \right)^2 = 0.0000309202
\]

The daily standard deviation of the returns for one call option is:

\[\sqrt{0.0000309202} = 0.00556059\]

The standard deviation for 100 of the call options is just 100 times the standard deviation for one call option:

\[100 \times 0.00556059 = 0.556059\]
Solution 22

Chapter 13, Frequency of Re-Hedging

The gamma the 100 calls is:

$$100 \times 0.0434 = 4.340$$

The return on the delta-hedged position is:

$$R_h = \frac{1}{2} S^2 \sigma^2 \Gamma(1 - x^2) h$$

We can solve for the random values that produce a profit greater than $1.00:

$$1.00 < \frac{1}{2} \times 60^2 \times 0.30^2 \times 4.34 \times (1 - x^2) \times \frac{1}{365}$$

$$1.00 < 1.926247(1 - x^2)$$

$$0.51914 < 1 - x^2$$

$$x^2 < 0.48086$$

$$x < \sqrt{0.48086} \quad \text{and} \quad x > -\sqrt{0.48086}$$

$$x < 0.69344 \quad \text{and} \quad x > -0.69344$$

Since $x$ is a standard normal random variable, the probability that both of the inequalities above are satisfied is:

$$N(0.69344) - [1 - N(0.69344)]$$

We have:

$$N(0.69344) = 0.75598$$

The probability of the profit exceeding $1.00 is therefore:

$$N(0.69344) - [1 - N(0.69344)] = 0.75598 - (1 - 0.75598) = 0.51193$$

Solution 23

Chapter 13, Frequency of Re-Hedging

The question does not mention re-hedging, so we assume that the market-maker does not re-hedge the position after purchasing it.

We do not need to know the stock price, volatility, gamma, or interval length to answer this question. Regardless of those parameters, a market-maker expects to have profits about two-thirds of the time.

The return on the delta-hedged position is:

$$R_h = \frac{1}{2} S^2 \sigma^2 \Gamma(1 - x^2) h$$
We can solve for the random values that produce a positive profit:

\[ 0.00 < \frac{1}{2} S^2 \sigma^2 \cdot (1 - x^2) h \]

\[ (x^2 - 1) < 0.00 \]

\[ x^2 < 1.00 \]

\[ x < 1.00 \quad \text{and} \quad x > -1.00 \]

Since \( x \) is a standard normal random variable, the probability that both of the inequalities above are satisfied is:

\[
N(1.0000) - N(-1.0000) = N(1.0000) - \left[ 1 - N(1.0000) \right] = 2 \times N(1.0000) - 1
\]

\[
= 2 \times 0.84134 - 1 = 0.68268
\]

Since 68.27% is approximately two-thirds, the solution to this question supports the comment made in the paragraph immediately preceding Section 13.5 in the textbook that, "The market-maker thus expects to make small profits about two-thirds of the time."

**Solution 24**

**B** Chapter 13, Frequency of Re-Hedging

The variance of the 2-day return of the delta-hedged position for one call option is:

\[
\Var \left[ R_h \right] = \frac{1}{n} \times \frac{1}{2} \left( S^2 \sigma^2 \Gamma h \right)^2
\]

\[
\Var \left[ \frac{R_{2\text{days}}}{\text{2 days}} \right] = \frac{1}{2} \times \frac{1}{2} \left( S^2 \sigma^2 \Gamma \frac{2}{365} \right)^2 = \frac{1}{2} \times \frac{1}{2} \left( 80^2 \times 0.30^2 \times 0.02058 \times \frac{2}{365} \right)^2
\]

\[ = 0.0010548 \]

The standard deviation of the 2-day return of the delta-hedged position for one call option is:

\[ \sqrt{0.0010548} = 0.032477 \]

The standard deviation for 100 of the call options is just 100 times the standard deviation for one call option:

\[ 100 \times 0.032477 = 3.2477 \]

**Solution 25**

**D** Chapter 13, Delta-Gamma Hedging

The gamma of the position to be hedged is:

\[ -100 \times 0.0521 = -5.21 \]
We can now solve for the quantity, $Q$, of the $55$-strike call option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[-5.21 + 0.0441Q = 0.00\]

\[Q = 118.1406\]

**Solution 26**

A Chapter 13, Delta-Gamma Hedging

The gamma of the position to be hedged is:

\[-100 \times 0.0521 = -5.21\]

We can solve for the quantity, $Q$, of the $55$-strike call option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[-5.21 + 0.0441Q = 0.00\]

\[Q = 118.1406\]

The delta of the position becomes:

\[-100 \times 0.5824 + 118.1406 \times 0.3769 = -13.7128\]

The quantity of underlying stock that must be purchased, $Q_S$, is the opposite of the delta of the position being hedged:

\[Q_S = 13.7128\]

**Solution 27**

C Chapter 13, Delta-Gamma Hedging

The gamma of the position to be hedged is:

\[-100 \times 0.0521 = -5.21\]

We can solve for the quantity, $Q$, of the $55$-strike call option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[-5.21 + 0.0441Q = 0.00\]

\[Q = 118.1406\]

The delta of the position becomes:

\[-100 \times 0.5824 + 118.1406 \times 0.3769 = -13.7128\]

The quantity of underlying stock that must be purchased, $Q_S$, is the opposite of the delta of the position being hedged:

\[Q_S = 13.7128\]
The cost of purchasing the $55-strike options and the stock is offset by the value of the $50-strike options that are sold. The resulting cost of establishing the delta-hedged position is:

\[ 13.7128 \times 50 + 118.1406 \times 2.05 - 100 \times 3.48 = 579.83 \]

**Solution 28**

**E** Chapter 13, Delta-Gamma Hedging

The gamma of the position to be hedged is:

\[-100 \times 0.0521 = -5.21\]

We can solve for the quantity, \( Q \), of the $55-strike call option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[-5.21 + 0.0441Q = 0.00\]

\[ Q = 118.1406 \]

The delta of the position becomes:

\[-100 \times 0.5824 + 118.1406 \times 0.3769 = -13.7128\]

The quantity of underlying stock that must be purchased, \( Q_S \), is the opposite of the delta of the position being hedged:

\[ Q_S = 13.7128 \]

The cost of purchasing the $55-strike options and the stock is offset by the value of the $50-strike options that are sold. The resulting cost of establishing the delta-hedged position is:

\[ 13.7128 \times 50 + 118.1406 \times 2.05 - 100 \times 3.48 = 579.83 \]

The cost of establishing the position earns the risk-free rate of interest, so we treat it as borrowing.

After 1 day, the value of the $50-strike options has changed by:

\[-100 \times (4.06 - 3.48) = -58.00\]

After 1 day, the value of the $55-strike options has changed by:

\[118.1406 \times (2.43 - 2.05) = 44.8934\]

After 1 day, the value of the shares of stock has changed by:

\[13.7128 \times (51.00 - 50.00) = 13.7128\]

After 1 day, the value of the funds that were borrowed at the risk-free rate has changed by:

\[-579.83 \left( e^{0.08/365} - 1 \right) = -0.1271\]
The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain $50-Strike Call</td>
<td>−58.0000</td>
</tr>
<tr>
<td>Gain on $55-Strike Call</td>
<td>44.8934</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>13.7128</td>
</tr>
<tr>
<td>Interest</td>
<td>−0.1271</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>0.4791</td>
</tr>
</tbody>
</table>

The overnight profit is $0.4791.

Solution 29

B Chapter 13, Delta-Gamma Hedging

The gamma of the position to be hedged is:

\[ 100 \times 0.05212 = 5.212 \]

We can solve for the quantity, \( Q \), of the $55-strike put option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[ 5.212 + 0.04860Q = 0.00 \]
\[ Q = −107.2428 \]

The delta of the position becomes:

\[ 100 \times 0.58240 - 107.2428 \times (−0.66576) = 129.6380 \]

The quantity of underlying stock that must be purchased, \( Q_S \), is the opposite of the delta of the position being hedged:

\[ Q_S = −129.6380 \]

The proceeds from selling the $55-strike put options and the stock more than cover the cost of the $50-strike call options, so the cost (i.e., investment) to create this position is negative:

\[ −129.6380 \times 50 - 107.2428 \times 5.44 + 100 \times 3.48 = −6,717.2991 \]

Solution 30

A Chapter 13, Delta-Rho Hedging

The rho of the position to be hedged is:

\[ −100 \times −0.08229 = 8.229 \]

We can now solve for the quantity, \( Q \), of the $60-strike call option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[ 8.229 + 0.10083Q = 0.00 \]
\[ Q = −81.6126 \]
Since \( Q \) is negative, 81.6126 of the calls are sold.

The delta of the position becomes:

\[
-100 \times (-0.45837) - 81.6126 \times 0.73855 = -14.43800
\]

The quantity of underlying stock that must be purchased, \( Q_S \), is the opposite of the delta of the position being hedged:

\[
Q_S = 14.43800
\]

**Solution 31**

Chapter 13, Delta-Rho Hedging

The rho of the position to be hedged is:

\[
-100 \times -0.08229 = 8.229
\]

We can solve for the quantity, \( Q \), of the $60-strike call option that must be purchased to bring the hedged portfolio’s rho to zero:

\[
8.229 + 0.10083Q = 0.00
\]

\[
Q = -81.6126
\]

Since \( Q \) is negative, 81.6126 of the calls are sold.

The delta of the position becomes:

\[
-100 \times (-0.45837) - 81.6126 \times 0.73855 = -14.4380
\]

The quantity of underlying stock that must be purchased, \( Q_S \), is the opposite of the delta of the position being hedged:

\[
Q_S = 14.4380
\]

The proceeds from selling the puts and the calls are greater than the cost of purchasing the stock. Therefore the following amount is available to lend at the risk-free rate:

\[
100 \times 3.67 + 81.6126 \times 6.83 - 14.4380 \times 64 = 0.3824
\]

After 1 day, the change in the value of the put options is:

\[
-100 \times (3.22 - 3.67) = 45.00
\]

After 1 day, the change in the value of the call options is:

\[
-81.6126 \times (7.55 - 6.83) = -58.7611
\]

After 1 day, the change in the value of the stock is:

\[
14.4380 \times (65.00 - 64.00) = 14.4380
\]

After 1 day, the change in the risk-free asset is:

\[
0.3824 \times \left( e^{0.08/365} - 1 \right) = 0.0001
\]
The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on $65-strike Put Options</td>
<td>45.0000</td>
</tr>
<tr>
<td>Gain on $60-strike Call Options</td>
<td>-58.7611</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>14.4380</td>
</tr>
<tr>
<td>Interest</td>
<td>0.0001</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>0.6770</td>
</tr>
</tbody>
</table>

The overnight profit is $0.6770.

Alternatively, we can find the value of the overnight profit by calculating the new value of the hedged position:

\[
\text{Overnight Profit} = -100 \times 3.22 - 81.6126 \times 7.55 + 14.4380 \times 65.00 + 0.3824 \times e^{0.08/365} = 0.6770
\]

**Solution 32**

C Chapter 13, Delta-Gamma-Rho Hedging

The rho of the position to be hedged is:

\[1 \times 0.08951 = 0.08951\]

The gamma of the position to be hedged is:

\[1 \times 0.03723 = 0.03723\]

We must solve for the quantity of the $80-strike call options, \(Q_{80}\), to purchase and the quantity of the $65-strike put options, \(Q_{65}\), to purchase in order to offset the rho and gamma of the $70-strike call option:

\[
0.11018Q_{80} - 0.21484Q_{65} = -0.08951 \\
0.02552Q_{80} + 0.01524Q_{65} = -0.03723
\]

The solution to this system of equations is:

\[Q_{80} = -1.307290 \]
\[Q_{65} = -0.253804\]

Since the values above are negative, both options are sold.

The delta of the position becomes:

\[-1.307290 \times 0.36647 - 0.253804 \times (-0.25344) + 1 \times 0.58240 = 0.167642\]

To offset this delta, the market-maker must sell 0.167642 shares. The value of these shares is:

\[0.167642 \times 70 = 11.7349\]
Solution 33

Chapter 13, Delta-Gamma-Vega Hedging

The vega of the position to be hedged is:

\[ 1 \times 0.13645 = 0.13645 \]

The gamma of the position to be hedged is:

\[ 1 \times 0.03723 = 0.03723 \]

We must solve for the quantity of the $80-strike call options, \( Q_{80} \), to purchase and the quantity of the $65-strike put options, \( Q_{65} \), to purchase in order offset the vega and gamma of the $70-strike call option:

\[
0.18502Q_{80} + 0.22406Q_{65} = -0.13645 \\
0.02552Q_{80} + 0.01524Q_{65} = -0.03723
\]

The solution to this system of equations is:

\[
Q_{80} = -2.160660 \\
Q_{65} = 1.175200
\]

The hedge is constructed by selling 2.160660 of the $80-strike call options and purchasing 1.175200 of the $65-strike put options.

The delta of the position becomes:

\[
-2.160660 \times 0.36647 + 1.175200 \times (-0.25344) + 1 \times 0.58240 = -0.507260
\]

To offset this delta, the market-maker must purchase 0.507260 shares.

Solution 34

Chapter 13, Delta-Gamma-Rho-Vega Hedging

European options on the same stock that have the same maturity date have the same ratio of gamma to vega. This implies that if the option to be hedged has the same time until maturity as the options used in the hedge, then hedging gamma is equivalent to hedging vega.

The rho of the position to be hedged is:

\[ -1 \times 0.1280 = -0.1280 \]

The vega of the position to be hedged is:

\[ -1 \times 0.1360 = -0.1360 \]

The gamma of the position to be hedged is:

\[ -1 \times 0.0272 = -0.0272 \]
We must solve for the quantity of the $85-strike call options, $Q_{85}$, to purchase and the quantity of the $90-strike put options, $Q_{90}$, to purchase in order to offset the rho and gamma of the $70-strike call option:

\[
\begin{align*}
&\text{rho:} & 0.1033Q_{85} - 0.1432Q_{90} = 0.1280 \\
&\text{vega:} & 0.1565Q_{85} + 0.1540Q_{90} = 0.1360 \\
&\text{gamma:} & 0.0313Q_{85} + 0.0308Q_{90} = 0.0272
\end{align*}
\]

We appear to have 3 equations and 2 unknowns in the system of equations, but notice that the second equation is 5 times the third equation. Therefore, any solution that satisfies the third equation also satisfies the second equation, so we can drop the second equation from the system:

\[
\begin{align*}
&\text{rho:} & 0.1033Q_{85} - 0.1432Q_{90} = 0.1280 \\
&\text{gamma:} & 0.0313Q_{85} + 0.0308Q_{90} = 0.0272
\end{align*}
\]

The solution to this system of equations is:

\[
\begin{align*}
Q_{85} &= 1.022657 \\
Q_{90} &= -0.156142
\end{align*}
\]

The hedge is constructed by purchasing 1.022657 of the $85-strike call options and selling 0.156142 of the $90-strike put options. The delta of the position becomes:

\[
1.022657 \times 0.5570 - 0.156142 \times (-0.5942) - 1 \times 0.7082 = -0.045800
\]

To offset this delta, the market-maker must purchase 0.045800 shares. The value of these shares is:

\[
0.045800 \times 84.18 = 3.86
\]

**Solution 35**

E Chapter 13, Delta-Gamma-Rho-Vega Hedging

The rho of the position to be hedged is:

\[-1 \times 0.1280 = -0.1280\]

The vega of the position to be hedged is:

\[-1 \times 0.1360 = -0.1360\]

The gamma of the position to be hedged is:

\[-1 \times 0.0272 = -0.0272\]
We must solve for the quantity of the $85-strike call options, $Q_{85}$, to purchase and the quantity of the $90-strike put options, $Q_{90}$, to purchase in order to offset the rho and gamma of the $70-strike call option:

- **rho:** \[0.1033Q_{85} - 0.1432Q_{90} = 0.1280\]
- **vega:** \[0.1565Q_{85} + 0.1540Q_{90} = 0.1360\]
- **gamma:** \[0.0313Q_{85} + 0.0308Q_{90} = 0.0272\]

We appear to have 3 equations and 2 unknowns in the system of equations, but notice that the second equation is 5 times the third equation. Therefore, any solution that satisfies the third equation also satisfies the second equation, so we can drop the second equation from the system:

\[0.1033Q_{85} - 0.1432Q_{90} = 0.1280\]
\[0.0313Q_{85} + 0.0308Q_{90} = 0.0272\]

The solution to this system of equations is:

\[Q_{85} = 1.022657\]
\[Q_{90} = -0.156142\]

The hedge is constructed by purchasing 1.022657 of the $85-strike call options and selling 0.156142 of the $90-strike put options. The delta of the position becomes:

\[1.022657 \times 0.5570 - 0.156142 \times (-0.5942) - 1 \times 0.7082 = -0.045800\]

To offset this delta, the market-maker must purchase 0.045800 shares.

The funds raised by selling the $80-strike call option and the $90-strike put option are a bit more than the funds required to buy the $85-strike call option and the stock:

\[1 \times 8.27 - 1.022657 \times 5.44 + 0.156142 \times 7.42 - 0.045800 \times 84.18 = 0.009876\]

Therefore, 0.009876 is lent at the risk-free rate.

After 1 day, the change in the value of the $80-strike call option is:

\[-1 \times (14.48 - 8.27) = -6.21\]

After 1 day, the change in the value of the 1.022657 $85-strike calls is:

\[1.022657 \times (10.64 - 5.44) = 5.3178\]

After 1 day, the change in the value of the –0.156142 $90-strike puts is:

\[-0.156142 \times (3.68 - 7.42) = 0.5840\]

After 1 day, the change in the value of the 0.045800 shares of stock is:

\[0.045800 \times (92.00 - 84.18) = 0.3582\]
After 1 day, the change in the risk-free asset is quite small:

\[
0.009876 \times \left(e^{0.08/365} - 1\right) = 0.000002
\]

The sum of these changes is the overnight profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on $80-strike call</td>
<td>−6.2100</td>
</tr>
<tr>
<td>Gain on $85-strike call</td>
<td>5.3178</td>
</tr>
<tr>
<td>Gain on $90-strike put</td>
<td>0.5840</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>0.3582</td>
</tr>
<tr>
<td>Interest</td>
<td>0.0000</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>0.0499</td>
</tr>
</tbody>
</table>

The overnight profit is $0.0499.

**Solution 36**

C Chapter 13, Delta-Hedging and the Black-Scholes Equation

*The page references below refer to the Second Edition of Derivatives Markets.*

Let’s consider each of the statements.

Statement A is true. It is a restatement of the first sentence of the “Delta-Hedging of American Options” section on page 430 of the textbook.

Statement B is true. It is a restatement of the sentence following the Example 13.3 on page 432.

Statement C is false. It is contradicted by the last sentence of the second paragraph of page 416. The graph on page 426 provides a nice illustration of the fact that delta underestimates the increase in the price of the option when the stock price increases.

Statement D is true. It is a restatement of the first sentence of the third paragraph on page 416.

Statement E is true. It is a restatement of the second paragraph on page 429.

**Solution 37**

E Chapter 13, Static Option Replication

*Static option replication is described in Section 13.5 of the textbook.*

Static option replication is a strategy that requires little if any rebalancing. Put-call parity tells us that a call option is equivalent to a share of stock, plus a put option, plus borrowing at the risk-free rate:

\[
C_{Eur} = S_0 + P_{Eur} - Ke^{-rT}
\]
Since the market-maker has written 100 calls, the components of a static hedge include purchasing 100 shares of the stock and 100 of the $85-strike put options. The market-maker may choose to borrow to fund this position or to invest its own funds.

Solution 38

A Chapter 13, Delta-Hedging

The table in the question is similar to Table 13.2 of the textbook.

For writing the 100 options, the market-maker receives:

\[ 100 \times 8.79 = 879.00 \]

The market-maker wrote 100 of the call options, so the initial delta of the position is:

\[ -100 \times 0.7493 = -74.93 \]

The market-maker hedges this position by purchasing 74.93 shares of the stock for:

\[ 74.93 \times 80.00 = 5,994.40 \]

Since the cost of the stock exceeds the price of the options, the market-maker borrows:

\[ 5,994.40 - 879.00 = 5,115.40 \]

At the end of Day 1, the change in the value of the options is:

\[ -100 \times (9.51 - 8.79) = -72.00 \]

At the end of Day 1, the change in the value of the stock is:

\[ 74.93 \times (81.00 - 80.00) = 74.93 \]

At the end of Day 1, the change in the value of the borrowed funds is:

\[ -5,115.40 \times \left( e^{0.10/365} - 1 \right) = -1.4017 \]

The first day’s profit is the sum of these changes:

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>-72.0000</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>74.9300</td>
</tr>
<tr>
<td>Interest</td>
<td>-1.4017</td>
</tr>
<tr>
<td>Daily Profit</td>
<td>1.5283</td>
</tr>
</tbody>
</table>

The first day’s profit is $1.5283.
Solution 39

Chapter 13, Delta-Hedging

The table in the question is similar to Table 13.2 of the textbook.

We can treat each day’s re-hedging as a liquidation of the position, followed by the recreation of the position. The liquidation produces each day’s profits or losses. The recreation of the position creates no cash flows at the outset since any funds needed are borrowed.

For writing the 100 options at the end of Day 4, the market-maker receives:

\[ 100 \times 10.20 = 1,020.00 \]

The market-maker wrote 100 of the call options, so the delta at the end of Day 4 is:

\[ -100 \times 0.8010 = -80.10 \]

The market-maker hedges this position by purchasing 80.10 shares of the stock for:

\[ 80.10 \times 82.00 = 6,568.20 \]

Since the cost of the stock exceeds the price of the options, the market-maker borrows:

\[ 6,568.20 - 1,020.00 = 5,548.20 \]

At the end of Day 5, the change in the value of the options is:

\[ -100 \times (10.98 - 10.20) = -78.00 \]

At the end of Day 5, the change in the value of the stock is:

\[ 80.10 \times (83.00 - 82.00) = 80.10 \]

At the end of Day 5, the change in the value of the borrowed funds is:

\[ -5,548.20 \times \left( e^{0.10/365} - 1 \right) = -1.5203 \]

The fifth day’s profit is the sum of these changes:

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>-78.0000</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>80.1000</td>
</tr>
<tr>
<td>Interest</td>
<td>-1.5203</td>
</tr>
<tr>
<td>Daily Profit</td>
<td>0.5797</td>
</tr>
</tbody>
</table>

The fifth day’s profit is $0.5797.
**Solution 40**

**C** Chapter 13, Delta-Gamma Hedging

A butterfly spread can be created by buying a $90-strike call, selling 2 $95-strike calls, and buying a $100-strike call. In this case, the market-maker writes (or sells) the butterfly spread, so the market-maker sells the $90-strike call, buys 2 $95-strike calls, and sells a $100-strike call.

The gamma of the position to be hedged is:

\[-1 \times 0.02502 + 2 \times 0.02807 - 1 \times 0.02774 = 0.00338\]

We can now solve for the quantity, \(Y\), of the 180-day call option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[0.00338 + 0.01956Y = 0.00\]

\[Y = -0.172802\]

The delta of the position becomes:

\[-1 \times 0.69088 + 2 \times 0.55464 - 1 \times 0.41878 - 0.172802 \times 0.59575 = -0.103327\]

The quantity of underlying stock that must be purchased, \(X\), is the opposite of the delta of the position being hedged:

\[X = 0.103327\]

The ratio is:

\[\frac{X}{Y} = \frac{0.103327}{-0.172802} = -0.597949\]

**Solution 41**

**B** Chapter 13, Delta-Hedging

*We are not told the interval of time over which the stock price increases. Instead, we are told that the stock price "quickly jumps." We can take this to mean that the stock price increase is instantaneous. Therefore there is no need to consider the impact from borrowing or lending at the risk-free rate.*

The delta of a call option covering 1 share is:

\[\Delta_{\text{Call}} = e^{-\delta T} N(d_1) = e^{-0.05 \times 1} N(0.41000) = e^{-0.05 \times 1} \times 0.65910 = 0.626955\]

The market-maker sold 100 call options, each of which covers 100 shares. Therefore, from the perspective of the market-maker, the delta of the position to be hedged is:

\[-100 \times 100 \times 0.626955 = -6,269.55\]
To delta-hedge the position, the market-maker purchases 6,269.55 shares of stock. When the stock price increases, the market-maker has a gain on the stock and a loss on the call options:

- Gain on Stock: \(6,269.55 \times (52 - 51) = 6,269.55\)
- Gain on Calls: \(-100 \times 63.90 = -6,390.00\)
- Net Profit: \(-120.45\)

The market-maker has a net loss of 120.45.

**Solution 42**

**Chapter 13, Market-Maker Profit**

The market-maker’s profit is zero if the stock price movement is one standard deviation:

One-standard-deviation move = \(\sigma S_t \sqrt{h}\)

To answer the question, we must determine the value of \(\sigma\). We can determine the value of \(N(d_1)\):

\[
\Delta_{Call} = e^{-\delta T} N(d_1)
\]

\[
0.591 = e^{0.0(0.25)} \times N(d_1)
\]

\[N(d_1) = 0.59100\]

From the normal distribution table, this implies that:

\[d_1 = 0.23012\]

The formula for \(d_1\) can be used to solve for the value of \(\sigma\):

\[d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}}\]

\[0.23012 = \frac{\ln(50/50) + (0.10 - 0 + 0.5\sigma^2)(0.25)}{\sigma \sqrt{0.25}}\]

\[0.11506\sigma = 0.025 + 0.125\sigma^2\]

\[0.125\sigma^2 - 0.11506\sigma + 0.025 = 0\]

We use the quadratic formula:

\[\sigma = \frac{0.11506 \pm \sqrt{(-0.11506)^2 - 4(0.125)(0.025)}}{2(0.125)}\]

\[\sigma = 0.568964 \quad \text{or} \quad \sigma = 0.351516\]

We are told that the volatility is less than 0.50, so \(\sigma = 0.351516\).
The price of the stock moves by:

$$\sigma S_t \sqrt{h} = 0.351516 \times 50 \times \sqrt{\frac{1}{365}} = 0.91996$$

The stock price moves either up or down by 0.92.

**Solution 43**

**E** Chapter 13, Frequency of Re-Hedging

If the market-maker re-hedges only once per year, then the variance of the annual profit is:

$$\text{Var}[R_1] = \frac{1}{2} \left(S^2 \sigma^2 \Gamma h\right)^2 = \frac{1}{2} \left(65^2 \times 0.14^2 \times 0.0834 \times 1\right)^2 = 23.8489$$

If the market-maker re-hedges $n$ times per year, then the variance of the annual profit is:

$$\frac{\text{Var}[R_1]}{n} = \frac{23.8489}{n}$$

The standard deviation of the annual profit does not exceed $2.184$:

$$\sqrt{\frac{23.8489}{n}} \leq 2.184$$

$$23.8489 \leq 2.184^2 n$$

$$4.9999 \leq n$$

Therefore rebalancing must occur at least 4.9999 times per year. The number of days between rebalancing is $X = \frac{365}{n}$. Since $n$ must be greater than or equal to 4.9999, $X$ must be less than or equal to 73:

$$4.9999 \leq n$$

$$\frac{1}{n} \leq \frac{1}{4.9999}$$

$$\frac{365}{n} \leq \frac{365}{4.9999}$$

$$\frac{365}{n} \leq 73$$

$$X \leq 73$$
Solution 44

D  Chapter 13, Frequency of Re-Hedging

If the re-hedging occurs once per month, then the variance of the monthly profit for one option is:

\[ \text{Var} \left( R_{\frac{1}{12}} \right) = \frac{1}{2} \left( S^2 \sigma^2 T \right)^2 = \frac{1}{2} \left( 65^2 \times 0.14^2 \times 0.0834 \times \frac{1}{12} \right)^2 = 0.165617 \]

If the market-maker re-hedges \( n \) times per month, then the variance of the monthly profit is for one option:

\[ \text{Var} \left( R_{\frac{1}{12}} \right) = \frac{1}{n} \times 0.165617 \]

The standard deviation of the monthly return for 100 of the options is 100 times the standard deviation of the monthly return for one option:

\[ 100 \times \sqrt{\frac{0.165617}{n}} = \frac{40.6961}{\sqrt{n}} \]

The market-maker chooses the re-hedging frequency so that the monthly returns have a standard deviation of no more than $18.20:

\[ \frac{40.6961}{\sqrt{n}} \leq 18.20 \]

\[ \frac{40.6961^2}{18.20^2} \leq n \]

\[ 4.9999 \leq n \]

Therefore rebalancing must occur at least 4.9999 times per month. The number of days between rebalancing is \( X = \frac{30}{n} \). Since \( n \) must be greater than or equal to 4.9999, \( X \) must be less than or equal to 6:

\[ 4.9999 \leq n \]

\[ \frac{1}{n} \leq \frac{1}{4.9999} \]

\[ \frac{30}{n} \leq \frac{30}{4.9999} \]

\[ \frac{30}{n} \leq 6 \]

\[ X \leq 6 \]
Solution 45

D  Chapter 13, Delta-Gamma Hedging

The gamma of the position to be hedged is:

\[-1,000 \times 0.0422 = -42.20\]

We can solve for the quantity, \( Q \), of the other call option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[-42.2 + 0.0414Q = 0.00\]

\[Q = 1,019.3\]

Since only choice D specifies the purchase of 1,019.3 units of Call-II, we already have enough information to see that Choice D must be the correct answer.

The delta of the position becomes:

\[-1,000 \times 0.4125 + 1,019.3 \times 0.6159 = 215.3\]

The quantity of underlying stock that must be purchased, \( Q_S \), is the opposite of the delta of the position being hedged:

\[Q_S = -215.3\]

Therefore, in order to delta-hedge and gamma-hedge the position, we must sell 215.3 units of stock and purchase 1,019.3 units of Call-II.

Solution 46

B  Chapter 13, Frequency of Re-Hedging

If the market-maker re-hedges only once per day, then the variance of the daily profit for one option is:

\[Var \left[ R_{\frac{1}{365}} \right] = \frac{1}{2} \left( \sigma^2 \Gamma h \right)^2 = \frac{1}{2} \left( 85^2 \times 0.25^2 \times 0.0187 \times \frac{1}{365} \right)^2 = 0.000267611\]

If the market-maker re-hedges \( n \) times per day, then the variance of the daily profit for one option is:

\[Var \left[ R_{\frac{1}{365}} \bigg| n \text{ re-hedgings} \right] = \frac{Var \left[ R_{\frac{1}{365}} \right]}{n} = \frac{0.000267611}{n}\]

If the market-maker re-hedges \( n \) times per day, then the standard deviation of the daily profit for 1,000 options is 1,000 times the standard deviation for one option:

\[1,000 \times \sqrt{\frac{0.000267611}{n}} = 16.3588\]
The market-maker chooses the re-hedging frequency so that the daily returns have a standard deviation of no more than $5.79:

\[
\frac{16.3588}{\sqrt{n}} \leq 5.79
\]

\[
\frac{16.3588^2}{5.79^2} \leq n
\]

\[
7.9826 \leq n
\]

Therefore rebalancing must occur at least 7.9826 times per day. The number of hours between rebalancing is \(X = \frac{24}{n}\). Since \(n\) must be greater than or equal to 7.9826, \(X\) must be less than or equal to 3.0065:

\[
7.9826 \leq n
\]

\[
1 \leq \frac{1}{n} \\ \leq \frac{7.9826}{24}
\]

\[
24 \leq \frac{24}{7.9826}
\]

\[
24 \leq 3.0065
\]

\[
X \leq 3.0065
\]

**Solution 47**

**A** Chapter 13, Market-Maker Profit

Let’s assume that the options expire at time \(T\) and that the current time is time \(t\). We can use put-call parity to obtain a system of 2 equations.

\[
8.85 \times Ke^{-r(T-t)} = 47 + 5.66
\]

\[
12.13 + Ke^{-rT} = 50 + 4.53
\]

This can be solved to find \(e^{rt}\):

\[
\left\{ \begin{array}{l}
Ke^{-r(T-t)} = 43.81 \\
Ke^{-rT} = 42.40
\end{array} \right. \\
\Rightarrow e^{rt} = \frac{43.81}{42.40} = 1.03325
\]

The market-maker sold 100 of the call options. From the market-maker’s perspective, the value of this short position was:

\[-100 \times 12.13 = -1,213.00\]

The delta of the position, from the perspective of the market-maker, was:

\[
\text{Delta of a short position in 100 calls} = -100 \times (0.726) = -72.6
\]
To delta-hedge the position, the market-maker purchased 72.6 shares of stock. The value of this position was:

\[ 72.6 \times 50 = 3,630.00 \]

Since the market-maker received only $1,213.00 from the sale of the calls, the differential was borrowed. The value of the loan, from the perspective of the market-maker, was:

\[ -3,630.00 + 1,213.00 = -2,417.00 \]

The initial position, from the perspective of the market-maker, was:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>-1,213.00</td>
</tr>
<tr>
<td>Shares</td>
<td>3,630.00</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>-2,417.00</td>
</tr>
<tr>
<td>Net</td>
<td>0.00</td>
</tr>
</tbody>
</table>

After \( t \) years elapsed, the value of the options changed by:

\[ -100 \times (8.85 - 12.13) = 328.00 \]

After \( t \) years elapsed, the value of the shares of stock changed by:

\[ 72.6 \times (47.00 - 50.00) = -217.80 \]

After \( t \) years elapsed, the value of the funds that were borrowed at the risk-free rate changed by:

\[ -2,417.00 (e^{rt} - 1) = -2,417.00 (1.03325 - 1) = -80.38 \]

The sum of these changes is the profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>328.00</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>-217.80</td>
</tr>
<tr>
<td>Interest</td>
<td>-80.38</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>29.82</td>
</tr>
</tbody>
</table>

The change in the value of the position is $29.82, and this is the profit.

**Solution 48**

**D** Chapter 13, Market-Maker Profit

We do not have enough information to use the standard formula for the market-maker’s profit:

\[
\text{Market-maker's profit} \approx -\frac{e^2}{2} \Gamma_t - h \theta_t - rh \left[ S_t \Delta_t - V(t) \right]
\]
We can use the Black-Scholes Equation to obtain an expression that is equal to zero:
\[ h \left[ \frac{\sigma^2 S_t^2}{2} - \Gamma_t + rS_t \Delta_t + \theta_t - rV(t) \right] = 0 \]

Let’s add this expression (which is equal to zero) to the Market-maker’s profit to obtain a new formula for the market-maker’s profit:

Market-maker’s profit \( = -\frac{\varepsilon^2}{2} \Gamma_t - h\theta_t - rh[S_t \Delta_t - V(t)] \)
\[ = h \left[ \frac{\sigma^2 S_t^2}{2} - \Gamma_t + rS_t \Delta_t + \theta_t - rV(t) \right] \]
\[ = \frac{\sigma^2 S_t^2}{2} \Gamma_t h - \frac{\varepsilon^2}{2} \Gamma_t \]

The market-maker’s profit is:

Market-maker's profit \( = \frac{\sigma^2 S_t^2}{2} \Gamma_t h - \frac{\varepsilon^2}{2} \Gamma_t \)
\[ = \frac{0.35^2 75^2}{2} (0.0296) \frac{1}{365} - \frac{2^2}{2} (0.0296) \]
\[ = -0.03126 \]

**Solution 49**

Let’s assume that “several months ago” was time 0. Further, let’s assume that the options expire at time \( T \) and that the current time is time \( t \). We can use put-call parity to obtain a system of 2 equations.

\[ 21.94 + Ke^{-rT} = 100 + 4.19 \]
\[ 27.82 + Ke^{-(T-t)} = 110 + 1.95 \]

This can be solved to find \( e^{rt} \):

\[
\begin{align*}
Ke^{-rT} &= 82.25 \\
Ke^{-(T-t)} &= 84.13
\end{align*}
\]

\[ \Rightarrow \quad e^{rt} = \frac{84.13}{82.25} \]

The market-maker sold 50 of the call options. From the market-maker’s perspective, the value of this short position was:

\[ -50 \times 21.94 = -1,097.00 \]

The delta of the position, from the perspective of the market-maker, was:

Delta of a short position in 50 calls \( = -50 \times 0.788 = -39.4 \)

To delta-hedge the position, the market-maker purchased 39.4 shares of stock. The value of this position was:

\[ 39.4 \times 100 = 3,940.00 \]
Since the market-maker received only $1,097 from the sale of the calls, the differential was borrowed. The value of the loan, from the perspective of the market-maker, was:

\[ 1,097.00 - 3,940.00 = -2,843.00 \]

The initial position, from the perspective of the market-maker, was:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>-1,097.00</td>
</tr>
<tr>
<td>Shares</td>
<td>3,940.00</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>-2,843.00</td>
</tr>
<tr>
<td>Net</td>
<td>0.00</td>
</tr>
</tbody>
</table>

After \( t \) years elapsed, the value of the options changed by:

\[ -50 \times (27.82 - 21.94) = -294.00 \]

After \( t \) years elapsed, the value of the shares of stock changed by:

\[ 39.4 \times (110.00 - 100.00) = 394.00 \]

After \( t \) years elapsed, the value of the funds that were borrowed at the risk-free rate changed by:

\[ -2,843.00 \left( e^{rt} - 1 \right) = -2,843.00 \left( \frac{84.13}{82.25} - 1 \right) = -64.9829 \]

The sum of these changes is the profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>-294.00</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>394.00</td>
</tr>
<tr>
<td>Interest</td>
<td>-64.98</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>35.02</td>
</tr>
</tbody>
</table>

The change in the value of the position is $35.02, and this is the profit.

**Solution 50**

C Chapter 13, Market-Maker Profit

*Question 47 of the Sample Exam used a constant risk-free interest rate, but in the solution provided by the Society of Actuaries a “remark” described how the question could still be answered if the risk-free interest rate is deterministic but not necessarily constant.*

Let’s assume that “several months ago” was time 0. Further, let’s assume that the options expire at time \( T \) and that the current time is time \( t \).

The interest rates are not necessarily constant, so they can vary over time. Therefore, we replace the usual discount factors as described below:

\[ e^{-rT} \] is replaced by \( e^{-\int_0^T r(s)ds} \)

\[ e^{-r(T-t)} \] is replaced by \( e^{-\int_t^T r(s)ds} \)
We can use put-call parity to obtain a system of 2 equations:

\[ 7.86 + Ke^{-\int_0^T r(s) ds} = 50 + 4.01 \]
\[ 14.26 + Ke^{-\int_T^T r(s) ds} = 60 + 1.31 \]

This can be solved to find \( e^{\int_T^T r(s) ds} \):

\[
Ke^{-\int_0^T r(s) ds} = 46.15 \\
Ke^{-\int_T^T r(s) ds} = 47.05 \\
\Rightarrow \\
Ke^{-\int_0^T r(s) ds} e^{-\int_0^T r(s) ds} = 46.15 \\
Ke^{-\int_T^T r(s) ds} = 47.05 \\
\Rightarrow \\
e^{\int_0^T r(s) ds} = \frac{47.05}{46.15}
\]

The market-maker purchased 100 of the put options. From the market-maker's perspective, the value of this position was:

\[ 100 \times 4.01 = 401.00 \]

The delta of the call option can be used to determine the delta of the put option:

\[ \Delta_{Put} = \Delta_{Call} - e^{-\delta T} = 0.662 - e^{-0 \times T} = 0.662 - 1 = -0.338 \]

The delta of the position, from the perspective of the market-maker, was:

Delta of a long position in 100 puts = 100 \times (-0.338) = -33.8

To delta-hedge the position, the market-maker purchased 33.8 shares of stock. The value of this position was:

\[ 33.8 \times 50 = 1,690.00 \]

The market maker borrowed to purchase the puts and the stock. The value of the loan, from the perspective of the market-maker, was:

\[ -401.00 - 1,690.00 = -2,091 \]

The initial position, from the perspective of the market-maker, was:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Options</td>
<td>401.00</td>
</tr>
<tr>
<td>Shares</td>
<td>1,690.00</td>
</tr>
<tr>
<td>Risk-Free Asset</td>
<td>-2,091.00</td>
</tr>
<tr>
<td>Net</td>
<td>0.00</td>
</tr>
</tbody>
</table>

After \( t \) years elapsed, the value of the options changed by:

\[ 100 \times (1.31 - 4.01) = -270.00 \]

After \( t \) years elapsed, the value of the shares of stock changed by:

\[ 33.8 \times (60.00 - 50.00) = 338.00 \]
After \( t \) years elapsed, the value of the funds that were borrowed at the risk-free rate changed by:

\[
-2,091.00 \left( e^{\int_0^t r(s)ds} - 1 \right) = -2,091.00 \left( \frac{47.05}{46.15} - 1 \right) = -40.7779
\]

The sum of these changes is the profit.

<table>
<thead>
<tr>
<th>Component</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain on Options</td>
<td>-270.00</td>
</tr>
<tr>
<td>Gain on Stock</td>
<td>338.00</td>
</tr>
<tr>
<td>Interest</td>
<td>-40.78</td>
</tr>
<tr>
<td>Overnight Profit</td>
<td>27.22</td>
</tr>
</tbody>
</table>

The change in the value of the position is $27.22, and this is the profit.

**Solution 51**

**C**  Chapter 13, Delta-Gamma Approximation

The delta-gamma approximation is:

\[
V(t + h) \approx V(t) + \varepsilon \Delta t + \frac{\varepsilon^2}{2} \Gamma_t
\]

The approximation above can be used to create a quadratic equation:

\[
6.08 = 5.92 - 0.323\varepsilon + \frac{\varepsilon^2}{2} (0.015)
\]

\[
0 = 0.0075\varepsilon^2 - 0.323\varepsilon - 0.16
\]

\[
\varepsilon = \frac{0.323 \pm \sqrt{(-0.323)^2 - 4(0.0075)(-0.16)}}{2(0.0075)}
\]

\[
\varepsilon = -0.4898 \quad \text{or} \quad \varepsilon = 43.5565
\]

The definition of \( \varepsilon \) can be rearranged to obtain an expression for \( S_0 \):

\[
\varepsilon = S_h - S_0 \quad \Rightarrow \quad S_0 = S_h - \varepsilon
\]

Since we have 2 possible values for \( \varepsilon \), we have 2 possible values for \( S_0 \):

\[
\varepsilon = -0.4898 \quad \Rightarrow \quad S_0 = 86 - (-0.4898) = 86.4898
\]

\[
\varepsilon = 43.5565 \quad \Rightarrow \quad S_0 = 86 - 43.5565 = 42.4435
\]

The value of 42.4435 is less than 75, and Statement (i) in the question tells us that the original stock price is greater than 75. Therefore, the other possibility must be correct, and the initial stock price is 86.4898.
Solution 52

Chapter 13, Delta-Gamma Hedging

The gamma of the position to be hedged is:

\[ -1,000 \times 0.0237 = -23.7 \]

We can solve for the quantity, \( Q \), of the other put option that must be purchased to bring the hedged portfolio’s gamma to zero:

\[ -23.7 + 0.0486Q = 0.00 \]

\[ Q = 487.6543 \]

Although the market-maker purchases stock to bring the delta of the position to zero, the vega of the stock is zero. Therefore, the vega of the position is not affected by the quantity of stock that the market-maker holds.

The vega of the position is determined by the quantities of Put-I and Put-II that the market-maker holds:

\[ -1,000 \times 0.1774 + 487.6543 \times 0.0909 = -133.0722 \]
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Chapter 14 – Solutions

Solution 1

B Chapter 14, Asian Options

The syllabus doesn’t include the version of the Black-Scholes formula that prices Asian options. This means that the examiners need to rely on the binomial model to calculate the price of an Asian option.

From the information provided, we have:

\[ S = 50 \]
\[ K = 50 \]
\[ r = 0.06 \]
\[ \sigma = 0.30 \]
\[ h = \frac{T}{n} = \frac{1}{2} = 0.5 \]
\[ \delta = 0 \]

This allows us to solve for \( u \) and \( d \):

\[ u = e^{(r-\delta)h + \sigma \sqrt{h}} = e^{(0.06-0.00)(0.5)+0.3\sqrt{0.5}} = 1.2740 \]
\[ d = e^{(r-\delta)h - \sigma \sqrt{h}} = e^{(0.06-0.00)(0.5)-0.3\sqrt{0.5}} = 0.8335 \]

We next solve for the risk-neutral probability of an upward movement:

\[ p^* = \frac{e^{(r-\delta)h - d}}{u - d} = \frac{e^{(0.06-0.00)0.5} - 0.8335}{1.2740 - 0.8335} = 0.4472 \]

The tree of stock prices below includes the payoff under each path:

<table>
<thead>
<tr>
<th>Arithmetic Avg Price</th>
<th>Call Payoff</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>81.15</td>
<td>72.42</td>
<td>22.42</td>
</tr>
<tr>
<td>63.70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>53.09</td>
<td>58.39</td>
<td>8.39</td>
</tr>
<tr>
<td>50.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>53.09</td>
<td>47.38</td>
<td>0.00</td>
</tr>
<tr>
<td>41.67</td>
<td>38.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Each average is calculated as an arithmetic average. For example, the first arithmetic average price is:

\[ \frac{63.70 + 81.15}{2} = 72.42 \]
Each call payoff is determined using the arithmetic average price. For example, the first call payoff value is:

$$\text{Max}[72.42 - 50, 0] = 22.42$$

Each probability is determined using the risk-neutral probabilities. For example, the first probability, which represents two upward movements, is:

$$0.4472 \times 0.4472 = 0.2000$$

The arithmetic average price exceeds 50 for only 2 of the 4 possible paths. The present value of the option is:

$$[22.42(0.2000) + 8.39(0.2472) + 0(0.2472) + 0(0.3056)]e^{-0.06 \times 1} = 6.18$$

### Solution 2

C Chapter 14, Asian Options

From the information provided, we have:

- $$S = 50$$
- $$r = 0.06$$
- $$\sigma = 0.30$$
- $$h = \frac{T}{n} = \frac{1}{2} = 0.5$$
- $$\delta = 0$$

This allows us to solve for $$u$$ and $$d$$:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.06-0.00)(0.5)+0.3 \sqrt{0.5}} = 1.2740$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.06-0.00)(0.5)-0.3 \sqrt{0.5}} = 0.8335$$

We next solve for the risk-neutral probability of an upward movement:

$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{e^{(0.06-0.00)0.5}-0.8335}{1.2740-0.8335} = 0.4472$$

The tree of stock prices below includes the payoff under each path:

<table>
<thead>
<tr>
<th>Geometric Avg. Strike</th>
<th>Call Payoff</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>81.15</td>
<td>71.90</td>
<td>9.25</td>
</tr>
<tr>
<td>63.70</td>
<td>53.09</td>
<td>58.15</td>
</tr>
<tr>
<td>50.00</td>
<td>53.09</td>
<td>47.04</td>
</tr>
<tr>
<td>41.67</td>
<td>34.74</td>
<td>38.05</td>
</tr>
</tbody>
</table>
Each average is calculated as a geometric average. For example, the first geometric average strike is:

\[(63.70 \times 81.15)^{0.5} = 71.90\]

Each call payoff is determined using the geometric average strike. For example, the first call payoff value is:

\[\text{Max}[81.15 - 71.90, 0]\]

The geometric average strike is less than the final stock price for only 2 of the 4 possible paths. The present value of the option is:

\[9.25(0.2000) + 0(0.2472) + 6.05(0.2472) + 0(0.3056)]e^{-0.06\times 1} = 3.15\]

**Solution 3**

A Chapter 14, Asian Options

From the information provided, we have:

\[S = 50\]
\[r = 0.06\]
\[\sigma = 0.30\]
\[h = \frac{T}{n} = \frac{1}{2} = 0.5\]
\[\delta = 0\]

This allows us to solve for \(u\) and \(d\):

\[u = e^{(r - \delta)h + \sigma \sqrt{h}} = e^{(0.06 - 0.00)(0.5) + 0.3\sqrt{0.5}} = 1.2740\]
\[d = e^{(r - \delta)h - \sigma \sqrt{h}} = e^{(0.06 - 0.00)(0.5) - 0.3\sqrt{0.5}} = 0.8335\]

We next solve for the risk-neutral probability of an upward movement:

\[p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.06 - 0.00)(0.5) - 0.8335}}{1.2740 - 0.8335} = 0.4472\]

The tree of stock prices below includes the payoff under each path:

<table>
<thead>
<tr>
<th>Geometric Avg. Strike</th>
<th>Call Payoff</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>81.15</td>
<td>71.90</td>
<td>9.25</td>
</tr>
<tr>
<td>63.70</td>
<td>53.09</td>
<td>58.15</td>
</tr>
<tr>
<td>50.00</td>
<td>53.09</td>
<td>47.04</td>
</tr>
<tr>
<td>41.67</td>
<td>34.74</td>
<td>38.05</td>
</tr>
</tbody>
</table>
The corresponding tree of option prices is:

<table>
<thead>
<tr>
<th></th>
<th>9.25</th>
<th>4.02</th>
<th>0.00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3.15</td>
<td>6.05</td>
<td>2.63</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.00</td>
</tr>
</tbody>
</table>

*It is not necessary to calculate the option price of 3.15, but we included it here for completeness.*

The option prices are obtained by working from right to left. For example, the option price after one upward movement is:

\[
e^{-0.06 \times 0.5} [0.4472 \times 9.25 + (1 - 0.4472) \times 0] = 4.02
\]

The value of delta is:

\[
\Delta(S,0) = e^{-\delta h} \frac{V_u - V_d}{S_u - S_d} = e^{-0 \times 0.5} \frac{4.02 - 2.63}{63.70 - 41.67} = 0.0630
\]

**Solution 4**

**B  Chapter 14, Barrier Options**

From the information provided, we have:

- \( S = 50 \)
- \( K = 50 \)
- \( r = 0.06 \)
- \( \sigma = 0.30 \)
- \( h = \frac{T}{n} = \frac{1}{2} = 0.5 \)
- \( \delta = 0 \)

This allows us to solve for \( u \) and \( d \):

\[
u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.06-0.00)(0.5)+0.3\sqrt{0.5}} = 1.2740
\]

\[
d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.06-0.00)(0.5)-0.3\sqrt{0.5}} = 0.8335
\]

We next solve for the risk-neutral probability of an upward movement:

\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.06-0.00)0.5} - 0.8335}{1.2740 - 0.8335} = 0.4472
\]
The tree of stock prices below includes the payoff under each path:

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>81.15</td>
<td>31.15</td>
</tr>
<tr>
<td>63.70</td>
<td>3.09</td>
</tr>
<tr>
<td>50.00</td>
<td>0.00</td>
</tr>
<tr>
<td>41.67</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The barrier of $60 is above the current stock price, so the call option is an up-and-in call option. Therefore the call option pays off only if the stock price exceeds $60, which occurs only if the stock moves up in the first period. For example, the first call payoff value is:

\[
\text{Max}[81.15 - 50.00, 0] = 31.15
\]

The present value of the option is:

\[
0.06 \times [31.15(0.2000) + 3.09(0.2472) + 0(0.2472) + 0(0.3056)] e^{-0.06 \times 1} = 6.59
\]

**Solution 5**

E Chapter 14, Asian options

*When calculating the averages, we do not use the initial stock price.*

The arithmetic average is:

\[
A(1.5) = \frac{5 + 7 + 8 + 12 + 4 + 3}{6} = 6.5
\]

The geometric average is:

\[
G(1.5) = (5 \times 7 \times 8 \times 12 \times 4 \times 3)^{1/6} = 5.86
\]

The payoff of the arithmetic average price Asian call is:

Average price Asian call option payoff = Max[\(A(T) - K\), 0] = 6.5 - 5 = 1.5

The payoff of the geometric average strike Asian put is:

Average strike Asian put option payoff = Max[\(G(T) - S_T\), 0] = 5.86 - 3 = 2.86

The total payoff on 7/1/2008 is:

\[1.5 + 2.86 = 4.36\]
### Solution 6

**D** Chapter 14, Asian Options

This question is similar to the example involving Euros in Section 14.2 of the textbook under the heading "An Asian Solution for XYZ."

Consider the payoffs of each of the available Asian options:

- **Arithmetic average price Asian call:** \( \text{Max}\left[A(T) - K, 0\right] \)
- **Geometric average price Asian call:** \( \text{Max}\left[G(T) - K, 0\right] \)
- **Arithmetic average strike Asian call:** \( \text{Max}\left[S_T - A(T), 0\right] \)
- **Arithmetic average price Asian put:** \( \text{Max}\left[K - A(T), 0\right] \)
- **Arithmetic average strike Asian put:** \( \text{Max}\left[A(T) - S_T, 0\right] \)

The arithmetic average price Asian call and the geometric average price Asian call are inappropriate since their payoffs decline if the yen falls in value.

The arithmetic average strike Asian call will not provide a positive payoff if the yen's value is low at the end of the 2 years.

The payoff of the arithmetic average price Asian put increases as the average value of the yen falls. Therefore, answer choice D is the correct answer.

The arithmetic average strike Asian put does not provide a positive payoff if the yen's value is low for most of the 2 years but rises at the very end.

### Solution 7

**E** Chapter 14, Barrier options

Only the first row of the table is needed to solve this problem. The rest of the table was included to make the question seem more difficult than it really is.

The stock price at issuance is less than $63, so the option is an up-and-in call option.

The up-and-in call option and the ordinary call option both have the same payoff:

\( \text{Max}\left[S_T - 65, 0\right] \)

Since a positive payoff requires that \( S_T \) be greater than $65, the barrier of $63 is reached in all of the positive payoff scenarios. Therefore, the barrier does not reduce the value of the up-and-in call relative to the ordinary call. So, the price of this call is $7.15.
Solution 8

C Chapter 14, Barrier Options

The payoffs of each of the options are:

<table>
<thead>
<tr>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: A knock-in call with a barrier of $105 and a strike of $100</td>
</tr>
<tr>
<td>B: A knock-in call with a barrier of $95 and a strike of $105</td>
</tr>
<tr>
<td>C: A knock-out call with a barrier of $95 and a strike of $105</td>
</tr>
<tr>
<td>D: A knock-in put with a barrier of $105 and a strike of $105</td>
</tr>
<tr>
<td>E: A knock-out put with a barrier of $95 and a strike of $120</td>
</tr>
</tbody>
</table>

Solution 9

E Chapter 14, Barrier Options

For a given barrier, the parity relationship can be used to find the value of the ordinary call option. Since we have the value of both the knock-in call and the knock-out call option when the barrier is $57:

\[
\text{Knock-in option} + \text{Knock-out option} = \text{Ordinary option}
\]

\[
10.35 + 5.14 = 15.49
\]

Therefore an ordinary call option with a strike price of $50 has a value of $15.49.

We can apply the parity relationship again, this time with the barrier set to $54:

\[
\text{Knock-in option} + \text{Knock-out option} = \text{Ordinary option}
\]

\[
6.40 + \text{Knock-out option} = 15.49
\]

Knock-out option = $15.49 - $6.40 = $9.09

Solution 10

D Chapter 14, Barrier Options

The first step is the use the Black-Scholes formula to find the value of an ordinary call option.

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln\left(\frac{S_0}{K} \times e^{-rT}\right) + \frac{\sigma^2T}{2}}{\sigma\sqrt{T}} = \frac{\ln(75e^{-0.08 x 1}) + 0.30^2 \times 1}{0.30\sqrt{1}} = 0.41667
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = 0.41667 - 0.3\sqrt{1} = 0.11667
\]
We have:

\[ N(d_1) = N(0.41667) = 0.66154 \]
\[ N(d_2) = N(0.11667) = 0.54644 \]

Using Black-Scholes, the value of the ordinary call option is:

\[ C_{Eur} = S e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \]
\[ = 75(0.66154) - 75e^{-0.08}(0.54644) = 11.7834 \]

The value of the knock-out call option can be found using the parity relation for barrier options:

Knock-in option + Knock-out option = Ordinary option

3.56 + Knock-out option = 11.7834
Knock-out option = 11.7834 – 3.56 = 8.22

Solution 11

A Chapter 14, Barrier Options

Since the current stock price of $75 is already above the barrier of $60, the up-and-in put is the same as an ordinary put option. Therefore, the ordinary put option has a price of $8.31.

We can now use the parity relationship for barrier options to find the value of the down-and-out put:

\[ \text{Down-and-in put} + \text{Down-and-out put} = \text{Ordinary put} \]
\[ 7.13 + \text{Down-and-out put} = 8.31 \]
\[ \text{Down-and-out put} = 8.31 – 7.13 = 1.18 \]

Solution 12

A Chapter 14, Barrier Options

*We don’t need the table to answer this question. The table was included to make the question seem more difficult than it really is.*

The stock price upon issuance of the option is less than the barrier of $63, so the option is an up-and-out call option.

For the up-and-out call option to have a positive payoff, the final stock price must exceed the strike price of $65. But that would imply that the final stock price is also above $63, meaning that the option has been “knocked-out.” Consequently, there is no possibility that the option will produce a positive payoff. Therefore, the price of the up-and-out call option is $0.
Solution 13
A Chapter 14, Compound Options

We are asked to find the value of a put on a put option. We can use put-call parity to find the value of the put on a put:

\[ \text{CallOnPut} - \text{PutOnPut} = P_{\text{Eur}} - xe^{-rt} \]

\[ 4.17 - \text{PutOnPut} = 6.51 - 3e^{-0.05(1)} \]

\[ \text{PutOnPut} = 0.51 \]

Solution 14
C Chapter 14, Compound Options

The value of the American call option is:

\[ C_{\text{Amer}}(S_0, K, T) = S_0 - Ke^{-rt} + \text{CallOnPut} \]

The call on a put expires at time \( t_1 \) and has a strike price of:

\[ x = D - K \left( 1 - e^{-r(T-t_1)} \right) = 5 - 83 \left( 1 - e^{-0.07(0.5-0.25)} \right) = 3.56 \]

The underlying put option has Stock Z as its underlying asset, a strike price of $83, and matures in 6 months.

From the table, we observe that the price of the call on the put is $4.04 when the strike is $3.56.

Therefore, the price of the American option is:

\[ C_{\text{Amer}}(S_0, K, T) = S_0 - Ke^{-rt} + \text{CallOnPut} = 80 - 83e^{-0.07(0.25)} + 4.04 = 2.48 \]

Solution 15
B Chapter 14, Compound Options

We know that the value of an American call option is:

\[ C_{\text{Amer}}(S_0, K, T) = S_0 - Ke^{-rt} + \text{CallOnPut} \]

The call on a put expires at time \( t_1 \) and has a strike price of:

\[ x = D - K \left( 1 - e^{-r(T-t_1)} \right) \]

In this case, we have:

\[ x = D - K \left( 1 - e^{-r(T-t_1)} \right) = 6 - 65 \left( 1 - e^{-0.08(1-0.25)} \right) = 2.215 \]
Therefore the value of the compound call with a strike price of $2.215 can be found as follows:

\[ C_{Amer}(S_0, K, T) = S_0 - Ke^{-rt_1} + CallOnPut \]

\[ 4.49 = 65 - 65e^{-0.08(0.25)} + CallOnPut \]

\[ CallOnPut = 3.20 \]

**Solution 16**

**D** Chapter 14, Compound Options

*Using the formula for an American call option, we can find the value of the call on a put. Then we can use compound option parity to find the value of the ordinary put option.*

We know that the value of an American call option is:

\[ C_{Amer}(S_0, K, T) = S_0 - Ke^{-rt_1} + CallOnPut \]

The call on a put expires at time \( t_1 \) and has a strike price of:

\[ x = D - K \left( 1 - e^{-r(T-t_1)} \right) \]

In this case, we have:

\[ x = D - K \left( 1 - e^{-r(T-t_1)} \right) = 6 - 57 \left( 1 - e^{-0.08(1-0.25)} \right) = 2.68 \]

Therefore the value of the compound call with a strike price of $2.68 can be found as follows:

\[ C_{Amer}(S_0, K, T) = S_0 - Ke^{-rt_1} + CallOnPut \]

\[ 9.85 = 65 - 57e^{-0.08(0.25)} + CallOnPut \]

\[ CallOnPut = 0.7213 \]

We can now use the parity relationship to find the value of the European put option:

\[ CallOnPut - PutOnPut = Put - xe^{-rt_1} \]

\[ 0.7213 - 0.88 = Put - 2.68e^{-0.08(0.25)} \]

\[ Put = 2.468 \]

**Solution 17**

**B** Chapter 14, Compound Options

We know that the value of an American call option is:

\[ C_{Amer}(S_0, K, T) = S_0 - Ke^{-rt_1} + CallOnPut \]
The call on a put expires at time $t_1$ and has a strike price of: 

$$x = D - K \left( 1 - e^{-r(T-t_1)} \right)$$

In this case, we have:

$$x = D - K \left( 1 - e^{-r(T-t_1)} \right) = 6 - 65 \left( 1 - e^{-0.08(1-0.25)} \right) = 2.215$$

Therefore the value of the compound call with a strike price of $2.215$ can be found as follows:

$$C_{Amer}(S_0, K, T) = S_0 - Ke^{-rT} + CallOnPut$$

$$4.49 = 65 - 65e^{-0.08(0.25)} + CallOnPut$$

$$CallOnPut = 3.2029$$

We can now use the parity relationship to find the value of the put on a put option:

$$CallOnPut - PutOnPut = Put - xe^{-rt_1}$$

$$3.2029 - PutOnPut = 5.21 - 2.215e^{-0.08(0.25)}$$

$$PutOnPut = 0.16$$

**Solution 18**

B Chapter 14, Compound Options

This question is potentially confusing because we might be tempted to try to apply the formula involving an American call option and a call on a put. But that formula is applicable only when there is a single discrete dividend, and Stock X does not pay any dividends.

At the end of 9 months, the American call option has 3 months until expiration. Since Stock X does not pay dividends, the value of the American call option is equal to the value of an otherwise equivalent European call option:

$$C_{Eur}(100,90,0.25) = 13.37$$

We can use put-call parity to find the value of the underlying European put option:

$$C_{Eur}(100,90,0.25) - P_{Eur}(100,90,0.25) = S_0e^{-\delta T} - Ke^{-rT}$$

$$13.37 - P_{Eur}(100,90,0.25) = 100 - 90e^{-0.08(0.25)}$$

$$P_{Eur}(100,90,0.25) = 1.588$$

The right to buy the European put option for $1 is worth:

$$1.588 - 1.00 = 0.588$$
Solution 19

The strike and trigger prices are:

\[ K_1 = \text{Strike price} = 29 \]
\[ K_2 = \text{Trigger price} = 31 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln \left( \frac{S e^{-\delta T}}{K_2 e^{-rT}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{30 e^{-0.03(0.5)}}{31 e^{-0.06(0.5)}} \right) + \frac{0.20^2(0.5)}{2}}{0.20 \sqrt{0.5}} = -0.05508
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = -0.05508 - 0.2 \sqrt{0.5} = -0.19650
\]

We have:

\[
N(d_1) = N(-0.05508) = 0.47804
\]
\[
N(d_2) = N(-0.19650) = 0.42211
\]

Using Black-Scholes, the value of the gap call option is:

\[
C_{Gap} = S e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)
\]
\[
= 30 e^{-0.03(0.5)}(0.47804) - 29 e^{-0.06(0.5)}(0.42211) = 2.2483
\]

Solution 20

Let’s consider each statement separately:

A  When the strike is equal to the trigger, the premium for a gap option is the same as for an ordinary option with the same strike.

True. When the strike is equal to the trigger, then the payoffs for the gap option are the same as the payoffs for an ordinary option. (See also the second sentence of the second paragraph before Section 14.6 of the textbook).

B  If the trigger exceeds the strike for a gap put, then increasing the trigger reduces the premium.

True. If the trigger exceeds the strike for a gap put, then negative payoffs are possible. Increasing the trigger makes even more negative payoffs possible.

C  If the trigger is fixed for a gap put, then increasing the strike price decreases the premium.

False. Increasing the strike for a gap put makes the payoffs larger, so it increases the premium.
D  If the strike is equal to the trigger, then increasing the trigger reduces the premium of a gap call.

*True.* *Increasing the trigger above the strike removes some of the positive payoffs that would otherwise occur when the final stock price is between the strike and the trigger.*

E  For any positive strike, there is a trigger that makes the premium for a gap put equal to zero.

*True.* *The premium is positive if the trigger is equal to the strike. From there, the trigger can be increased until the premium falls to zero. (See also the first sentence of the paragraph immediately preceding Section 14.6 of the textbook).*

Since statement C is false, answer choice C is the answer to the question.

**Solution 21**

A  Chapter 14, Gap Options

Option A must have a higher price than Option B because if $50 < S_T < 60$, then Option A has positive payoffs while Option B has zero payoffs.

Option A must have a higher price than Option C because if $S_T > 50$, then the payoffs to Option C are $10$ less than the payoffs the Option A.

We can use put-call parity to show that the price of Option A must be higher than the price of Option E. Notice that in the put-call parity relationship below, Option A is an ordinary call and Option E is an ordinary put:

\[
C_{Eur} - P_{Eur} = S_0 e^{-\delta T} - Ke^{-r T}
\]

\[
(Option A) - (Option E) = 50 e^{-0.04} - 50 e^{-0.08} = 1.88
\]

\[
(Option A) - (Option E) > 0
\]

Option E must have a higher price than Option D because if $40 < S_T < 50$, then Option E has positive payoffs while Option D has zero payoffs.

Since we have already shown that Option A has a higher price than Option E, Option A must have the highest price.

**Solution 22**

B  Chapter 14, Gap Options

*Gap options are not path-dependent, so we can ignore all of the prices except for the final one. If these options had been either Asian or Barrier options, then we would have needed to know the path to answer the question.*
We recall that the gap call option payoff is the final stock price less the strike price if the final stock price is above the trigger, but the gap call option payoff is zero if the final stock price is less than or equal to the trigger:

\[
\text{Gap call option payoff} = \begin{cases} S_T - K_1 & \text{if } S_T > K_2 \\ 0 & \text{if } S_T \leq K_2 \end{cases}
\]

Similarly, the gap put option payoff is the strike price less the final stock price if the final stock price is less than the trigger, but the gap put option payoff is zero if the final stock price is greater than or equal to the trigger:

\[
\text{Gap put option payoff} = \begin{cases} K_1 - S_T & \text{if } S_T < K_2 \\ 0 & \text{if } S_T \geq K_2 \end{cases}
\]

Based on the final stock price of $55, the payoffs to each option are shown in the rightmost column below:

<table>
<thead>
<tr>
<th>Option</th>
<th>Type</th>
<th>Strike</th>
<th>Trigger</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Gap call</td>
<td>50</td>
<td>50</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>Gap call</td>
<td>50</td>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Gap call</td>
<td>60</td>
<td>50</td>
<td>-5</td>
</tr>
<tr>
<td>4</td>
<td>Gap put</td>
<td>50</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>Gap put</td>
<td>50</td>
<td>60</td>
<td>-5</td>
</tr>
</tbody>
</table>

The total payoff is:

\[5 + 0 - 5 + 0 - 5 = -5\]

**Solution 23**

Chapter 14, Gap Options

This problem isn’t as time-consuming as it may first appear because we can use the same values of \(N(d_1)\) and \(N(d_2)\) for each of the options.

Since each of the options has the same trigger \((K_2 = 82)\), we will use the same values of \(N(d_1)\) and \(N(d_2)\) for all three options.

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln \left( \frac{S_0 e^{-\delta T}}{K_2 e^{-rT}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{80e^{-0.01 \times 1}}{82e^{-0.08 \times 1}} \right) + \frac{0.41^2 \times 1}{2}}{0.41 \sqrt{1}} = 0.33990
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.33990 - 0.41\sqrt{1} = -0.07010
\]
We have:
\[ N(d_1) = N(0.33990) = 0.63303 \]
\[ N(d_2) = N(-0.07010) = 0.47206 \]

The value of the European call option is:
\[
C(80,82,82) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)
\]
\[
= 80(0.63303) - 82e^{-0.08}(0.47206) = 14.9096
\]

The value of the gap option having a strike price of $84 is:
\[
C(80,84,82) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)
\]
\[
= 80(0.63303) - 84e^{-0.08}(0.47206) = 14.0380
\]

The value of the gap option having a strike price of $120 is:
\[
C(80,120,82) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)
\]
\[
= 80(0.63303) - 120e^{-0.08}(0.47206) = -1.6496
\]

The total value of the three options is:
\[ 14.9096 + 14.0380 - 1.6496 = 27.2980 \]

**Solution 24**

C  Chapter 14, Gap Options

The strike and trigger prices are:
\[ K_1 = \text{Strike price} = 1.00 \]
\[ K_2 = \text{Trigger price} = 0.90 \]

The values of \( d_1 \) and \( d_2 \) are:
\[
d_1 = \frac{\ln \left( \frac{Se^{-\delta T}}{K_2 e^{-rT}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{0.90e^{-0.03(1)}}{0.90e^{-0.06(1)}} \right) + 0.10^2(1)}{0.10 \sqrt{1}} = 0.35000
\]
\[ d_2 = d_1 - \sigma \sqrt{T} = 0.35000 - 0.1 \sqrt{1} = 0.25000 \]

The values of \( N(d_1) \) and \( N(d_2) \) are:
\[ N(d_1) = N(0.35000) = 0.63683 \]
\[ N(d_2) = N(0.25000) = 0.59871 \]

The value of the gap call option is:
\[
C_{\text{Gap}} = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)
\]
\[
= 0.90e^{-0.03(1)}(0.63683) - 1.00e^{-0.06(1)}(0.59871) = -0.007636
\]
Solution 25

**D** Chapter 14, Gap Options

The price of a gap option is linearly related to its strike price. This means that if the only variable changing is the strike price, then we can use linear interpolation (or extrapolation when appropriate) to find the new value of a gap option corresponding to a new strike price.

First, let’s use linear interpolation to solve this problem.

Since the only difference between the three gap call options is their strike price, we can use linear interpolation to find the value of the third option:

\[
4.98 + \frac{50 - 48}{56 - 48} \times (1.90 - 4.98) = 4.21
\]

Second, there is another (somewhat longer) way to solve this problem.

The formula for the price of a gap call option is:

\[
C_{Gap} = S e^{-\delta t} N(d_1) - K_1 e^{-rT} N(d_2)
\]

This suggests the following system of 2 equations:

\[
\begin{align*}
4.98 &= S e^{-\delta t} N(d_1) - 48 e^{-rT} N(d_2) \\
1.90 &= S e^{-\delta t} N(d_1) - 56 e^{-rT} N(d_2)
\end{align*}
\]

Subtracting the second equation from the first equation, we have:

\[
3.08 = 8 e^{-rT} N(d_2) \\
e^{-rT} N(d_2) = 0.385
\]

This allows us to solve for the first term of the pricing formula:

\[
4.98 = S e^{-\delta t} N(d_1) - 48(0.385) \\
S e^{-\delta t} N(d_1) = 23.46
\]

We can now find the price of the gap call option having a strike price of $50:

\[
C_{Gap} = S e^{-\delta t} N(d_1) - K_1 e^{-rT} N(d_2) \\
= 23.46 - (50)(0.385) = 4.21
\]

Solution 26

**C** Chapter 14, Gap Options

The strike and trigger prices are:

\[
\begin{align*}
K_1 &= \text{Strike price} = 65 \\
K_2 &= \text{Trigger price} = 52
\end{align*}
\]
The values of $d_1$ and $d_2$ are:

$$
d_1 = \frac{\ln \left( \frac{Se^{-\delta T}}{K_{2}e^{-rT}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{50e^{-0.04(0.25)}}{52e^{-0.07(0.25)}} \right) + \frac{0.40^2(0.25)}{2}}{0.40 \sqrt{0.25}} = -0.05860
$$

$$
d_2 = d_1 - \sigma \sqrt{T} = -0.05860 - 0.4 \sqrt{0.25} = -0.25860
$$

We have:

$$
N(d_1) = N(-0.05860) = 0.47664
$$

$$
N(d_2) = N(-0.25860) = 0.39797
$$

The value of the gap call option is:

$$
C_{Gap} = Se^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)
$$

$$
= 50e^{-0.04(0.25)}(0.47664) - 65e^{-0.07(0.25)}(0.39797) = -1.8244
$$

Solution 27

**E Chapter 14, Exchange Options**

The underlying asset is one share of Stock A. The strike asset is 4 shares of Stock B. Therefore, we have:

$$
S = 42
$$

$$
K = 4 \times 10 = 40
$$

The volatility of $\ln(S/K)$ is:

$$
\sigma = \sqrt{\frac{\sigma_S^2}{S} + \frac{\sigma_K^2}{K} - 2\rho \sigma_S \sigma_K} = \sqrt{0.3^2 + 0.5^2 - 2(0.4)(0.3)(0.5)} = \sqrt{0.22} = 0.46904
$$

The values of $d_1$ and $d_2$ are:

$$
d_1 = \frac{\ln \left( \frac{Se^{-\delta_S T}}{K_{2}e^{-\delta_K T}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{42e^{-0.0(1)}}{40e^{-0.04(1)}} \right) + \frac{0.46904^2(1)}{2}}{0.46904(1)} = 0.42382
$$

$$
d_2 = d_1 - \sigma \sqrt{T} = 0.42382 - 0.46904 = -0.04522
$$

We have:

$$
N(d_1) = N(0.42382) = 0.66415
$$

$$
N(d_2) = N(-0.04522) = 0.48197
$$
The value of the exchange call is:

\[
\text{ExchangeCallPrice} = S e^{-\delta S^T} N(d_1) - K e^{-\delta K^T} N(d_2) = 42(0.66415) - 40e^{-0.04(1)}(0.48197) = 9.3714
\]

**Solution 28**

The underlying asset is one share of Stock R. The strike asset is 1 share of Stock Q. Therefore, we have:

\[
S = 75 \\
K = 75
\]

The volatility of \(\ln(S/K)\) is:

\[
\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho \sigma_S \sigma_K} = \sqrt{0.3^2 + 0.25^2 - 2(1)(0.3)(0.25)} = \sqrt{0.0025} = 0.05
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln\left(\frac{Se^{-\delta S^T}}{Ke^{-\delta K^T}}\right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln\left(\frac{75}{75}\right) + \frac{0.05^2(1)}{2}}{0.05(1)} = 0.02500
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.02500 - 0.05(1) = -0.02500
\]

We have:

\[
N(d_1) = N(0.02500) = 0.50997 \\
N(d_2) = N(-0.02500) = 0.49003
\]

The value of the exchange call is:

\[
\text{ExchangeCallPrice} = S e^{-\delta S^T} N(d_1) - K e^{-\delta K^T} N(d_2) = 75(0.50997) - 75(0.49003) = 1.4955
\]

**Solution 29**

The underlying asset is one share of Stock D. The strike asset is 0.5 shares of Stock C. Therefore, we have:

\[
S = 40 \\
K = 0.5 \times 70 = 35
\]
The volatility of $\ln(S/K)$ is:
\[
\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2 \rho \sigma_S \sigma_K} = \sqrt{0.3^2 + 0.5^2 - 2(-0.4)(0.3)(0.5)} = \sqrt{0.46} = 0.67823
\]
The values of $d_1$ and $d_2$ are:
\[
d_1 = \frac{\ln \left( \frac{S e^{-\delta T}}{K e^{-\delta T}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{40 e^{-0.0(0.75)}}{35 e^{-0.03(0.75)}} \right) + \frac{0.67823^2(0.75)}{2}}{0.67823 \sqrt{0.75}} = 0.55933
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = 0.55933 - 0.67823 \sqrt{0.75} = -0.02804
\]
We have:
\[
N(d_1) = N(0.55933) = 0.71203
\]
\[
N(d_2) = N(-0.02804) = 0.48882
\]
The value of the exchange call is:
\[
\text{ExchangeCallPrice} = S e^{-\delta T} N(d_1) - K e^{-\delta T} N(d_2)
\]
\[
= 40(0.71203) - 35 e^{-0.03(0.75)}(0.48882) = 11.7531
\]

**Solution 30**

**A**

**Chapter 14, Exchange Options**

*We are asked to find the price of an exchange call option having one share of Stock Y as the underlying asset and 1/3 share of Stock X as the strike asset. We can find this price using the standard formula:*

\[
\text{ExchangeCallPrice} = S e^{-\delta T} N(d_1) - K e^{-\delta T} N(d_2)
\]

*But it is quicker to use put-call parity.*

Let's define the underlying asset as 1 share of Stock X and the strike asset as 3 shares of Stock Y.

The first option in the question is then an exchange call option. It gives its owner the right to acquire 1 share of Stock X for 3 shares of Stock Y.

Using put-call parity, we have:
\[
C(S_t,K_t,T-t) - P(S_t,K_t,T-t) = F_{t,T}^P(S) - F_{t,T}^P(K)
\]
\[
17.97 - P(S_t,K_t,T-t) = 100 e^{-0.04(0.75)} - 3(30) e^{-0.03(0.75)}
\]
\[
P(S_t,K_t,T-t) = 8.923
\]

The value of an option giving its owner the right to give up 1 share of Stock X for 3 shares of Stock Y is therefore $8.923.$
Since the option to exchange 1 share of Stock X for 3 shares of Stock Y has a value of $8.923, the value of an option allowing its owner to exchange 1/3 shares of Stock X for 1 share of Stock Y is:

\[
\frac{8.923}{3} = 2.97
\]

**Solution 31**

E Chapter 14, Exchange Options

Let’s define Stock E as the strike asset and Stock F as the underlying asset.

The first option described in the question gives its owner the right to acquire stock F in exchange for giving up Stock E, so this exchange option is a call option.

We are asked to find the value of an exchange option giving its owner the right to give up Stock F in exchange for Stock E. Since we defined Stock F as the underlying asset, this exchange option is a put option.

Since both the call and put options have the same strike asset and same underlying asset, we can use put-call parity to find the value of the put option:

\[
C(S_t, K_t, T - t) - P(S_t, K_t, T - t) = F^P_{t,T}(S) - F^P_{t,T}(K)
\]

\[
2.16 - P(S_t, K_t, T - t) = 50e^{-0.07(1)} - 55
\]

\[
P(S_t, K_t, T - t) = 10.54
\]

**Solution 32**

B Chapter 14, Exchange Options

Let’s define the underlying asset to be Stock Y and the strike asset to be 4 shares of Stock X. This makes the option an exchange call option with:

\[
S = 42
\]

\[
K = 4 \times 10 = 40
\]

The volatility of ln(S/K) is:

\[
\sigma = \sqrt{\text{Var}[\ln(S_t)] + \text{Var}[\ln(K_t)] - 2 \text{Cov}[\ln(S_t), \ln(K_t)]} = \sqrt{0.4^2 + 0.3^2 - 2(0.06)}
\]

\[
= \sqrt{0.13} = 0.36056
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{Ke^{-\delta K T}}\right) + \sigma^2T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{42e^{-0.02(1)}}{40e^{-0.00(1)}}\right) + 0.36056^2(1)}{0.36056\sqrt{1}} = 0.26013
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = 0.26013 - 0.36056\sqrt{1} = -0.10043
\]
We have:
\[ N(d_1) = N(0.26013) = 0.60262 \]
\[ N(d_2) = N(-0.10043) = 0.46000 \]

The value of the exchange call is:
\[
\text{ExchangeCallPrice} = Se^{-\delta S T} N(d_1) - Ke^{-\delta K T} N(d_2)
\]
\[ = 42e^{-0.02(1)}(0.60262) - 40(0.46000) = 6.4089 \]

Solution 33
C Chapter 14, Exchange Options

Let’s define the underlying asset to be 1 euro and the strike asset to be 1.4 Canadian dollars. This makes the option an exchange call option with:
\[ S = 1.25 \]
\[ K = 1.4 \times 0.9 = 1.26 \]

The volatility of \( \ln(S/K) \) is:
\[
\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho \sigma_S \sigma_K} = \sqrt{0.2^2 + 0.3^2 - 2(0.605)(0.2)(0.3)} = \sqrt{0.0574} = 0.23958
\]

The values of \( d_1 \) and \( d_2 \) are:
\[
d_1 = \frac{\ln \left( \frac{Se^{-\delta S T}}{Ke^{-\delta K T}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{1.25e^{-0.06(1)}}{1.26e^{-0.08(1)}} \right) + \frac{0.23958^2 (1)}{2}}{0.23958\sqrt{1}} = 0.17001
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = 0.17001 - 0.23958\sqrt{1} = -0.06957
\]

We have:
\[ N(d_1) = N(0.17001) = 0.56750 \]
\[ N(d_2) = N(-0.06957) = 0.47227 \]

The value of the exchange call is:
\[
\text{ExchangeCallPrice} = Se^{-\delta S T} N(d_1) - Ke^{-\delta K T} N(d_2)
\]
\[ = 1.25e^{-0.06(1)}(0.56750) - 1.26e^{-0.08(1)}(0.47227) = 0.11875 \]
Solution 34

D  Chapter 14, Exchange Options

For questions like this, it is a good idea to scan all the answer choices first, and then begin where you are most confident either choosing one of the answer choices as the correct answer or eliminating answer choices until the correct answer choice remains.

Answer choice C is the option to give up 1 share of Stock X in exchange for 0.5 shares of Stock Y. Answer choice D is the option to give up 2 shares of Stock X for 1 Share of Stock Y. Therefore, D is essentially 2 of C. So, the value of D is twice that of C, and the answer cannot be C.

Answer choice B is the option to give up 0.5 shares of Stock Y in exchange for 1 share of Stock X. Answer choice E is the option to give up 1 share of Stock Y in exchange for 2 shares of Stock X. Therefore, E is essentially 2 of B. So, the value of E is twice that of B, and the answer cannot be B.

At this point, the possible answers are A, D, and E. Let’s compare A with E. The underlying asset is 1 share of Stock Y, and the strike asset is 2 shares of Stock X:

\[ S = 80 \]
\[ K = 2 \times 40 = 80 \]

From put call parity, we have:

\[ C(S, K, T - t) - P(S, K, T - t) = F_{t, T}^P (S) - F_{t, T}^P (K) \]
\[ C(S, K, T - t) - P(S, K, T - t) = 80 - 80e^{-0.04} \]
\[ C(S, K, T - t) - P(S, K, T - t) = 3.137 \]

So the value of the call option is greater than the value of the put option by $3.137. That means that the value of D is $3.137 greater than E, so E cannot be the correct answer.

Since the value of the put option cannot be less than zero, the value of D must be at least $3.137. Therefore the value of D is more than $3.10, and A cannot be the correct answer.

By the process of elimination, the value of D is highest.

Solution 35

D  Chapter 14, Exchange Options

Let’s define 0.5 shares of Stock N as the strike asset and 1 share of Stock M as the underlying asset:

\[ S = 35 \]
\[ K = 0.5 \times 80 = 40 \]

The first option described in the question gives its owner the right to acquire Stock M in exchange for giving up 0.5 shares of Stock N, so this exchange option is a call option. The value of this call option is:

\[ C(35, 40, 0.5) = 0.0353(35) = 1.2355 \]
The value of an exchange put option giving its owner the right to give up 1 share of Stock M in exchange for 0.5 shares of Stock N can be found using put-call parity:

\[
C(S_t, K_t, T - t) - P(S_t, K_t, T - t) = F_{t,T}^P(S) - F_{t,T}^P(K)
\]

\[
1.2355 - P(35,40,0.5) = 35e^{-0.02(0.5)} - 40e^{-0.04(0.5)}
\]

\[
P(35,40,0.5) = 5.7917
\]

Two of these exchange put options give their owner the right to give up 2 shares of Stock M in exchange for 1 share of Stock N, so the solution is:

\[
2 \times 5.7917 = 11.5834
\]

**Solution 36**

**A** Chapter 14, Barrier Options

Deferred rebate options are barrier options, and barrier options are path-dependent since their payoff depends on whether the barrier is reached.

**Solution 37**

**D** Chapter 14, Gap Options

A gap option must be exercised if the trigger condition is met. This can result in negative payoffs. As negative payoffs become more likely, the premium falls, and it can even fall below zero.

**Solution 38**

**E** Chapter 14, Asian options

*When calculating the averages, we do not use the initial stock price.*

The arithmetic average is:

\[
A(1.5) = \frac{8 + 5 + 2 + 4 + 3 + 2}{6} = 4
\]

The geometric average is:

\[
G(1.5) = (8 \times 5 \times 2 \times 4 \times 3 \times 2)^{1/6} = 3.5255
\]

The payoff of the arithmetic average strike Asian put is:

Average strike Asian put option payoff = Max\[A(T) - S_T, 0]\] = 4 - 2 = 2.0

The payoff of the geometric average price Asian put is:

Average price Asian put option payoff = Max\[K - G(T), 0]\] = 5 - 3.5255 = 1.4745

The total payoff is:

2.0 + 1.4745 = 3.4745
Solution 39
A Chapter 14, Compound Options

The dividend on Stock Y does not occur until a year has elapsed. By then, the options have expired, so the dividend does not affect our valuation of the options.

Since the American call option expires before any dividends are paid, it is equivalent to a European call option expiring in 6 months:

\[ C_{Eur} = 6.75 \]

We are asked to find the value of a put on a call option. We can use put-call parity to find the value of the put on a call:

\[ \text{CallOnCall} - \text{PutOnCall} = C_{Eur} - xe^{-rt_1} \]

\[ 4.13 - \text{PutOnCall} = 6.75 - 3.45e^{-0.08(0.25)} \]

\[ \text{PutOnCall} = 0.76 \]

Solution 40
B Chapter 14, Asian Options

From the information provided, we have:

\[ S = 50 \]
\[ r = 0.06 \]
\[ \sigma = 0.30 \]
\[ h = \frac{T}{n} = \frac{1}{2} = 0.5 \]
\[ \delta = 0 \]

This allows us to solve for \( u \) and \( d \):

\[ u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.06-0.00)(0.5)+0.3\sqrt{0.5}} = 1.2740 \]
\[ d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.06-0.00)(0.5)-0.3\sqrt{0.5}} = 0.8335 \]

We next solve for the risk-neutral probability of an upward movement:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.06-0.00)0.5} - 0.8335}{1.2740 - 0.8335} = 0.4472 \]
The tree of stock prices below includes the payoff under each path:

<table>
<thead>
<tr>
<th>Geometric Avg. Strike</th>
<th>Put Payoff</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>81.15</td>
<td>71.90</td>
<td>0.00 0.2000</td>
</tr>
<tr>
<td>63.70</td>
<td>58.15</td>
<td>5.06 0.2472</td>
</tr>
<tr>
<td>50.00</td>
<td>47.04</td>
<td>0.00 0.2472</td>
</tr>
<tr>
<td>41.67</td>
<td>38.05</td>
<td>3.31 0.3056</td>
</tr>
</tbody>
</table>

Each average is calculated as a geometric average. For example, the first geometric average strike is:

\[
(63.70 \times 81.15)^{0.5} = 71.90
\]

Each put payoff is determined using the geometric average strike. For example, the first put payoff is:

\[
\text{Max}[71.90 - 81.15, 0] = 0
\]

The geometric average strike is less than the final stock price for only 2 of the 4 possible paths. The present value of the option is:

\[
[0(0.2000) + 5.06(0.2472) + 0(0.2472) + 3.31(0.3056)]e^{-0.06 \times 1} = 2.13
\]

**Solution 41**

C Chapter 14, Barrier Options

All of the options have the same strike price. We can use the parity relationship for barrier options to find the value of a regular option. Making use of the down-and-out and down-and-in options that have a barrier of $90, we have:

\[
\text{Knock-in option} + \text{Knock-out option} = \text{Ordinary option}
\]

\[
9.61 + 5.70 = 15.31
\]

Therefore, the value of the ordinary option is $15.31. We can use the parity relationship again, this time with the up-and-in and up-and-out options that have a barrier of $125:

\[
\text{Knock-in option} + \text{Knock-out option} = \text{Ordinary option}
\]

\[
X + 1.48 = 15.31
\]

\[
X = 13.83
\]

The value of the up-and-in option with a barrier of $125 is $13.83.
Solution 42

B Chapter 14, Gap Options

For the gap option, we have:

\[ K_1 = 100 \]
\[ K_2 = 110 \]

The delta of the regular call option is:

\[ \Delta_{Call} = e^{-\delta T} N(d_1) = 0.1563 \]

For the regular European call option, we have:

\[ 1.32 = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \]
\[ 1.32 = 85(0.1563) - 110e^{-rT} N(d_2) \]
\[ e^{-rT} N(d_2) = 0.108777 \]

Since the regular call option and the gap call option have the same values for \( d_1 \) and \( d_2 \), we can substitute the final line above into the equation for the value of the gap call option:

\[ \text{Gap call price} = Se^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2) \]
\[ = 85(0.1563) - 100(0.108777) \]
\[ = 2.41 \]

Solution 43

E Chapter 14, Asian Options

Statement I is true. If there were only one sampling period, then an average strike option would be certain to pay zero. As the number of sampling periods increases, the strike price has a greater chance of being different from the final price of the stock, thereby increasing the value of the average strike Asian call.

Statement II is false. As the frequency of the sampling is increased, the average price exhibits less volatility, thereby decreasing the value of the average price Asian call.

Statement III is true. The arithmetic average is always greater than or equal to the geometric average, so the value of a geometric average price Asian call option is always less than the value of an otherwise equivalent arithmetic average price Asian call option.
Solution 44

Chapter 14, Compound Options

Since Stock Y does not pay dividends, an American call option on Stock Y has the same value as a European call option on Stock Y. Therefore, we can use the Black-Scholes formula to find the value of the call option at the end of 3 months, at which time the call option has 9 months remaining until it matures.

The first step is to calculate $d_1$ and $d_2$ at the end of time 3 months:

$$
d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(87/90) + (0.07 - 0.00 + 0.5 \times 0.30^2) \times 0.75}{0.30\sqrt{0.75}} = 0.20149
$$

$$
d_2 = d_1 - \sigma\sqrt{T} = 0.20149 - 0.30\sqrt{0.75} = -0.05832
$$

We have:

$$
N(d_1) = N(0.20149) = 0.57984
$$

$$
N(d_2) = N(-0.05832) = 0.47675
$$

The value of the call option is:

$$
C_{Eur} = S e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) = 87e^{-0.00(0.75)} \times 0.57984 - 90e^{-0.07(0.75)} \times 0.47675 = 9.73311
$$

The payoff of the put on a call is:

$$
\text{Compound put payoff} = \max\left[ x - V(S_{t_1}, K, T-t_1), 0 \right] = \max[12 - 9.73311, 0] = 2.2669
$$

Solution 45

Chapter 14, Compound Options

The strike price of the call on a put is:

$$
x = D - K \left( 1 - e^{-r(T-t_1)} \right) = 7 - 90 \left( 1 - e^{-0.08(152-91)/365} \right) = 5.80
$$

Therefore, the decision to exercise the call on a put option is equivalent to deciding not to exercise the American call option:

Don't Exercise American call option at time $t_1$ $\iff$ Exercise call on put at time $t_1$
The American call option is not exercised at the end of 91 days if the payoff from early exercise of the American call option is less than the value of the unexercised option, which is a European call option that expires in 61 days:

Don’t exercise American call early if \( S_{t_1} + D - K < C_{Eur}(S_{t_1}, K, T - t_1) \)

The table below summarizes the calculations and the resulting decisions regarding exercise of the options:

<table>
<thead>
<tr>
<th>Ex-Dividend Price: ( S_{9_1} )</th>
<th>( S_{t_1} + D - K )</th>
<th>European Call Maturing in 61 days: ( C_{Eur}(S_{t_1}, K, T - t_1) )</th>
<th>Exercise American Call early?</th>
<th>Exercise Call on Put?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$86</td>
<td>86 + 7 - 90 = 3</td>
<td>$3.02</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$87</td>
<td>87 + 7 - 90 = 4</td>
<td>$3.46</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$88</td>
<td>88 + 7 - 90 = 5</td>
<td>$3.93</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$89</td>
<td>89 + 7 - 90 = 6</td>
<td>$4.45</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$90</td>
<td>90 + 7 - 90 = 7</td>
<td>$5.00</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

The call on a put is exercised if the ex-dividend stock price is $86 in 91 days, but if the stock price is higher than $86, then the call on a put expires unexercised.

Although we have shown the full table above, it is not necessary to check all of the ex-dividend prices to answer this question. If it is optimal to exercise the American call option for some stock price, then it is also optimal to exercise the American call option at higher prices.

Alternate Solution:

Another approach is to use put-call parity to find the value of the put option at the end of 91 days. The call on the put is exercised only if the put option is worth more than $5.80.
Although the full table is provided above, it is not necessary to calculate the value of the put option for every stock price. The value of the put option declines as the stock price increases, so since the put option’s value is less than $5.80 when the stock price is $87, we can conclude that the put option’s value is also less than $5.80 if the stock price is greater than $87.

Solution 46

C  Chapter 14, Path-Dependent Options

The arithmetic average price is:

\[
\bar{S} = \frac{95 + 120 + 115 + 110 + 115 + 110 + 100 + 95 + 120 + 125 + 110 + 105}{12} = 110
\]

The payoff of Option (i) is:

\[
\text{Max}[K - \bar{S}, 0] = \text{Max}[100 - 110, 0] = 0
\]

The barrier of 130 for Option (ii) is not reached, so the option is not knocked out. Therefore, the payoff of Option (ii) is:

\[
\text{Max}[K - S, 0] = \text{Max}[120 - 105, 0] = 15
\]

The barrier of 120 for Option (iii) is reached on February 28, 2009, so the option is knocked in. Therefore, the payoff of Option (iii) is:

\[
\text{Max}[K - S, 0] = \text{Max}[115 - 105, 0] = 10
\]

The barrier of 125 for Option (iv) is reached on October 31, 2009, so the option is knocked out. Therefore, the payoff for Option (iv) is 0.

The highest payoff is 15 from Option (ii). The lowest payoff is 0 from Options (i) and (iv). The difference is:

\[
15 - 0 = 15
\]

Solution 47

D  Chapter 14,Chooser Options

The formula for the value of the chooser option is:

\[
\text{Price of Chooser Option} = C_{\text{Eur}}(S_0, K, T) + e^{-\delta(T-t_1)}P_{\text{Eur}}\left(S_0, Ke^{-(r-\delta)(T-t_1)}, t_1\right)
\]

Since \( r \) and \( \delta \) are both equal to zero, this formula can be simplified and used to find the value of a 2-year European put option:

\[
42 = 30 + P_{\text{Eur}}(100, 90, 2)
\]

\[
P_{\text{Eur}}(100, 90, 2) = 12
\]
We can now use put-call parity to find the value of the 2-year European call option:

\[
C_{Eur} (100,90,2) + Ke^{-2r} = S_0e^{-2\delta} + P_{Eur} (100,90,2)
\]

\[
C_{Eur} (100,90,2) + 90e^{-2\times0} = 100e^{-2\times0} + 12
\]

\[
C_{Eur} (100,90,2) = 22
\]

**Solution 48**

**D** Chapter 14, Chooser Options

The formula for the value of the chooser option is:

\[
\text{Price of Chooser Option} = C_{Eur} (S_0, K, T) + e^{-\delta(T-t_1)}P_{Eur} \left( S_0, Ke^{-(r-\delta)(T-t_1)}, t_1 \right)
\]

In the question, we are given the first term to the right of the equal sign:

\[
C_{Eur} (S_0, K, T) = C(50,55,0.75) = 4.00
\]

The strike price of the put option in the second term is:

\[
Ke^{-(r-\delta)(T-t_1)} = 55e^{-(0.10-0.04)(0.25)} = 54.1812
\]

We must use the Black-Scholes formula to find the price of the 6-month put option. The first step is to calculate \( d_1 \) and \( d_2 \):

\[
d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(50/54.1812) + (0.10 - 0.04 + 0.5 \times 0.30^2) \times 0.5}{0.30\sqrt{0.5}}
\]

\[
d_1 = -0.13110
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = -0.13110 - 0.30\sqrt{0.5} = -0.34323
\]

We have:

\[
N(-d_1) = N(0.13110) = 0.55215
\]

\[
N(-d_2) = N(0.34323) = 0.63429
\]

The value of the European put option is:

\[
P_{Eur} = 54.1812e^{-rT}N(-d_2) - 50e^{-\delta T}N(-d_1)
\]

\[
= 54.1812e^{-0.10(0.5)} \times 0.63429 - 50e^{-0.04(0.5)} \times 0.55215
\]

\[
= 5.62965
\]

We can now find the price of the chooser option:

\[
\text{Price of Chooser Option} = C_{Eur} (S_0, K, T) + e^{-\delta(T-t_1)}P_{Eur} \left( S_0, Ke^{-(r-\delta)(T-t_1)}, t_1 \right)
\]

\[
= 4.00 + e^{-0.04(0.75-0.50)} \times 5.62965
\]

\[
= 9.5736
\]
Solution 49

A Chapter 14, Forward Start Option

In one year, the value of the call option will be:

\[ C_{Eur}(S_1) = S_1 e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \]
\[ = S_1 e^{-0.02} N(d_1) - S_1 e^{-0.10} N(d_2) \]

In one year, the values of \( d_1 \) and \( d_2 \) will be:

\[ d_1 = \frac{\ln(S/K) + (r-\delta+0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(S_1/K_0) + (0.10 - 0.02 + 0.5 \times 0.30^2)(1)}{0.30\sqrt{1}} \]
\[ = 0.41667 \]
\[ d_2 = d_1 - \sigma\sqrt{T} = 0.41667 - 0.30\sqrt{1} = 0.11667 \]

From the normal distribution table:

\[ N(d_1) = N(0.41667) = 0.66154 \]
\[ N(d_2) = N(0.11667) = 0.54644 \]

In one year, the value of the call option can be expressed in terms of the stock price at that time:

\[ C_{Eur}(S_1) = S_1 e^{-0.02} N(d_1) - S_1 e^{-0.10} N(d_2) \]
\[ = S_1 e^{-0.02} \times 0.66154 - S_1 e^{-0.10} \times 0.54644 \]
\[ = 0.154001 \times S_1 \]

In one year, the call option will be worth 0.154001 shares of stock.

The prepaid forward price of one share of stock is:

\[ F_{0,T}^P(S) = e^{-rT} F_{0,T}(S) \]
\[ F_{0,1}^P(S) = e^{-0.10} F_{0,1}(S) \]
\[ F_{0,1}(S) = e^{-0.10} \times 50 \]
\[ F_{0,1}(S) = 45.2419 \]

The value today of 0.1540 shares of stock in one year is:

\[ 0.154001 \times 45.2419 = 6.9673 \]
Solution 50

D Chapter 14, Forward Start Option

In one year, the value of the call option will be:

\[ C_{Eur}(S_1) = S_1e^{-\delta T}N(d_1) - Ke^{-r T}N(d_2) \]
\[ = S_1e^{-0.02}N(d_1) - S_1e^{-0.10}N(d_2) \]

In one year, the values of \( d_1 \) and \( d_2 \) will be:

\[ d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(S_1 / S_1) + (0.10 - 0.02 + 0.5 \times 0.30^2)(1)}{0.30\sqrt{1}} \]
\[ = 0.41667 \]
\[ d_2 = d_1 - \sigma \sqrt{T} = 0.41667 - 0.30 \sqrt{1} = 0.11667 \]

From the normal distribution table:

\[ N(d_1) = N(0.41667) = 0.66154 \]
\[ N(d_2) = N(0.11667) = 0.54644 \]

In one year, the value of the call option can be expressed in terms of the stock price at that time:

\[ C_{Eur}(S_1) = S_1e^{-0.02}N(d_1) - S_1e^{-0.10}N(d_2) \]
\[ = S_1e^{-0.02} \times 0.66154 - S_1e^{-0.10} \times 0.54644 \]
\[ = 0.154001 \times S_1 \]

In one year, the call option will be worth 0.154001 shares of stock. One share of stock in 1 year is equivalent to \( e^{-\delta} \) shares of stock now. Therefore, 0.154001 shares of stock in 1 year is equivalent to the following quantity of shares now:

\[ 0.154001e^{-\delta} = 0.154001e^{-0.02} = 0.150952 \]

The delta of 1 share of stock is 1.0, so the delta of 0.150952 shares of stock is:

\[ 0.150952 \times 1.0 = 0.150952 \]

Solution 51

A Chapter 14, Forward Start Option

In one year, the value of the put option will be:

\[ P_{Eur}(S_1) = Ke^{-r T}N(-d_2) - Se^{-\delta T}N(-d_1) \]
\[ = S_1e^{-0.10}N(-d_2) - S_1e^{-0.02}N(-d_1) \]
In one year, the values of $d_1$ and $d_2$ will be:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln(S_1 / S_1) + (0.10 - 0.02 + 0.5 \times 0.30^2)(1)}{0.30\sqrt{1}}$$

$$= 0.41667$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.41667 - 0.30\sqrt{1} = 0.11667$$

From the normal distribution table:

$$N(-d_1) = N(-0.41667) = 0.33846$$

$$N(-d_2) = N(-0.11667) = 0.45356$$

In one year, the value of the put option can be expressed in terms of the stock price at that time:

$$P_{Eur} (S_1) = S_1 e^{-rT} N(-d_2) - S_1 e^{-\delta T} N(-d_1)$$

$$= S_1 e^{-0.10} \times 0.45356 - S_1 e^{-0.02} \times 0.33846$$

$$= 0.078640S_1$$

In one year, the put option will be worth 0.078640 shares of stock.

The prepaid forward price of one share of stock is:

$$F_{0,T}^P(S) = e^{-rT} F_{0,T}(S)$$

$$F_{0,1}^P(S) = e^{-0.10} F_{0,1}(S)$$

$$F_{0,1}^P(S) = e^{-0.10} \times 50$$

$$F_{0,1}^P(S) = 45.2419$$

The value today of 0.078640 shares of stock in one year is:

$$0.078640 \times 45.2419 = 3.55782$$

**Solution 52**

**A**  Chapter 14, Forward Start Option

At time 1, the value of the option is:

$$C_{Eur}(S_1, 1.1S_1, 0.5) = S_1 e^{-0.5\delta} N(d_1) - 1.10S_1 e^{-0.5r} N(d_2)$$

$$= S_1 \left[ e^{-0.5(0.03)} N(d_1) - 1.10 e^{-0.5(0.11)} N(d_2) \right]$$
where:

\[
d_1 = \frac{\ln \left( \frac{S_1}{1.10S_1} \right) + (0.11 - 0.03 + 0.5\sigma^2)(0.5)}{\sigma \sqrt{0.5}} = \frac{-\ln(1.10) + (0.08 + 0.5\sigma^2)(0.5)}{\sigma \sqrt{0.5}}
\]

\[
d_2 = d_1 - \sigma \sqrt{0.5}
\]

We would need to know the value of \( \sigma \) to determine the values of \( N(d_1) \) and \( N(d_2) \), but notice that these values are of independent of \( S_1 \). Therefore, the ratio of the value of the call option at time 1 to the value of the underlying stock at time 1 is constant across all possible values of \( S_1 \):

\[
\frac{C_{Eur}(S_1, 1.1S_1, 0.5)}{S_1} = e^{-0.5(0.03)}N(d_1) - 1.10e^{-0.5(0.11)}N(d_2)
\]

Let’s consider one particular possible value of \( S_1 \). If the stock price at time 1 is $150, then the strike price at time 1 will be 110% \times $150 = $165.

We can use put-call parity to find the value of a 6-month European call option with a strike price of $165 when the stock price is $150.

\[
C(S_t, K_t, T-t) - P(S_t, K_t, T-t) = F_{t,T}^P(S) - F_{t,T}^P(K)
\]

\[
C(150,165,0.5) = 17.47 = 150e^{-0.03 \times 0.5} - 165e^{-0.11 \times 0.5}
\]

\[
C(150,165,0.5) = 9.0667
\]

We can now determine the ratio of the call price to the stock price at time 1:

\[
\frac{C_{Eur}(150,165,0.5)}{150} = e^{-0.5(0.03)}N(d_1) - 1.10e^{-0.5(0.11)}N(d_2)
\]

\[
\frac{9.0667}{150} = e^{-0.5(0.03)}N(d_1) - 1.10e^{-0.5(0.11)}N(d_2)
\]

\[
0.06044 = e^{-0.5(0.03)}N(d_1) - 1.10e^{-0.5(0.11)}N(d_2)
\]

Therefore 0.06044 shares of stock at time 1 will have the same value as the call option at time 1. The prepaid forward price of 0.06044 shares delivered at time 1 is therefore equal to the value of the forward start option:

\[
0.06044e^{-\delta}S_0 = 0.06044e^{-0.03 \times 100} = 5.866
\]

**Solution 53**

E Chapter 14, Forward Start Option

*The strategies described in this question are introduced in Problem 14.22 at the end of the textbook chapter.*
The cost of Ron’s strategy is the cost of a 1-year European put option with a strike price of $95. The values of $d_1$ and $d_2$ are:

\[ d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2\sqrt{T})}{\sigma\sqrt{T}} = \frac{\ln(100/95) + [0.10 - 0 + 0.5(0.25)^2] \times 1}{0.25\sqrt{1}} \]

\[ = 0.73017 \]

\[ d_2 = d_1 - \sigma\sqrt{T} = 0.73017 - 0.25\sqrt{1} = 0.48017 \]

From the normal distribution table:

\[ N(-d_1) = N(-0.73017) = 0.23264 \]
\[ N(-d_2) = N(-0.48017) = 0.31555 \]

The cost of the 1-year European put option is:

\[ P_{Eur}(100,95,1) = 95e^{-0.10}(0.31555) - 100(0.23264) = 3.86054 \]

Now let’s consider the cost of Wally’s strategy. We must compute new values of $d_1$ and $d_2$ for the 1-month option:

\[ d_1 = \frac{\ln(S/0.95S) + (r - \delta + 0.5\sigma^2\sqrt{T})}{\sigma\sqrt{T}} = \frac{-\ln(0.95) + (0.10 - 0.00 + 0.5 \times 0.25^2)(1/12)}{0.25\sqrt{1/12}} \]

\[ = 0.86230 \]

\[ d_2 = d_1 - \sigma\sqrt{T} = 0.86230 - 0.25\sqrt{1/12} = 0.79013 \]

From the normal distribution table:

\[ N(-d_1) = N(-0.86230) = 0.19426 \]
\[ N(-d_2) = N(-0.79013) = 0.21473 \]

The cost of a 1-month put option that has a strike price of 95% of the current stock price is a function of the then-current stock price:

\[ P_{Eur}(S,0.95S,1/12) = 0.95Se^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1) \]
\[ = 0.95Se^{-0.10/12}N(-d_2) - Se^{-0.00/12}N(-d_1) \]
\[ = S\left[ 0.95e^{-0.10/12}(0.21473) - 0.19426 \right] \]
\[ = 0.00804S \]
Therefore, each of the 1-month put options can be purchased with 0.00804 shares of stock. The cost of purchasing all 12 of the options now is equal to the cost of acquiring $12 \times 0.00804 = 0.096487$ shares of stock. Since the current stock price is $100, Wally’s cost is:

$$0.096487 \times 100 = 9.6487$$

Wally’s cost therefore exceeds Ron’s cost by:

$$9.6487 - 3.8605 = 5.7882$$

**Solution 54**

C Chapter 14, Chooser Options

The price of the chooser option can be expressed in terms of a call option and put option:

$$\text{Price of Chooser Option} = C_{Eur} (S_0, K, T) + e^{-\delta(T-t_1)} P_{Eur} \left( S_0, Ke^{-(r-\delta)(T-t_1)}, t_1 \right)$$

The risk-free interest rate and the dividend yield are equal, so the put option has the same strike price as the call option:

$$\text{Price of Chooser Option} = C_{Eur} (S_0, K, T) + e^{-\delta(T-t_1)} P_{Eur} \left( S_0, Ke^{-(r-\delta)(T-t_1)}, t_1 \right)$$

$$\text{Price of Chooser Option} = C_{Eur} (95,100,3) + e^{-0.05(2)} P_{Eur} (95,100,1)$$

We can use put-call parity to find the value of the 1-year put option:

$$C_{Eur} (95,100,1) + Ke^{-0.05(1)} = e^{-0.05(1)}S_0 + P_{Eur} (95,100,1)$$

$$9.20 + 100e^{-0.05(1)} = 95e^{-0.05(1)} + P_{Eur} (95,100,1)$$

$$P_{Eur} (95,100,1) = 13.96$$

We can now solve for the price of the 3-year call option:

$$\text{Price of Chooser Option} = C_{Eur} (95,100,3) + e^{-0.05(2)} P_{Eur} (95,100,1)$$

$$28.32 = C_{Eur} (95,100,3) + 13.96e^{-0.05(2)}$$

$$C_{Eur} (95,100,3) = 15.69$$

**Solution 55**

E Chapter 14, Chooser Options and Delta

The price of the chooser option can be expressed in terms of a call option and put option:

$$\text{Price of Chooser Option} = C_{Eur} (S_0, K, T) + e^{-\delta(T-t_1)} P_{Eur} \left( S_0, Ke^{-(r-\delta)(T-t_1)}, t_1 \right)$$

$$\text{Price of Chooser Option} = C_{Eur} (50,45,4) + e^{-0.08(4-2)} P_{Eur} \left( 50,45e^{-(0.05-0.08)(4-2)}, 2 \right)$$

$$\text{Price of Chooser Option} = C_{Eur} (50,45,4) + 0.8521P_{Eur} (50,47.7826,2)$$
From the equation above, we observe that owning a chooser option is equivalent to owning the 4-year call option and 0.8521 of the 2-year put options. To find the delta of the call option, we must calculate $d_1$ with a 4-year maturity and $K = 45$:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(50/45) + (0.05 - 0.08 + 0.5(0.3)^2)4}{0.3\sqrt{4}} = 0.27560$$

$N(d_1) = N(0.27560) = 0.60857$

$$\Delta_{Call} = e^{-\delta T}N(d_1) = e^{-(0.08)4} \times 0.60857 = 0.44191$$

To find the delta of the put option, we must calculate $d_1$ with a 2-year maturity and $K = 47.7826$:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln(50/47.7826) + (0.05 - 0.08 + 0.5(0.3)^2)2}{0.3\sqrt{2}} = 0.17763$$

$N(-d_1) = N(-0.17763) = 0.42951$

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{-(0.08)2} \times 0.42951 = -0.36600$$

Now we can calculate the delta of a portfolio consisting of one of the call options and 0.8521 of the put options. This is the delta of the chooser option:

$$\Delta_{Chooser} = 1 \times \Delta_{Call} + 0.8521 \Delta_{Put} = 0.44191 + 0.8521 \times (-0.36600) = 0.13002$$

Since we wrote 100 of the chooser options, the number of shares of stock that we must purchase to delta-hedge is:

$$100 \times 0.13002 = 13.00$$

**Solution 56**

**C** Chapter 14, Gap Options

We can use put-call parity for gap options to answer this question:

$$GapCall + K_1e^{-rT} = Se^{-\delta T} + GapPut$$

$$2.7 \times GapPut + 90e^{-0.08(1)} = 100e^{-0.03(1)} + GapPut$$

$$2.7 \times GapPut - GapPut = 13.9641$$

$$GapPut = 8.2142$$

**Solution 57**

**D** Chapter 14, Chooser Options

A formula for the value of the chooser option is:

$$\text{Price of Chooser Option} = P_{Eur} \left(S_0, K, T\right) + e^{-\delta(T-t_1)}C_{Eur} \left(S_0, Ke^{(r-\delta)(T-t_1)}, t_1\right)$$
Note that:

\[ K e^{-(r-\delta)(T-t_i)} = 90 e^{-(0.08-0.02)(1-0.75)} = 88.66 \]

The price of the chooser option is:

\[
\text{Price of Chooser Option} = P_{\text{Eur}}(85,90,1) + e^{-0.02(1-0.75)} C_{\text{Eur}}(85,88.66,0.75) \\
= 9.80 + e^{-0.02(0.25)} 8.76 = 18.5163
\]

**Solution 58**

**B**  Chapter 14, Exchange Options

Let’s treat this option as a call option. The underlying asset is one share of Stock A. The strike asset is \( \frac{a}{\beta} \) shares of Stock B. Therefore, we have:

\[ S = \alpha \]
\[ K = \frac{a}{\beta} \times \beta = \alpha \]

The volatility of \( \ln(S/K) \) is:

\[ \sigma = \sqrt{\sigma_s^2 + \sigma_K^2 - 2\rho \sigma_s \sigma_K} = \sqrt{0.3^2 + 0.5^2 - 2(0.4)(0.3)(0.5)} = \sqrt{0.22} = 0.46904 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
d_1 = \frac{\ln \left( \frac{Se^{-\delta_s T}}{Ke^{-\delta_K T}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{\alpha}{\alpha} \right) + 0.46904^2(1)}{0.46904(1)} = 0.23452
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = 0.23452 - 0.46904 = -0.23452
\]

We have:

\[ N(d_1) = N(0.23452) = 0.59271 \]
\[ N(d_2) = N(-0.23452) = 0.40729 \]

The value of the exchange call is:

\[
\text{ExchangeCallPrice} = Se^{-\delta_s T} N(d_1) - Ke^{-\delta_K T} N(d_2) \\
= \alpha(0.59271) - \alpha(0.40729) \\
= 0.18542 \alpha
\]
Solution 59
D Chapter 14, Forward Start Options

The values of \( d_1 \) and \( d_2 \) for a 6-month put option with a strike price that is 85% of the current stock price are:

\[
d_1 = \frac{\ln\left(\frac{S}{0.85S}\right) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{-\ln(0.85) + (0.06 - 0.00 + 0.5 \times 0.4^2)(0.5)}{0.4\sqrt{0.5}} = 0.82208
\]
\[
d_2 = d_1 - \sigma\sqrt{T} = 0.82208 - 0.4\sqrt{0.5} = 0.53924
\]

From the normal distribution table:
\[
N(-d_1) = N(-0.82208) = 0.20552
\]
\[
N(-d_2) = N(-0.53924) = 0.29486
\]

The cost of a 6-month put option that has a strike price of 85% of the current stock price is a function of the then-current stock price:

\[
P_{\text{Eur}}(S, 0.85S, 0.5) = 0.85Se^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)
\]
\[
= 0.85Se^{-0.5 \times 0.06} N(-d_2) - Se^{-0.00} N(-d_1)
\]
\[
= S\left[0.85e^{-0.03}(0.29486) - 0.20552\right]
\]
\[
= 0.03770S
\]

Therefore, each of the 6-month put options can be purchased with 0.03770 shares of stock. The cost of purchasing both of the options now is equal to the cost of acquiring 2 \times 0.03770 = 0.07541 shares of stock. Since the current stock price is $75, this cost is:

\[
0.07541 \times 75 = 5.65556
\]

Solution 60
C Chapter 14, Forward Start Options

The values of \( d_1 \) and \( d_2 \) for a 6-month put option with a strike price that is 85% of the current stock price are:

\[
d_1 = \frac{\ln\left(\frac{S}{0.85S}\right) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{-\ln(0.85) + (0.06 - 0.05 + 0.5 \times 0.4^2)(0.5)}{0.4\sqrt{0.5}} = 0.73369
\]
\[
d_2 = d_1 - \sigma\sqrt{T} = 0.73369 - 0.4\sqrt{0.5} = 0.45085
\]

From the normal distribution table:
\[
N(-d_1) = N(-0.73369) = 0.23157
\]
\[
N(-d_2) = N(-0.45085) = 0.32605
\]
The cost of a 6-month put option that has a strike price of 85% of the current stock price is a function of the then-current stock price:

\[
P_{Eur}(S, 0.85S, 0.5) = 0.85Se^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)
\]

\[
= 0.85Se^{-0.5\times0.06}N(-d_2) - Se^{-0.5\times0.05}N(-d_1)
\]

\[
= S \left[ 0.85e^{-0.03}(0.32605) - e^{-0.025}(0.23157) \right]
\]

\[
= 0.043099S
\]

Therefore, the first put option can be purchased with 0.043099 shares of stock now, and the value of this purchase price is:

\[
0.043099S = 0.043099 \times 75 = 3.23244
\]

The second option can be purchased with 0.043099 shares of stock in 6 months. The current value of a share of stock 6 months from now is its prepaid forward price, so the time 0 value of this purchase price is:

\[
0.043099F_{0.0.5}(S) = 0.043099 \times 75e^{-0.5\times0.05} = 3.15263
\]

The broker is paid the sum of the two prices:

\[
3.23244 + 3.15263 = 6.38507
\]

**Solution 61**

**C Chapter 14, Barrier Options**

As the barrier of an up-and-out call approaches infinity, the price of the up-and-out call approaches the price of a regular call.

Therefore, a regular call option with a strike price of $85 can be read from the far right cell of the first table:

\[
C(85) = 12.6264
\]

The value of the corresponding up-and-in calls can be determined with the parity relationship for barrier options:

\[
\text{Up-and-in call} + \text{Up-and-out call} = \text{Ordinary call}
\]

For the $95 and $105 barriers, we have:

<table>
<thead>
<tr>
<th>H</th>
<th>Up-and-out Call</th>
<th>Up-and-in Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>0.0239</td>
<td>12.6264 - 0.0239 = 12.6025</td>
</tr>
<tr>
<td>105</td>
<td>0.2072</td>
<td>12.6264 - 0.2072 = 12.4192</td>
</tr>
</tbody>
</table>

Let’s use \( \text{Max}(1) \) to denote the running maximum of the stock price from time 0 to time 1:

\[
\text{Max}(1) = \max_{0 \leq t \leq 1} S(t)
\]
The payoff to the special option depends on the running maximum of the stock price:

\[ 95 \leq Max(1) < 105 \quad \Rightarrow \quad \text{Payoff} = 3 \times Max[S(1) - 85, 0] \]
\[ 105 \leq Max(1) \quad \Rightarrow \quad \text{Payoff} = 2 \times Max[S(1) - 85, 0] \]

If the running maximum is between $95 and $105, then the payoff is the same as the payoff of 3 calls. Over this range, we can replicate the payoff by purchasing 3 $95-barrier up-and-in calls. But if the running maximum reaches $105, then the payoff is the same as 2 calls. So we need to reduce the payoff by one call if the running maximum reaches $105. We can do this by selling a $105-barrier up-and-in call.

Therefore, the special option consists of 3 of the $95-barrier up-and-in calls and a short position in 1 of the $105-barrier up-and-in calls. The price of the special option is equal to the cost of establishing this position:

\[ 3 \times 12.6025 - 1 \times 12.4192 = 25.3883 \]

**Solution 62**

C Chapter 14, Barrier Options

Since the current price of the stock is $65, the first four knock-out puts are down-and-out puts. The remaining put (with \( H \) of \( \infty \) and knock-out put price of 8.4825) is an up-and-out put.

When the barrier of a knock-out option is infinity, it is certain that the barrier will not be reached and the option will not be knocked out. Therefore, the price of a regular put option with a strike price of $60 can be read from the far right cell of the first table:

\[ P(60) = 8.4825 \]

The value of the corresponding down-and-in puts can be determined with the parity relationship for barrier options:

\[ \text{Down-and-in put} + \text{Down-and-out put} = \text{Ordinary put} \]

For the $30 and $40 barriers, we have:

<table>
<thead>
<tr>
<th>( H )</th>
<th>Down-and-out Put</th>
<th>Down-and-in Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>4.1830</td>
<td>8.4825 - 4.1830 = 4.2995</td>
</tr>
<tr>
<td>40</td>
<td>1.1175</td>
<td>8.4825 - 1.1175 = 7.3650</td>
</tr>
</tbody>
</table>

Let’s use \( Min(1) \) to denote the running minimum of the stock price from time 0 to time 1:

\[ Min(1) = \min_{0 \leq t \leq 1} S(t) \]
The payoff to the special option depends on the running minimum of the stock price at the end of one year:

\[
30 < \text{Min}(1) \leq 40 \Rightarrow \text{Payoff} = 5 \times \text{Max}[60 - S(1), 0]
\]

\[
\text{Min}(1) \leq 30 \Rightarrow \text{Payoff} = 3 \times \text{Max}[60 - S(1), 0]
\]

If the running minimum is between $30 and $40, then the payoff is the same as the payoff of 5 puts. Over this range, we can replicate the payoff by purchasing 5 $40-barrier down-and-in puts. But if the running minimum is less than or equal to $30, then the payoff is the same as 3 puts. So we need to reduce the payoff by two puts if the running minimum is less than $30. We can do this by selling two $30-barrier down-and-in puts.

Therefore, the special option consists of 5 of the $40-barrier down-and-in puts and a short position in 2 of the $30-barrier down-and-in puts. The price of the special option is equal to the cost of establishing this position:

\[
5 \times 7.3650 - 2 \times 4.2995 = 28.2260
\]

Solution 63

Consider an asset, which we will call Asset X, which has a payoff of:

\[X_1 = \text{Min}[S_1(1), 2S_2(1)]\]

From statement (vi), we know that the price of a European call option on Asset X, with a strike price of 22, is 0.237. The payoff of this call option is:

\[\text{Max}\{\text{Min}[S_1(1), 2S_2(1)] - 22, 0\} = \text{Max}\{X_1 - 22, 0\}\]

We are asked to find the price of an option that has the payoff described below:

\[\text{Max}\{22 - \text{Min}[S_1(1), 2S_2(1)], 0\} = \text{Max}\{22 - X_1, 0\}\]

We recognize this as a European put option on Asset X. From put-call parity, we have:

\[C_{\text{Eur}} + Ke^{-r(T-t)} = S_t + P_{\text{Eur}}\]

\[0.237 + 22e^{-0.06\times1} = X_0 + P_{\text{Eur}}\]

If we can find the current value of Asset X, then we can solve for the value of the put option. Asset X can be replicated by purchasing 1 share of Stock 1, and selling an exchange call option that allows its owner to exchange 2 shares of Stock 2 for 1 share of Stock 1:

\[X_1 = \text{Min}[S_1(1), 2S_2(1)] = S_1(1) - \text{Max}[S_1(1) - 2S_2(1), 0]\]

The current value of Asset X is the value of Stock 1 minus the value of the exchange call:

\[X_0 = F_{0,1}^P(\text{Min}[S_1(1), 2S_2(1)]) = F_{0,1}^P(S_1(0)) - C_{\text{Eur}}(S_1(0), 2S_2(0), 1)\]

\[= S_1(0) - \text{ExchangeCallPrice}\]
The exchange call option is an exchange call option with 1 share of Stock 1 as the underlying asset and 2 shares of Stock 2 as the strike asset. To find the value of this exchange call option, we find the appropriate volatility parameter:

\[
\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho \sigma_S \sigma_K} = \sqrt{0.15^2 + 0.20^2 - 2(-0.30)(0.15)(0.20)} = 0.28373
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln \left( \frac{S \cdot e^{-\delta_S T}}{K \cdot e^{-\delta_K T}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{20e^{-0.01}}{(2 \times 11)e^{-0.01}} \right) + 0.28373^2 \times 1}{0.28373 \sqrt{1}} = -0.19406
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = -0.19406 - 0.28373 \sqrt{1} = -0.47779
\]

From the normal table, we have:

\[
N(d_1) = N(-0.19406) = 0.42306
\]

\[
N(d_2) = N(-0.47779) = 0.31640
\]

The value of the exchange call option is:

\[
\text{ExchangeCallPrice} = S e^{-\delta_S T} N(d_1) - K e^{-\delta_K T} N(d_2)
\]

\[
= 20e^{0.01}(0.42306) - 2 \times 11e^{0.01}(0.31640)
\]

\[
= 1.5004
\]

The current value of Asset X is:

\[
X_0 = S_1(0) - \text{ExchangeCallPrice} = 20 - 1.5004 = 18.4996
\]

We can now use put-call parity to find the value of the put option:

\[
0.237 + 22e^{-0.06 \times 1} = X_0 + P_{\text{Eur}}
\]

\[
0.237 + 22e^{-0.06} = 18.4996 + P_{\text{Eur}}
\]

\[
P_{\text{Eur}} = 2.4562
\]

**Solution 64**

**D**  
Chapter 14, Exchange Options

Consider an asset, which we will call Asset X, which has a payoff of:

\[
X_1 = \text{Max}[S_1(1), 2S_2(1)]
\]

From statement (vi), we know that the price of a European put option on Asset X, with a strike price of 22, is 0.29. The payoff of this put option is:

\[
\text{Max}[22 - \text{Max}[S_1(1), 2S_2(1)], 0] = \text{Max}[22 - X_1, 0]
\]

We are asked to find the price of an option that has the payoff described below:

\[
\text{Max}\{\text{Max}[S_1(1), 2S_2(1)] - 22, 0\} = \text{Max}\{X_1 - 22, 0\}
\]
We recognize this as a European call option on Asset X. From put-call parity, we have:

\[ C_{Eur} + Ke^{-r(T-t)} = S_t + P_{Eur} \]
\[ C_{Eur} + 22e^{-0.06 \times 1} = X_0 + 0.29 \]

If we can find the current value of Asset X, then we can solve for the value of the call option. Asset X can be replicated by purchasing 2 shares of Stock 2, and buying an exchange call option that allows its owner to exchange 2 shares of Stock 2 for 1 share of Stock 1:

\[ X_1 = \text{Max}[S_1(1), 2S_2(1)] = 2S_2(1) + \text{Max}[S_1(1) - 2S_2(1), 0] \]

The current value of Asset X is the value of 2 shares of Stock 2 plus the value of the exchange call:

\[ X_0 = F_{0,1}^P(\text{Max}[S_1(1), 2S_2(1)]) = F_{0,1}^P(2S_2(0)) + C_{Eur}(S_1(0), 2S_2(0), 1) \]
\[ = 2S_2(0) + \text{ExchangeCallPrice} \]

The exchange call option is an exchange call option with 1 share of Stock 1 as the underlying asset and 2 shares of Stock 2 as the strike asset. To find the value of this exchange call option, we find the appropriate volatility parameter:

\[ \sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho \sigma_S \sigma_K} = \sqrt{0.15^2 + 0.20^2 - 2(-0.30)(0.15)(0.20)} = 0.28373 \]

The values of \( d_1 \) and \( d_2 \) are:

\[ d_1 = \frac{\ln \left( \frac{Se^{-\delta S^T}}{Ke^{-\delta K^T}} \right) + \frac{\sigma^2 T}{2}}{\frac{\sigma \sqrt{T}}{2}} = \frac{\ln \left( \frac{20e^{-0 \times 1}}{(2 \times 11)e^{-0 \times 1}} \right) + \frac{0.28373^2 \times 1}{2}}{0.28373 \sqrt{1}} = -0.19406 \]
\[ d_2 = d_1 - \sigma \sqrt{T} = -0.19406 - 0.28373 \sqrt{1} = -0.47779 \]

From the normal table, we have:

\[ N(d_1) = N(-0.19406) = 0.42306 \]
\[ N(d_2) = N(-0.47779) = 0.31640 \]

The value of the exchange call option is:

\[ \text{ExchangeCallPrice} = Se^{-\delta S^T} N(d_1) - Ke^{-\delta K^T} N(d_2) \]
\[ = 20e^{0 \times 1}(0.42306) - 2 \times 11e^{0 \times 1}(0.31640) \]
\[ = 1.5004 \]

The current value of Asset X is:

\[ X_0 = 2S_2(0) + \text{ExchangeCallPrice} = 2 \times 11 + 1.5004 = 23.5004 \]
We can now use put-call parity to find the value of the call option:

\[ C_{Eur} + 22e^{-0.06 \times 1} = X_0 + 0.29 \]
\[ C_{Eur} + 22e^{-0.06} = 23.5004 + 0.29 \]
\[ C_{Eur} = 3.0716 \]

**Solution 65**

E Chapter 14, Exchange Options

Consider an asset, which we will call Asset X, which has a payoff of:

\[ X_1 = \text{Max}[S_1(1), 2S_2(1)] \]

From statement (vi), we know that the price of a European put option on Asset X, with a strike price of 22, is 0.29. The payoff of this put option is:

\[ \text{Max}\{22 - \text{Max}[S_1(1), 2S_2(1)], 0\} = \text{Max}[22 - X_1, 0] \]

We are asked to find the price of an option that has the payoff described below:

\[ \text{Max}[S_1(1), 2S_2(1), 22] = \text{Max}[X_1, 22] \]

This payoff can be obtained by owning an asset that has a payoff of \( X_1 \) and the put option:

\[ \text{Max}[X_1, 22] = X_1 + \text{Max}[22 - X_1, 0] \]

The present value of this payoff is the current value of Asset X plus the value of the European put option on Asset X:

\[ F_{0,1}^P (\text{Max}[X_1, 22]) = F_{0,1}^P (X_1) + F_{0,1}^P (\text{Max}[22 - X_1, 0]) = X_0 + 0.29 \]

If we can find the current value of Asset X, then we can solve for the value of the rainbow option. Asset X can be replicated by purchasing 2 shares of Stock 2, and buying an exchange call option that allows its owner to exchange 2 shares of Stock 2 for 1 share of Stock 1:

\[ X_1 = \text{Max}[S_1(1), 2S_2(1)] = 2S_2(1) + \text{Max}[S_1(1) - 2S_2(1), 0] \]

The current value of Asset X is the value of 2 shares of Stock 2 plus the value of the exchange call:

\[ X_0 = F_{0,1}^P (\text{Max}[S_1(1), 2S_2(1)]) = F_{0,1}^P (2S_2(0)) + C_{Eur} (S_1(0), 2S_2(0), 1) = 2S_2(0) + \text{ExchangeCallPrice} \]
The exchange call option is an exchange call option with 1 share of Stock 1 as the underlying asset and 2 shares of Stock 2 as the strike asset. To find the value of this exchange call option, we find the appropriate volatility parameter:

$$
\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K} = \sqrt{0.15^2 + 0.20^2 - 2(-0.30)(0.15)(0.20)} = 0.28373
$$

The values of \(d_1\) and \(d_2\) are:

$$
d_1 = \frac{\ln \left( \frac{S_1 e^{-\delta S T}}{K e^{-\delta K T}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \ln \left( \frac{20e^{-0.1}}{(2 \times 11)e^{-0.1}} \right) + \frac{0.28373^2 \times 1}{2} = -0.19406
$$

$$
d_2 = d_1 - \sigma \sqrt{T} = -0.19406 - 0.28373 \sqrt{1} = -0.47779
$$

From the normal table, we have:

\[\begin{align*}
N(d_1) &= N(-0.19406) = 0.42306 \\
N(d_1) &= N(-0.47779) = 0.31640
\end{align*}\]

The value of the exchange call option is:

$$
\text{ExchangeCallPrice} = S_1 e^{-\delta S T} N(d_1) - K e^{-\delta K T} N(d_2)
$$

$$
= 20e^{-0.1} (0.42306) - 2 \times 11 e^{0.1} (0.31640)
$$

$$
= 1.5004
$$

The current value of Asset X is:

$$
X_0 = 2S_2(0) + \text{ExchangeCallPrice} = 2 \times 11 + 1.5004 = 23.5004
$$

The value of the rainbow option is:

$$
F_{0,1}^P \left( \text{Max} \left[ X_1, 22 \right] \right) = X_0 + 0.29 = 23.5004 + 0.29 = 23.7904
$$

**Solution 66**

A Chapter 14, American Call on Dividend-Paying Stock

*The trick to this question is to recognize that the American call option is never called early, and therefore its value must be the same as that of a corresponding European call option.*

At time 0.5, the American call is exercised only if the cum-dividend stock price minus the strike price is greater than the value of retaining the option as a European call option. That is, if the American call option is exercised at time 0.5, then the following condition must be satisfied:

$$
S_{t_1} + D - K > C_{Eur}(S_{t_1}, K, T - t_1)
$$
Let’s put in the values provided in the question and make use of put-call parity to rewrite the value of the European call option at time 0.5:

\[
S_{0.5} + 2 - 82 > C_{Eur}(S_{t_1}, 82, 0.5)
\]
\[
S_{0.5} + 2 - 82 > -82e^{-0.05 	imes 0.5} + S_{0.5} + P_{Eur}(S_{t_1}, 82, 0.5)
\]
\[
-0.02459 > P_{Eur}(S_{t_1}, 82, 0.5)
\]

Since the minimum possible value of the put option at time 0.5 is zero, it is impossible for the inequality above to be satisfied at time 0.5. Therefore, the American call option is never called early, and its value is the same as that of a corresponding European call option.

Here’s another way to reach the conclusion that the American call option will not be exercised early. At time 0.5, the present value of the dividends is less than the present value of the interest cost of paying the strike price early:

\[
PV_{0.5, 1}(\text{div}) < K - Ke^{-r(T-t)}
\]
\[
2 < 82 - 82e^{-0.05(0.5)}
\]
\[
2 < 2.0246
\]

Therefore, the following necessary but not sufficient condition for early exercise (which appears in the Chapter 9 Review Note) is not satisfied:

\[
PV_{t, T}(\text{div}) > K - Ke^{-r(T-t)}
\]

The prepaid forward price of the stock is:

\[
F_{0,T}^P(S) = S_0 - PV_{0,T}(\text{Div}) = 80 - 2e^{-(-0.05 	imes 0.5)} = 78.04938
\]

The volatility of the prepaid forward price is equal to the volatility of the forward price:

\[
\sigma_{PF} = \sqrt{\frac{\text{Var} \left( \ln \left( F_{t,1}^P(S) \right) \right)}{t}} = \sqrt{\frac{\text{Var} \left[ \ln F_{t,1}^P \right]}{t}} = \sqrt{\frac{0.0625t}{t}} = \sqrt{0.0625} = 0.25
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma_{PF}^2T}{\sigma_{PF}\sqrt{T}} = \frac{\ln \left( \frac{78.04938}{82e^{-0.05 	imes 1}} \right) + 0.5 \times 0.25^2 \times 1}{0.25 \sqrt{1}} = 0.12749
\]
\[
d_2 = d_1 - \sigma_{PF}\sqrt{T} = 0.12749 - 0.25\sqrt{1} = -0.12251
\]

We have:

\[
N(d_1) = N(0.12749) = 0.55072
\]
\[
N(d_2) = N(-0.12251) = 0.45125
\]
The value of the call option is:
\[
C_{Eur} = F_{0,T}^P(S)N(d_1) - F_{0,T}^P(K)N(d_2) = 78.04938 \times 0.55072 - 82e^{-0.05} \times 0.45125 \\
= 7.7855
\]

**Solution 67**

**B Chapter 14, Barrier Options**

The value of the corresponding up-and-in calls can be determined with the parity relationship for barrier options:

\[
\text{Up-and-in call} + \text{Up-and-out call} = \text{Ordinary call}
\]

Let's use \(C_{Eur}\) to denote the price of an ordinary call. For the $95, $105, and $115 barriers, we have the following prices:

<table>
<thead>
<tr>
<th>H</th>
<th>Up-and-out Call</th>
<th>Up-and-in Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>0.0239</td>
<td>(C_{Eur} - 0.0239)</td>
</tr>
<tr>
<td>105</td>
<td>0.2072</td>
<td>(C_{Eur} - 0.2072)</td>
</tr>
<tr>
<td>115</td>
<td>0.6650</td>
<td>(C_{Eur} - 0.6650)</td>
</tr>
</tbody>
</table>

Let's use \(\text{Max}(1)\) to denote the running maximum of the stock price from time 0 to time 1:

\[
\text{Max}(1) = \max_{0 \leq t \leq 1} S(t)
\]

The payoff to the special option depends on the running maximum of the stock price:

\[
\begin{align*}
95 \leq \text{Max}(1) < 105 & \quad \Rightarrow \quad \text{Payoff} = 3 \times \text{Max}[S(1) - 85, 0] \\
105 \leq \text{Max}(1) < 115 & \quad \Rightarrow \quad \text{Payoff} = 2 \times \text{Max}[S(1) - 85, 0] \\
115 \leq \text{Max}(1) & \quad \Rightarrow \quad \text{Payoff} = 0
\end{align*}
\]

Let's consider each possibility for the running maximum:

- If the running maximum is between $95 and $105, then the payoff is the same as the payoff of 3 calls. Over this range, we can replicate the payoff by purchasing 3 $95-barrier up-and-in calls.

- If the running maximum is between $105 and $115, then the payoff is the same as 2 calls. So we need to reduce the payoff obtained above by one call if the running maximum is between $105 and $115. We can do this by selling a $105-barrier up-and-in call.

- If the running maximum is above $115, then the payoff is zero. So we need to reduce the payoff obtained above by 2 calls if the running maximum is above $115. We can do this by selling 2 $115-barrier up-and-in calls.
Therefore, the special option consists of 3 of the $95-barrier up-and-in calls, a short position in 1 of the $105-barrier up-and-in calls, and a short position in 2 of the $115-barrier up-and-in calls. The price of the special option is equal to the cost of establishing this position:

\[
3 \times (C_{Eur} - 0.0239) - 1 \times (C_{Eur} - 0.2072) - 2 \times (C_{Eur} - 0.6650) = 1.4655
\]

**Solution 68**

A Chapter 14, Asian Options

Newly issued average price Asian options are worth less than otherwise equivalent ordinary options because the average stock price exhibits less volatility than the final stock price. Therefore, we can rule out answer choices C, D, and E.

Making use of put-call parity, the ordinary call option is worth more than the ordinary put option:

\[
C_{Eur} - P_{Eur} = S_0 e^{-\delta T} - Ke^{-r T}
\]

\[
C_{Eur} - P_{Eur} = 100 - 100e^{-0.03(2)}
\]

\[
C_{Eur} - P_{Eur} = 5.82 \Rightarrow C_{Eur} > P_{Eur}
\]

**Solution 69**

C Chapter 14, Put-Call Parity For Gap Options

Option B must have a higher price than Option A, because although they have the same trigger price, Option B has a higher strike price. Therefore Option A cannot have the highest price.

Option C must have a higher price than Option B, because although they have the same strike price, if \(84 < S_T < 85\), then Option C has a positive payoff while Option B has a zero payoff. Therefore, Option B cannot have the highest price.

Option D must have a higher price than Option E, because although they have the same trigger price, Option D has a lower strike price. Therefore, Option E cannot have the highest price.

The correct answer must be either Option C or Option D. We can use gap put-call parity to compare the two options:

\[
GapCall + K_1 e^{-r T} = Se^{-\delta T} + GapPut
\]

\[
GapCall - GapPut = Se^{-\delta T} - K_1 e^{-r T}
\]

\[
(\text{Option D}) - (\text{Option C}) = 85e^{-0.04(1)} - 86e^{-0.05(1)}
\]

\[
(\text{Option D}) - (\text{Option C}) = -0.1386
\]

Since the difference above is negative, Option C must have a higher price than Option D.
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Chapter 18 – Solutions

Solution 1

C Chapter 18, The Lognormal Distribution and its Parameters

Choice A is false. The observed mean is determined by the difference between the beginning and ending stock price, so more frequent observations do not change the observed mean. (See page 607 of the textbook.)

Choice B is false. Observations taken over a longer period of time are more likely to reflect the true mean and therefore they increase the precision of the estimate. (See page 607 of the textbook.)

Choice C is true. More frequent observations allow us to better observe the extent to which the returns vary. (See page 607 of the textbook.)

Choice D is false. The median price is less than the expected price:

\[ \text{Median of } S_T = S_0 e^{(\alpha - \delta - 0.5\sigma^2)T} \]

By definition, the return is less than or equal to the median return 50% of the time. Since the median return is less than the expected return, this implies that the return is less than or equal to the mean return more than 50% of the time. (See page 597 of the textbook.)

Choice E is false. The product of lognormal random variables is lognormal. Suppose that \( y_1 \) and \( y_2 \) are lognormal, and \( x_1 \) and \( x_2 \) are normal. Suppose further that:

\[ y_1 = e^{x_1} \quad y_2 = e^{x_2} \]

The product of the two lognormal random variables is lognormal, because the sum of the two normal random variables is normal:

\[ y_1 \times y_2 = e^{x_1}e^{x_2} = e^{x_1+x_2} \]

(See page 593 of the textbook.)

Solution 2

C Chapter 18, Converting to a Standard Normal Random Variable

We use the following relationship to convert the draws into their standard normal equivalents:

\[ z = \frac{x - \mu}{\sigma} \]

The 3 standard normal equivalents are:

\[ z = \frac{-4 - 3}{\sqrt{10}} = -2.21 \quad z = \frac{5 - 3}{\sqrt{10}} = 0.63 \quad z = \frac{3 - 3}{\sqrt{10}} = 0.00 \]
Solution 3

A  Chapter 18, Sums of Normal Random Variables

There is no need to calculate the expected value to answer this question, because all of the answer choices have the same expected value. For the sake of thoroughness, we calculate the expected value below.

The expected value of the linear combination is:

\[
E[x_1 - x_2 + 3x_3] = 2 - (-1) + 3 \times 4 = 15
\]

The variance of the linear combination is:

\[
\text{Var}(ax_1 + bx_2 + cx_3) = a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2 + 2ab \sigma_{1,2} + 2bc \sigma_{2,3} + 2ac \sigma_{1,3}
\]

\[
= 4 + 3 + 9 \times 6 - 2 \times 0.3 \times \sqrt{4 \times 3} - 2 \times 3 \times 0.4 \times \sqrt{3} \times 6 + 2 \times 3 \times (-0.5) \times \sqrt{4 \times 6}
\]

\[
= 34.04
\]

Solution 4

D  Chapter 18, Median Stock Price

The formulas for the expected value and the median are:

\[
E[S_T] = S_t e^{(\alpha - \delta)(T-t)}
\]

\[
\text{Median} = S_t e^{(\alpha - \delta - 0.5 \sigma^2)(T-t)}
\]

We can substitute the expected value into the formula for the median:

\[
\text{Median} = S_t e^{(\alpha - \delta - 0.5 \sigma^2)(T-t)} = S_t e^{(\alpha - \delta)(T-t)} e^{-0.5 \sigma^2(T-t)} = E[S_T] e^{-0.5 \sigma^2(T-t)}
\]

The median price of the stock in 3 years is:

\[
\text{Median} = E[S_T] e^{-0.5 \sigma^2(T-t)} = 135 e^{-0.5(0.3)^2(3-0)} = 117.95
\]

Solution 5

A  Chapter 18, Expected Value

We can use the expected value of the stock price to determine \((\alpha - \delta)\):

\[
E[S_T | S_t] = S_t e^{(\alpha - \delta)(T-t)}
\]

\[
1.27125 = 1 e^{(\alpha - \delta)(4-0)}
\]

\[(\alpha - \delta) = 0.0600
\]

Since the Black-Scholes framework holds, we know that:

\[
\ln \left( \frac{S_T}{S_t} \right) - N \left[ (\alpha - \delta - 0.5 \sigma^2)(T-t), \sigma^2(T-t) \right] \quad T > t
\]
The variance of the natural log of the stock price can be used to determine \( \sigma \):

\[
Var[\ln S(4)|S(0)] = 0.49 \\
\sigma^2(4 - 0) = 0.49 \\
\sigma = 0.35
\]

The expected value of the natural log of the time-8 stock price is:

\[
E[\ln S(8)|S(0)] = S(0)E\left[ \frac{\ln S(8)}{S(0)} \right] = 1 \times (\alpha - \delta - 0.5\sigma^2)(8 - 0) \\
= (0.0600 - 0.5 \times 0.35^2)8 = -0.01
\]

**Solution 6**

**B** Chapters 12 and 18, Elasticity and Risk Premium

Since the Black-Scholes framework applies, we have:

\[
\ln \left( \frac{S_T}{S_t} \right) = N\left( (\alpha - \delta - 0.5\sigma^2)(T - t), \sigma^2(T - t) \right)
\]

Therefore, from (iii), we have:

\[
\alpha - \delta - 0.5\sigma^2 = 0.14
\]

Under the risk-neutral probability measure, the expected return is \( r \), and so from (iv), we have:

\[
(r - \delta - 0.5\sigma^2)(4 - 2) = 0.10 \\
r - \delta - 0.5\sigma^2 = 0.05
\]

Subtracting the expression above from the one obtained from (iii), we have the risk premium for the stock:

\[
\alpha - r = 0.14 - 0.05 = 0.09
\]

This question is modified from the MFE/3F Sample Exam, so the somewhat confusing language in (vii) is the same as it was written there. In (vii), we are not told if the put option being hedged has been purchased or sold. Therefore, we know that one of the following is true:

\[
SA = 42.43 \quad \text{or} \quad -SA = 42.43
\]

Since the \( \Delta \) of a put is negative, the second expression above must be true. Therefore, we have:

\[
SA = -42.43
\]

We can now find the elasticity of the put option:

\[
\Omega = \frac{SA}{V} = \frac{-42.43}{15} = -2.8287
\]
The formula for the risk premium of the option allows us to solve for the absolute value of its expected return:

\[
\gamma - r = \Omega \times (\alpha - r) \\
\gamma - 0.07 = -2.8287 \times (0.09) \\
\gamma = -0.18458 \\
|\gamma| = 0.18458
\]

**Solution 7**

**B Chapter 18, Stock Price Probabilities**

The probability that the future stock price exceeds a specified amount can be found with the following formula:

\[
\text{Prob}(S_T > K) = \Phi(\hat{d}_2) \\
\text{where: } \hat{d}_2 = \frac{\ln \left( \frac{S_t}{K} \right) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}}
\]

We begin by finding:

\[
\hat{d}_2 = \frac{\ln \left( \frac{50}{65} \right) + (0.10 - 0 - 0.5(0.25)^2)(2 - 0)}{0.25\sqrt{2 - 0}} = -0.35317
\]

We can now find the probability that \( S_t \) is greater than $65:

\[
\text{Prob}(S_T > 65) = \Phi(-0.35317) = 0.36198
\]

**Solution 8**

**E Chapter 18, Conditional Expectation**

The conditional expected value is:

\[
E[S_T | S_T > K] = \frac{S_0 e^{(\alpha - \delta)(T - t)} N(\hat{d}_1)}{N(\hat{d}_2)}
\]
The values of $\hat{d}_1$ and $\hat{d}_2$ are:

$$\hat{d}_1 = \frac{\ln \left( \frac{S_t}{K} \right) + (\alpha - \delta + 0.5\sigma^2)(T-t)}{\sigma \sqrt{T-t}} = \frac{\ln \left( \frac{50}{65} \right) + (0.10 - 0 + 0.5(0.25)^2)(2-0)}{0.25\sqrt{2}-0}$$
$$\hat{d}_1 = 0.00038$$

$$\hat{d}_2 = \frac{\ln \left( \frac{S_t}{K} \right) + (\alpha - \delta - 0.5\sigma^2)(T-t)}{\sigma \sqrt{T-t}} = \frac{\ln \left( \frac{50}{65} \right) + (0.10 - 0 - 0.5(0.25)^2)(2-0)}{0.25\sqrt{2}-0}$$
$$\hat{d}_2 = -0.35317$$

The values of $N(\hat{d}_1)$ and $N(\hat{d}_2)$ are:

$$N(\hat{d}_1) = N(0.00038) = 0.50015$$
$$N(\hat{d}_2) = N(-0.35317) = 0.36198$$

The conditional expectation is:

$$E\left[ S_2 | S_2 > 65 \right] = \frac{50e^{(0.10-0)(2-0)} \times 0.50015}{0.36198} = 84.3810$$

**Solution 9**

**E Chapter 18, Effect of Increasing Time Until Maturity**

*If we can find an exception to a statement, then we know that the statement is not true.*

Let’s begin by considering the probability that the time $t$ stock price is less than $K$:

$$\text{Prob}(S_t < K) = N(-\hat{d}_2) = N\left( -\frac{\ln \left( \frac{S_0}{K} \right) + (\alpha - \delta - 0.5\sigma^2)t}{\sigma \sqrt{t}} \right)$$

If we can find parameters for $S_0$, $K$, $\alpha$, $\delta$, and $\sigma$ such that an increase in $t$ leads to a decrease in $\text{Prob}(S_t < K)$, then Choice A must be false. To simplify the expression, let’s choose $S_0 = K$, so that we have:

$$\text{Prob}(S_t < K) = N\left( -\frac{\ln (1) + (\alpha - \delta - 0.5\sigma^2)t}{\sigma \sqrt{t}} \right) = N\left( -\frac{-\alpha + \delta + 0.5\sigma^2}{\sigma} \right) \sqrt{t}$$

From the above expression we can see that the probability decreases as $t$ increases when the numerator of the expression at right above is negative. We can make the numerator negative by setting the parameters so that:

$$\alpha > \delta + 0.5\sigma^2$$
Therefore, Choice A is false.

Furthermore, we can see that the probability increases as \( t \) increases when the numerator of the expression is positive. We can make the numerator positive by setting the parameters so that:

\[
\alpha < \delta + 0.5\sigma^2
\]

Therefore, Choice B is false.

For Choice C, the conditional expectation is:

\[
E\left[ S_t \mid S_t < K \right] = \frac{S_0 e^{(\alpha - \delta)t} N(\hat{d}_1)}{N(\hat{d}_2)} = \frac{S_0 e^{(\alpha - \delta)t} N(\hat{d}_1)}{\sigma \sqrt{t}} \left( \frac{\ln \left( \frac{S_0}{K} \right) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma \sqrt{t}} \right)
\]

To simplify the expression, let’s choose \( S_0 = K \) and \( \alpha = \delta \), so that we have:

\[
E\left[ S_t \mid S_t < K \right] = S_0 \frac{N\left( -\frac{\ln \left( 1 + (0.5\sigma^2)t \right)}{\sigma \sqrt{t}} \right)}{N\left( \frac{0.5\sigma \sqrt{t}}{\sigma \sqrt{t}} \right)} = S_0 \frac{1 - N\left( 0.5\sigma \sqrt{t} \right)}{N\left( 0.5\sigma \sqrt{t} \right)} = S_0 \frac{1}{N\left( 0.5\sigma \sqrt{t} \right)} - S_0
\]

From the expression above, we can see that the conditional expectation decreases when \( t \) increases. Therefore, Choice C is false.

In fact, \( E\left[ S_t \mid S_t < K \right] \) always monotonically decreases as \( t \) increases, regardless of our choices for \( S_0, K, \alpha, \delta \), and \( \sigma \). This fact is mentioned (but not proven) in Problem 18.13 at the end of the textbook chapter.

For Choice D, the partial expectation is:

\[
PE\left[ S_t \mid S_t < K \right] = S_0 e^{(\alpha - \delta)t} N(\hat{d}_1) - S_0 e^{(\alpha - \delta)t} N\left( \frac{\ln \left( \frac{S_0}{K} \right) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma \sqrt{t}} \right)
\]
To simplify the expression, let’s choose \( S_0 = K \) and \( \alpha = \delta \), so that we have:

\[
PE[S_t \mid S_t < K] = S_0 N\left(-\frac{\ln(1) + 0.5\sigma^2 t}{\sigma \sqrt{t}}\right) = S_0 N\left(-0.5\sigma \sqrt{t}\right)
\]

From the expression above, we can see that the partial expectation decreases when \( t \) increases. Therefore, Choice D is false.

Since Choices A, B, C, and D are false, Choice E is the correct answer.

**Solution 10**

**D**  Chapter 18, Confidence Intervals

For a 90% lognormal confidence interval, we set \( p = 10\% \) in the expression below:

\[
S^U_T = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T-t) + \sigma z^U \sqrt{T-t}}
\]

where: \( P\left(z > z^U\right) = \frac{P}{2} \)

First, we use the standard normal calculator to determine \( z^U \):

\[
\begin{align*}
P\left(z > z^U\right) &= \frac{0.10}{2} \\
P\left(z > z^U\right) &= 0.05 \\
1 - P\left(z < z^U\right) &= 0.05 \\
P\left(z < z^U\right) &= 0.95 \\
z^U &= 1.64485
\end{align*}
\]

The upper bound is:

\[
S^U_T = 75 e^{(0.16 - 0 - 0.5(0.30)^2)(0.5 - 0) + 0.3(1.64485)\sqrt{0.5 - 0}} = 112.608
\]

**Solution 11**

**C**  Chapter 18, Probability that Stock Price is Greater Than \( K \)

Let \( F \) be the fund amount. The time-5 cash flow to the company that sold the put option is:

\[
F(1.05)^5 - \text{Max}[0.50 - S_5]
\]

For this cash flow to be negative, the put option must be in-the-money, meaning that \( S_5 \) must be less than 50. Furthermore, the put option’s nonzero payoff must be greater than \( F(1.05)^5 \). That is, the following must be negative:

\[
F(1.05)^5 - (50 - S_5)
\]
The company wants the probability of a nonnegative cash flow to be 98%, so:

\[ P\left[F(1.05)^5 - (50 - S_5) > 0 \right] = 0.98 \]
\[ P\left[F(1.05)^5 - 50 + S_5 > 0 \right] = 0.98 \]
\[ P\left[S_5 > 50 - F(1.05)^5 \right] = 0.98 \]

We make use of the following formula for the probability of the price of a stock exceeding a threshold:

\[ \text{Prob}(S_T > K) = N(d_2) \quad \text{with} \quad K = 50 - F(1.05)^5 \]

We can use the normal distribution table to find \( d_2 \):

\[ N(d_2) = 0.98000 \quad \Rightarrow \quad d_2 = 2.05375 \]

We have:

\[
d_2 = \frac{\ln \left( \frac{S_T}{K} \right) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}}
\]

\[
2.05375 = \frac{\ln \left( \frac{50}{50 - F(1.05)^5} \right) + (0.14 - 0.02 - 0.5 \times 0.25^2)(5 - 0)}{0.25 \sqrt{5 - 0}}
\]

\[
0.704331 = \ln \left( \frac{50}{50 - F(1.05)^5} \right)
\]

\[
e^{-0.704331} = \frac{50 - F(1.05)^5}{50}
\]

\[ F = 19.3703 \]

**Solution 12**

C  Chapter 18, Expected Value of Future Stock Price

From the distribution provided, we have:

\[ \sigma = \sqrt{0.09} = 0.30 \]

\[ \alpha - \delta - 0.5\sigma^2 = 0.035 \quad \Rightarrow \quad \alpha = 0.035 + \delta + 0.5\sigma^2 = 0.035 + 0.03 + 0.5(0.09) = 0.11 \]

The expected value of the stock price at time 4 is:

\[ E[S_t] = S_0 e^{(\alpha - \delta)t} = 75e^{(0.11 - 0.03) \times 4} = 103.28 \]
Solution 13
A Chapter 18, Median Value of Future Stock Price

From the distribution provided, we have:
\[ \alpha - \delta - 0.5\sigma^2 = 0.035 \]

The median value of the stock price at time 4 is:
\[ \text{Median} = S_0 e^{(\alpha - \delta - 0.5\sigma^2) t} = 75 e^{0.035 \times 4} = 86.27 \]

Solution 14
B Chapter 18, Probability of Future Stock Price

From the distribution provided, we have:
\[ \sigma = \sqrt{0.09} = 0.30 \]
\[ \alpha - \delta - 0.5\sigma^2 = 0.035 \]

The probability that the stock price at the end of 4 years is less than $75 is:
\[ \text{Prob}[S(4) < 75] = N(-\hat{d}_2) \]

The value of \( \hat{d}_2 \) is:
\[ \hat{d}_2 = \frac{\ln \left( \frac{S_0}{K} \right) + (\alpha - \delta - 0.5\sigma^2) t}{\sigma \sqrt{t}} = \frac{\ln \left( \frac{75}{75} \right) + 0.035 \times 4}{0.30 \sqrt{4}} = 0.23333 \]

The probability is:
\[ N(-\hat{d}_2) = N(-0.23333) = 0.40775 \]

Solution 15
B Chapter 18, Median Value of Future Stock Price

From the distribution provided, we have:
\[ \alpha - \delta - 0.5\sigma^2 = 0.035 \]

The median value of the stock price at time 4 is:
\[ \text{Median} = S_0 e^{(\alpha - \delta - 0.5\sigma^2) t} = 75 e^{0.035 \times 4} = 86.27 \]

Since the investor reinvests the dividends, the investor has the following quantity of shares at the end of 4 years:
\[ e^{\delta t} = e^{0.03 \times 4} = 1.1275 \]
The median value of the investor’s position at the end of 4 years is the quantity of shares owned times the median stock price:

\[ 1.1275 \times 86.27 = 97.27 \]

**Solution 16**

**Chapter 18, Probability that Stock Price is Less Than \( K \)**

The probability that the stock price at the end of \( T \) years is lower than \( K \) is:

\[ \Pr(S_T < K) = N(-\hat{d}_2) \]

Since for \( K = S_0 \) this probability is greater than 50%, the value of \( \hat{d}_2 \) must be negative:

\[ N(-\hat{d}_2) > 50\% \quad \Rightarrow \quad -\hat{d}_2 > 0 \quad \Rightarrow \quad \hat{d}_2 < 0 \]

Setting \( K \) equal to the initial price, we have the following value of \( \hat{d}_2 \):

\[ \hat{d}_2 = \frac{\ln\left(\frac{S_0}{S_T}\right) + (\alpha - \delta - 0.5\sigma^2)(T)}{\sigma \sqrt{T}} = \frac{(\alpha - \delta - 0.5\sigma^2)}{\sigma} \sqrt{T} \]

Since \( \sqrt{T} \) is positive, for \( \hat{d}_2 \) to be negative, the following expression must be negative:

\[ \frac{(\alpha - \delta - 0.5\sigma^2)}{\sigma} \]

Let’s evaluate the expression for each of the choices:

- **Choice A:** \( (0.10 - 0.00 - 0.5 \times 0.40^2)/0.40 = 0.05 \)
- **Choice B:** \( (0.11 - 0.02 - 0.5 \times 0.40^2)/0.40 = 0.025 \)
- **Choice C:** \( (0.12 - 0.02 - 0.5 \times 0.30^2)/0.30 = 0.1833 \)
- **Choice D:** \( (0.13 - 0.04 - 0.5 \times 0.40^2)/0.40 = 0.025 \)
- **Choice E:** \( (0.14 - 0.05 - 0.5 \times 0.45^2)/0.45 = -0.025 \)

Since the expression is negative for Choice E, the parameterization in Choice E implies the probability that the time \( T \) stock price is less than the initial price is greater than 50%.

**Solution 17**

**Chapter 18, Median Value of Future Stock Price**

The expected stock price is:

\[ E[S_T] = S_0 e^{(\alpha - \delta)t} \]
The median stock price is:

\[ \text{Median} = S_0 e^{(\alpha - \delta - 0.5\sigma^2)t} \]

The median stock price can be written in terms of the expected stock price:

\[ \text{Median} = S_0 e^{(\alpha - \delta)t} e^{-0.5\sigma^2 t} = E[S_t] e^{-0.5\sigma^2 t} \]

Setting \( t = 5 \), the median is:

\[ \text{Median} = E[S_5] e^{-0.5(0.16)^5} = 130 \times e^{-0.40} = 87.14 \]

**Solution 18**

C  Chapter 18, One Standard Deviation Move

The stock price can be expressed in terms of a standard normal random variable \( z \):

\[ S_T = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T - t) + \sigma z \sqrt{T - t}} \]

where: \( z \sim N(0,1) \)

A one standard deviation move up is obtained by setting \( z = 1 \):

One standard deviation move up for \( S_T = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T - t) + \sigma \sqrt{T - t}} \)

\[ = 100 e^{(0.13 - 0.02 - 0.5 \times 0.25^2)0.5 + 0.25 \sqrt{0.5}} \]

\[ = 124.13 \]

**Solution 19**

B  Chapter 18, One Standard Deviation Move

The stock price can be expressed in terms of a standard normal random variable \( z \):

\[ S_T = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T - t) + \sigma z \sqrt{T - t}} \]

where: \( z \sim N(0,1) \)

We can use the prices from the table to solve for \( \alpha \) and \( \sigma \). Let’s use the prices from the bottom row that correspond to 1 standard deviation moves up and down:

\[ 74.39 = 100 e^{(\alpha - 0.5\sigma^2)5 - \sigma \sqrt{5}} \]

\[ 284.57 = 100 e^{(\alpha - 0.5\sigma^2)5 + \sigma \sqrt{5}} \]

Dividing the second equation by the first equation, we can solve for \( \sigma \) and then for \( \alpha \):

\[ \frac{284.57}{74.39} = e^{2\sigma \sqrt{5}} \Rightarrow \sigma = 0.30 \]

\[ 74.39 = 100 e^{(\alpha - 0.5 \times 3^2)5 - 3 \sqrt{5}} \Rightarrow \alpha = 0.12 \]
A one standard deviation move down over 1 year is obtained by setting $z = -1$ and $(T - t) = 1$:

One standard deviation move down for $S_T = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T - t) - \sigma \sqrt{T - t}}$

$$= 100 e^{(0.12 - 0.00 - 0.5 \times 0.30^2)1 - 0.30 \sqrt{1}}$$

$$= 79.85$$

Solution 20

Chapter 18, Two Standard Deviation Move

Notice that 1 month is expressed as 0.0849 of one year. For Exam MFE/3F we usually express one month as the following quantity of years:

$$\frac{1}{12} = 0.0833$$

But in this question, we use:

$$\frac{31}{365} = 0.0849$$

The stock price can be expressed in terms of a standard normal random variable $z$:

$$S_T = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T - t) + \sigma z \sqrt{T - t}}$$

where: $z \sim N(0,1)$

We can use the prices from the table to solve for $\alpha$ and $\sigma$. Let’s use the prices from the bottom row that correspond to 1 standard deviation moves up and down:

$$74.39 = 100 e^{(\alpha - 0.5\sigma^2)5 - \sigma \sqrt{5}}$$

$$284.57 = 100 e^{(\alpha - 0.5\sigma^2)5 + \sigma \sqrt{5}}$$

Dividing the second equation by the first equation, we can solve for $\sigma$ and then for $\alpha$:

$$\frac{284.57}{74.39} = e^{2\sigma \sqrt{5}} \Rightarrow \sigma = 0.30$$

$$74.39 = 100 e^{(\alpha - 0.5 \times 0.30^2)5 - 0.3\sqrt{5}} \Rightarrow \alpha = 0.12$$

A two standard deviation move down over one month is obtained by setting $z = -2$ and $(T - t) = 31/365$:

Two standard deviation move down for $S_T = S_t e^{(\alpha - \delta - 0.5\sigma^2)(T - t) - 2\sigma \sqrt{T - t}}$

$$= 100 e^{(0.12 - 0.00 - 0.5 \times 0.30^2) \times \frac{31}{365} - 2 \times 0.30 \sqrt{\frac{31}{365}}}$$

$$= 84.49$$
Solution 21

D  Chapter 18, Two Standard Deviation Move

From the table we observe that if a 2 standard deviation move down occurs after one day, then the new stock price will be $96.93 at the end of one day.

For the stock price to be less than $96.93 at the end of one day, the stock price must move by more than 2 standard deviations down. This is equivalent to $z$ moving more than 2 standard deviations down in the following expression:

$$S_t + \frac{1}{365} = S_t e^{(\alpha - \delta - 2\times 0.5\sigma^2)\frac{1}{365}} + \sigma z\sqrt{\frac{1}{365}}$$

where: $z \sim N(0,1)$

The probability of $z$ moving more than two standard deviations down is:

$$\text{Prob}(z < -2.00000) = N(-2.00000) = 0.02275$$

Solution 22

E  Chapter 18, Effect of Increasing Time to Maturity

The probability that the put option expires in the money is:

$$\text{Prob}(S_T < K) = N(-\hat{d}_2) \quad \text{where: } K = S_0 e^{rT}$$

The value of $\hat{d}_2$ is:

$$\hat{d}_2 = \frac{\ln \left( \frac{S_0}{K} \right) + (\alpha - \delta - 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{S_0}{S_0 e^{rT}} \right) + (\alpha - 0 - 0.5\sigma^2)T}{\sigma \sqrt{T}}$$

$$\hat{d}_2 = \frac{[(\alpha - r) - 0.5\sigma^2]\sqrt{T}}{\sigma}$$

The probability that the put option expires in the money is:

$$\text{Prob}(S_T < S_0 e^{rT}) = N(-\hat{d}_2) = N\left(\frac{0.5\sigma^2 - (\alpha - r)}{\sigma \sqrt{T}}\right)$$

The question tells us that the risk premium is greater than 0.5\sigma^2:

$$0.5\sigma^2 < \alpha - r \quad \Rightarrow \quad 0.5\sigma^2 - (\alpha - r) < 0$$

Therefore $-\hat{d}_2$ is negative:

$$\frac{0.5\sigma^2 - (\alpha - r)}{\sigma \sqrt{T}} < 0$$

This implies that $N(-\hat{d}_2)$ is less than 50%. Therefore, Statement E is false.

We could stop here, but let’s go through the other choices as well.
We have established that the numerator in the fraction below is negative:

\[
\text{Prob}\left[ S_T < S_0e^{rT} \right] = N\left( \frac{0.5\sigma^2 - (\alpha - r)\sqrt{T}}{\sigma} \right)
\]

Therefore, the value in the parentheses becomes more negative as \( T \) increases, meaning that an increase in \( T \) decreases the probability that the put option expires in the money. Therefore Statement A is true.

The value in the parenthesis becomes more negative as \( \alpha \) increases, so an increase in \( \alpha \) decreases the probability that the put option expires in the money. Therefore, Statement D is true.

The risk-neutral probability is obtained by substituting \( r \) for \( \alpha \):

\[
N\left( \frac{0.5\sigma^2 - (r - r)\sqrt{T}}{\sigma} \right) = N\left( \frac{0.5\sigma^2}{\sigma} \right)
\]

The value in the parentheses is now positive, meaning that an increase in \( T \) increases the risk-neutral probability that the put option expires in the money. Therefore, Statement C is true.

Statement B is true because options on non-dividend-paying stocks increase in value as their time to maturity increases when the strike price grows at the risk-free interest rate. \( \text{This was covered in the Chapter 9 Review Note.} \)

**Solution 23**

**A  
Chapter 18, Comparing Stock with a Risk-free Bond**

The probability that Pete's investment outperforms Jim's investment is equal to the probability that a risk-free investment outperforms the stock:

\[
\text{Prob}\left[ S_T < K \right] = N(-\hat{d}_2) \quad \text{where:} \quad K = 100e^{rT}
\]

The value of \( \hat{d}_2 \) is:

\[
\hat{d}_2 = \frac{\ln \left( \frac{S_0}{K} \right) + (\alpha - \delta - 0.5\sigma^2)T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{100}{100e^{rT}} \right) + (\alpha - 0 - 0.5\sigma^2)T}{\sigma \sqrt{T}}
\]

\[
= \frac{[(\alpha - r) - 0.5\sigma^2] \sqrt{T}}{\sigma}
\]

The probability that Pete's investment outperforms Jim's investment is:

\[
\text{Prob}\left[ S_T < S_0e^{rT} \right] = N(-\hat{d}_2) = N\left( \frac{0.5\sigma^2 - (\alpha - r)\sqrt{T}}{\sigma} \right)
\]
The question tells us that this probability is equal to 50%:

\[ N \left( \frac{0.5\sigma^2 - (\alpha - r) \sqrt{T}}{\sigma} \right) = 50\% \]

This implies that the value in parentheses above is 0:

\[ \frac{0.5\sigma^2 - (\alpha - r) \sqrt{T}}{\sigma} = 0 \]
\[ 0.5\sigma^2 - (\alpha - r) = 0 \]
\[ 0.5\sigma^2 - (0.10 - 0.08) = 0 \]
\[ \sigma^2 = 0.04 \]
\[ \sigma = 0.20 \]

**Solution 24**

**D** Chapter 18, Conditional and Partial Expectations

The expected value of the call option at time 1 is:

\[ E[\text{Call}_1] = \left\{ E[S_1 \g S_1 > 102] - 102 \right\} \times \text{Prob}(S_1 > 102) \]

The probability that the final stock price is above 102 is found using the relationship below:

\[ \text{Prob}(S_1 > 102) = \frac{PE[S_T \g S_T > K]}{E[S_1 \g S_1 > 102]} \]
\[ = \frac{68.98}{132.69} = 0.51986 \]

The expected value of the call at time 1 is therefore:

\[ E[\text{Call}_1] = \left\{ E[S_1 \g S_1 > 102] - 102 \right\} \times \text{Prob}(S_1 > 102) = (132.69 - 102) \times 0.51986 \]
\[ = 15.9545 \]

The initial value of the call is:

\[ \text{Call}_0 = E[\text{Call}_1]e^{-0.258} = 15.9545 \times e^{-0.258} = 12.33 \]
Solution 25

E Chapter 18, Conditional and Partial Expectations

The risk-neutral partial expectation of the stock price, conditional on the stock price being below 102 is:

\[ PE^* \left[ S_T \mid S_T < K \right] = \int_0^{102} S_T g^* (S_T; S_t) dS_T = \int_0^{102} S_1 g^* (S_1; S_{t0.5}) dS_1 = 45.94 \]

The risk-neutral probability that the final stock price is below 102 is found using the relationship below:

\[ \frac{PE^* \left[ S_T \mid S_T < K \right]}{Prob^* (S_T < K)} = \frac{PE^* \left[ S_1 \mid S_1 < 102 \right]}{E^* \left[ S_1 \mid S_1 < 102 \right]} = \frac{45.94}{86.27} = 0.5325 \]

The risk-neutral expected value of the put at time 1 is therefore:

\[ E^* [\text{Put Payoff}] = \left\{ K - E^* \left[ S_T \mid S_T < K \right] \right\} \times \text{Prob}^* (S_T < K) \]

\[ = \{102 - 86.27\} \times 0.5325 = 8.3764 \]

Since we used the risk-neutral probability to obtain this expected value, we can use the risk-free rate of return to discount it back to time 0.5:

\[ 8.3764 \times e^{-0.09 \times 0.5} = 8.01 \]

Solution 26

B Chapter 18, Partial Expectation

The partial expectation of the stock price, conditional on the stock price being at least $48 is:

\[ PE \left[ S_T \mid S_T > K \right] = S_t e^{(\alpha - \delta)(T-t)} N(d_1) = 50 e^{(0.13 - 0.04) \times 0.5} N(d_1) = 52.3014 N(d_1) \]

The value of \( d_1 \) (rounded to two places) is:

\[ d_1 = \frac{\ln \left( \frac{S_t}{K} \right) + (\alpha - \delta + 0.5\sigma^2)(T-t)}{\sigma \sqrt{T-t}} = \frac{\ln \left( \frac{50}{48} \right) + (0.13 - 0.04 + 0.5(0.3)^2)(0.5)}{0.3 \sqrt{0.5}} = 0.51063 \]

The value of \( N(d_1) \) is:

\[ N(d_1) = N(0.51063) = 0.69519 \]
The partial expectation is:

\[
52.3014 N(\hat{d}_1) = 52.3014 \times 0.69519 = 36.3594
\]

**Solution 27**

**B  Chapter 18, Conditional Expectation**

The values of \( \hat{d}_1 \) (rounded to two places) and \( N(\hat{d}_1) \) are:

\[
\hat{d}_1 = \frac{\ln(S_t/K) + (\alpha - \delta + 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}} = \frac{\ln(50/48) + (0.13 - 0.04 + 0.5(0.3)^2)(0.5)}{0.3\sqrt{0.5}} = 0.51063
\]
\[N(\hat{d}_1) = N(0.51063) = 0.69519\]

The values of \( \hat{d}_2 \) and \( N(\hat{d}_2) \) are:

\[
\hat{d}_2 = \frac{\ln(S_t/K) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}} = \frac{\ln(50/48) + (0.13 - 0.04 - 0.5(0.3)^2)(0.5)}{0.3\sqrt{0.5}} = 0.29850
\]
\[N(\hat{d}_2) = N(0.29850) = 0.61734\]

The conditional expectation is:

\[
E[S_T | S_T > 48] = \frac{S_t e^{(\alpha - \delta)(T-t)} N(\hat{d}_1)}{N(\hat{d}_2)} = \frac{50e^{0.13-0.04} \times 0.5 \times 0.69519}{0.61734} = 58.8969
\]

**Solution 28**

**C  Chapter 18, Partial Expectation**

The values of \( \hat{d}_1 \) (rounded to two places) and \( N(\hat{d}_1) \) are:

\[
\hat{d}_1 = \frac{\ln(S_t/K) + (\alpha - \delta + 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}} = \frac{\ln(S_0/S_0) + (\alpha - \delta + 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}} = \frac{0 + (0.09 - 0.03 + 0.5(0.25)^2)(3)}{0.25\sqrt{3}} = 0.63220
\]
\[N(\hat{d}_1) = N(0.63220) = 0.73637\]

We can use the formula for the partial expectation to solve for the current stock price:

\[
PE[S_T | S_T > K] = S_t e^{(\alpha - \delta)(T-t)} N(\hat{d}_1)
\]
\[
PE[S_3 | S_2 > S_0] = S_0 e^{(0.09-0.03)3} N(\hat{d}_1)
\]
\[85.88 = S_0 e^{0.09-0.03} \times 0.73637\]
\[S_0 = 97.3668\]
Solution 29

A  Chapter 18, Conditional and Partial Expectations

The partial expectation of the stock price in 3 months, conditional on the stock price being less than $K$ is the conditional expectation times the probability of the condition:

$$PE[S_{0.25} | S_{0.25} < K] = E[S_{0.25} | S_{0.25} < K] \times \text{Prob}(S_{0.25} < K) = 67.69 \times (1 - 0.60) = 27.076$$

Since the conditions are mutually exclusive and exhaustive, we can add the two partial expectations to obtain the expected value of the stock price in 3 months:

$$E[S_T] = \frac{E[S_T | S_T < K] \times \text{Prob}(S_T < K) + E[S_T | S_T > K] \times \text{Prob}(S_T > K)}{\text{Partial expectation conditional on } S_T < K + \text{Partial expectation conditional on } S_T > K}$$

$$= 27.076 + 52.01$$

$$= 79.086$$

The expected return on the stock is 17%, so:

$$e^{0.17 \times 0.25} = \frac{e^{0.02 \times 0.25} E[S_T]}{S_0}$$

$$S_0 = e^{-0.0375} E[S_T]$$

$$S_0 = e^{-0.0375} \times 79.086$$

$$S_0 = 76.18$$

Solution 30

D  Chapter 18, Order Statistics

The order statistics are the result of sorting the data. The order statistics are $\{2, 3, 5, 8, 10\}$. The fourth order statistic is 8.

Solution 31

A  Chapter 18, Quantiles

The second draw from $\{8, 2, 10, 5, 3\}$ is 2.

The order statistics are the result of sorting the data. The order statistics are $\{2, 3, 5, 8, 10\}$.

We have 5 observed data points, so the probability ranges are of the following size:

$$\frac{100\%}{5} = 20\%$$
The ranges are therefore 0 – 20%, 20% – 40%, 40% – 60%, 60% – 80%, and 80% – 100%. We use the midpoint of each range as a cumulative probability and assign it to its corresponding order statistic:

<table>
<thead>
<tr>
<th>Order Statistics (Quantiles)</th>
<th>Cumulative Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10%</td>
</tr>
<tr>
<td>3</td>
<td>30%</td>
</tr>
<tr>
<td>5</td>
<td>50%</td>
</tr>
<tr>
<td>8</td>
<td>70%</td>
</tr>
<tr>
<td>10</td>
<td>90%</td>
</tr>
</tbody>
</table>

Although the quantiles are the order statistics, the textbook uses the term “quantile value” to refer to the probability associated with a quantile. Therefore, the quantile value associated with 2 is 10%.

Solution 32

E  Chapter 18, Quantiles

The order statistics are the result of sorting the data. The order statistics are \{2, 3, 5, 8, 10\}.

We have 5 observed data points, so the probability ranges are of the following size:

\[
\frac{100\%}{5} = 20\%
\]

The ranges are therefore 0 – 20%, 20% – 40%, 40% – 60%, 60% – 80%, and 80% – 100%. We use the midpoint of each range as a cumulative probability and assign it to its corresponding order statistic:

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<td>3</td>
<td>30%</td>
</tr>
<tr>
<td>5</td>
<td>50%</td>
</tr>
<tr>
<td>8</td>
<td>70%</td>
</tr>
<tr>
<td>10</td>
<td>90%</td>
</tr>
</tbody>
</table>

The 90% quantile is 10.

Solution 33

D  Chapter 18, The Lognormal Distribution

Products of lognormal random variables are lognormally distributed. The sums of lognormal random variables are not lognormally distributed.
Solution 34

E  Chapter 18, The Normal Distribution

In the binomial model, the continuously compounded return approaches normality as the number of steps becomes large.

Solution 35

C  Chapter 18, Black-Scholes Model

Statement II is false. There are too many large returns in histograms and normal probability plots for daily and weekly compounded returns when compared to normally distributed returns.

Solution 36

B  Chapter 18, Probability of Future Stock Price

The stock will expire in the money if its price exceeds $125 in 2 years. The probability that the call will expire in the money is:

\[
P(S_t > K) = N(\hat{d}_2) = N\left(\frac{\ln(S_t / K) + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)
\]

\[
= N\left[\ln(100/125) + (0.15 - 0.03 - 0.5(0.3)^2)2\right]
\]

\[
= N[-0.17240]
\]

\[
= 0.43156
\]

Solution 37

A  Chapter 18, Quantiles

The 20% quantile is the value such that there is a 20% chance that a draw from the standard normal distribution is less than that number.

This question can be answered very quickly using the online normal distribution calculator. We see that \( P(z < x) = N^{-1}(x) = 20\% \), so the 20% quantile is \( x = -0.84162 \).

Using the printed normal distribution table, we notice that the bottom of the MFE/3F standard normal table provides values of \( z \) for selected values of \( P(Z < z) \). From the bottom of the table, we see that \( P(Z < 0.8420) = 80\% \), so the 80% quantile is 0.8420. Due to the symmetric nature of the normal distribution, we have \( P(Z < -0.8420) = 20\% \), so the 20% quantile is \(-0.8420\).
Alternatively, we can also determine the answer from the body of the MFE/3F printed standard normal table. Scanning the body of the table for 20%, we see that only values greater then 50% are provided. Since the normal distribution is symmetric, we look for 80% instead. We see that the closest we can get to 80% in the table is 0.7995, which corresponds with a \( z \) of 0.84. This means that \( P(Z < 0.84) = 79.95\% \), i.e., the 79.95\% quantile is 0.84. Due to symmetry, \( P(Z < -0.84) = 1 - 0.7995 = 0.2005 \). So the 20.05\% quantile is \(-0.84\), which matches one of the answer choices.

**Solution 38**

**D**  Chapter 18, Conditional Expectation

Given that the call is in the money at expiration, the expected stock price is:

\[
\sum_{S_t > 100} \text{Prob}(S_t) \times S_t = \frac{3}{8} \times 105 + \frac{1}{8} \times 125 = 55
\]

This is the partial expectation of the stock price conditional upon the stock price at expiration being greater than $100. The conditional expectation of the stock price assuming the call is in the money at expiration is:

\[
\frac{1}{\text{Prob}(S_t > 100)} \times \sum_{S_t > 100} \text{Prob}(S_t) \times S_t = \frac{1}{0.5} \times 55 = 110
\]

**Solution 39**

**B**  Chapter 18, Confidence Intervals

Since \( t = 0.5 \), the semiannual continuously compounded mean return is:

\[
\left( \alpha - \delta - 0.5\sigma^2 \right) t = \left( 0.12 - 0.0 - 0.5 \times 0.25^2 \right) 0.5 = 0.044375
\]

The semiannual standard deviation is:

\[
\sigma \sqrt{t} = 0.25 \times \sqrt{0.5} = 0.176777
\]

We use the standard normal calculator to determine \( z^U \):

\[
P \left( z > z^U \right) = \frac{0.05}{2}
\]

\[
P \left( z > z^U \right) = 0.025
\]

\[
1 - P \left( z < z^U \right) = 0.025
\]

\[
P \left( z < z^U \right) = 0.975
\]

\[
z^U = 1.95996
\]
For a standard normal distribution, there is a 95% chance of drawing a number in the interval \((-1.95996, 1.95996)\). Over a six month horizon, there is a 95% chance that the return will be:

\[ 0.044375 \pm 1.95996 \times 0.176777 = -0.302100 \text{ and } 0.390850 \]

The 6-month return has a 95% confidence interval of \((-0.302100, 0.390850)\). We need to determine the 95% confidence interval for the stock price in 6 months, so we exponentiate these returns and multiply them by the initial stock price of $75:

\[ -0.302100 \leq \frac{\ln(S_{0.5})}{75} \leq 0.390850 \]

\[ 75e^{-0.302100} \leq S_{0.5} \leq 75e^{0.390850} \]

\[ 55.4448 \leq S_{0.5} \leq 110.8678 \]

**Solution 40**

**A** Chapter 18, Covariance of \( S_t \) and \( S_T \)

To answer this question, we use the following formulas:

\[
E\left[ \frac{S_T}{S_t} \right] = e^{(\alpha-\delta)(T-t)}
\]

\[
\text{Var}\left[ S_T \middle| S_t \right] = S_t^2 e^{2(\alpha-\delta)(T-t)} \left( e^{\sigma^2(T-t)} - 1 \right)
\]

\[
\text{Cov}[S_t, S_T] = \text{E}\left[ \frac{S_T}{S_t} \right] \text{Var}[S_t \middle| S_0]
\]

The variance of \( A(2) \) is:

\[
\text{Var}[A(2)] = \frac{1}{4} \{ \text{Var}[S(1)] + \text{Var}[S(2)] + 2\text{Cov}[S(1), S(2)] \}
\]

The variances of \( S(1) \) and \( S(2) \) are:

\[
\text{Var}\left[ S_1 \middle| S_0 \right] = 10^2 e^{2(0.04)(1-0)} \left( e^{0.4^2(1-0)} - 1 \right) = 18.7962
\]

\[
\text{Var}\left[ S_2 \middle| S_0 \right] = 10^2 e^{2(0.04)(2-0)} \left( e^{0.4^2(2-0)} - 1 \right) = 44.2564
\]

The covariance is:

\[
\text{Cov}[S_1, S_2] = \text{E}\left[ \frac{S_2}{S_1} \right] \text{Var}[S_1 \middle| S_0] = e^{0.04(2-1)} \times 18.7962 = 19.5633
\]
We can now find the variance of \( A(2) \):

\[
\text{Var}[A(2)] = \frac{1}{4} \left\{ \text{Var}[S(1) + \text{Var}[S(2)] + 2\text{Cov}[S(1), S(2))] \right\} \\
= \frac{1}{4} \left\{ 18.7962 + 44.2564 + 2 \times 19.5633 \right\} = 25.5448
\]

**Solution 41**

**B** Chapter 18, Covariance of \( S_t \) and \( S_T \)

To answer this question, we use the following formulas:

\[
E \left[ \frac{S_T}{S_t} \right] = e^{(\alpha-\delta)(T-t)}
\]

\[
\text{Var}[S_T | S_t] = S_t^2 e^{2(\alpha-\delta)(T-t)} \left( e^{\sigma^2(T-t)} - 1 \right)
\]

\[
\text{Cov}[S_t, S_T] = E \left[ \frac{S_T}{S_t} \right] \text{Var}[S_t | S_0]
\]

The variance of \( X \) is:

\[
\text{Var}[X] = 16 \text{Var}[S(2)] + \text{Var}[S(3)] + 2 \times 4 \times (-1) \text{Cov}[S(2), S(3)]
\]

The variances of \( S(2) \) and \( S(3) \) are:

\[
\text{Var}[S_2 | S_0] = 2^2 e^{2(0.06)(2-0)} \left( e^{0.3^2(2-0)} - 1 \right) = 1.0028
\]

\[
\text{Var}[S_3 | S_0] = 2^2 e^{2(0.06)(3-0)} \left( e^{0.3^2(3-0)} - 1 \right) = 1.7771
\]

The covariance is:

\[
\text{Cov}[S_2, S_3] = E \left[ \frac{S_3}{S_2} \right] \text{Var}[S_2 | S_0] = e^{0.06(3-2)} \times 1.0028 = 1.0649
\]

We can now find the variance of \( X \):

\[
\text{Var}[X] = 16 \text{Var}[S(2)] + \text{Var}[S(3)] + 2 \times 4 \times (-1) \text{Cov}[S(2), S(3)]
\]

\[
= 16 \times 1.0028 + 1.7771 - 8 \times 1.0659 = 9.3038
\]
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Chapter 19 – Solutions

Solution 1

C  Chapter 19, Standard Deviation of Monte Carlo Estimate

The standard deviation of the Monte Carlo estimate is the standard deviation of the individual price estimates divided by the square root of the number of paths:

\[
\sigma_{V(n)} = \frac{\sigma_V}{\sqrt{n}} = \frac{3.46}{\sqrt{10,000}} = 0.0346
\]

Solution 2

B  Chapter 19, Standard Deviation of Monte Carlo Estimate

The standard deviation of the Monte Carlo estimate is the standard deviation of the individual price estimates divided by the square root of the number of paths:

\[
\sigma_{V(n)} = \frac{\sigma_V}{\sqrt{n}} = \frac{3.46}{\sqrt{n}}
\]

If we were given the actual price of the option, we would use it, but since we do not have the actual price, we use the Monte Carlo estimate as a reasonable approximation:

\[
\frac{3.46}{\sqrt{n}} = 0.02 \implies n = 4,906
\]

Solution 3

D  Chapter 19, Forward Price with Monte Carlo Valuation

The Monte Carlo estimate for the current price of the derivative is:

\[
\bar{V} = \frac{1}{n} \times \sum_{i=1}^{n} V_i(t) = 24.43
\]

In the Chapter 9 Review Note, we saw that the forward price for an asset that does not pay dividends is its current price accrued forward at the risk-free rate of return. Therefore the forward price of the derivative is:

\[
24.43 \times e^{0.08 \times 0.25} = 24.92
\]

Alternatively, we can obtain the same answer by finding the average of the undiscounted derivative prices. These prices are in the second column from the right side of the table, and the bottom row tells us that their average is $24.93.

The small difference between $24.92 obtained using the first method and $24.93 obtained using the second method is attributable to rounding.
Solution 4

B Chapter 19, Path Discount Rate

We begin by calculating the realistic probability of an up and down movement:

$$p = \frac{e^{(\alpha - \delta)h} - d}{u - d} = \frac{e^{(0.15 - 0) \times 0.25} - 0.8825}{1.1912 - 0.8825} = 0.5044 \quad 1 - p = 0.4956$$

We need to find the expected return on the option at the initial node, after one up movement, and after an up-down movement. We use the following formula:

$$V = e^{-\gamma h} \left[ (p)V_u + (1 - p)V_d \right]$$

At the initial node, the expected return on the option can be found using the formula above:

$$5.21 = e^{-\gamma \times 0.25} \left[ (0.5044)9.457 + (0.4956)1.799 \right]$$

$$\gamma = 0.333$$

After one up movement, the new option price is 9.457:

$$9.457 = e^{-\gamma \times 0.25} \left[ (0.5044)16.327 + (0.4956)3.987 \right]$$

$$\gamma = 0.307$$

After one up movement and one down movement, the new option price is 3.987:

$$3.987 = e^{-\gamma \times 0.25} \left[ (0.5044)8.837 + (0.4956)0.000 \right]$$

$$\gamma = 0.446$$

The path discount rate is the arithmetic average of the 3 rates:

$$\frac{0.333 + 0.307 + 0.446}{3} = 0.362$$

We can take the arithmetic average because it is equivalent to the following process for finding the discount factor for the path:

The discount factor for the path is:

$$e^{-0.333 \times 0.25}e^{-0.307 \times 0.25}e^{-0.446 \times 0.25}$$

We convert this into one interest rate for the path, which we denote $$\bar{\gamma}$$:

$$e^{-0.333 \times 0.25}e^{-0.307 \times 0.25}e^{-0.446 \times 0.25} = e^{-0.75\bar{\gamma}}$$

$$0.333 + 0.307 + 0.446 \times 0.25 = 0.75\bar{\gamma}$$

$$\bar{\gamma} = \frac{0.333 + 0.307 + 0.446}{3}$$

$$\bar{\gamma} = 0.362$$
Solution 5
D Chapter 19, Path Discount Rate
We begin by calculating the realistic probability of an up and down movement:

\[ p = \frac{e^{(\alpha - \delta)h} - d}{u - d} = \frac{e^{0.15 - 0.25} - 0.8825}{1.1912 - 0.8825} = 0.5044 \quad 1 - p = 0.4956 \]

We need to find the expected return on the option at the initial node, after one down movement, and after a down-up movement. We use the following formula:

\[ V = e^{-\gamma h} [(p) V_u + (1 - p) V_d] \]

At the initial node, the expected return on the option can be found using the formula above:

\[ 5.21 = e^{-\gamma \times 0.25} [(0.5044) 9.457 + (0.4956) 1.799] \]

\[ \gamma = 0.333 \]

After one down movement, the new option price is 1.799:

\[ 1.799 = e^{-\gamma \times 0.25} [(0.5044) 3.987 + (0.4956) 0.000] \]

\[ \gamma = 0.446 \]

After one down movement and one up movement, the new option price is 3.987:

\[ 3.987 = e^{-\gamma \times 0.25} [(0.5044) 8.837 + (0.4956) 0.000] \]

\[ \gamma = 0.446 \]

The path discount rate is the arithmetic average of the 3 rates:

\[ \frac{0.333 + 0.446 + 0.446}{3} = 0.408 \]

Solution 6
B Chapter 19, Monte Carlo Valuation in Binomial Model
The risk-neutral probability of up and down movements is:

\[ p^* = \frac{e^{(r - \delta)h} - d}{u - d} = \frac{e^{(0.06 - 0.05) \times 0.5} - 0.92770}{1.08877 - 0.92770} = 0.4800 \quad 1 - p^* = 0.5200 \]

The risk-neutral probabilities of two ups, an up and a down, and two downs are:

uu: \[ 0.4800 \times 0.4800 = 0.2304 \]
ud or du: \[ 2 \times 0.4800 \times 0.5200 = 0.4992 \]
dd: \[ 0.5200 \times 0.5200 = 0.2704 \]
The tree of stock prices, option payoffs, the associated probabilities, and the associated colors is:

<table>
<thead>
<tr>
<th>Option Payoff</th>
<th>Risk-Neutral Probability</th>
<th>Color</th>
</tr>
</thead>
<tbody>
<tr>
<td>59.2710</td>
<td>0.0000</td>
<td>Red</td>
</tr>
<tr>
<td>54.4385</td>
<td>0.2304</td>
<td></td>
</tr>
<tr>
<td>50.0000</td>
<td>0.0000</td>
<td>Blue</td>
</tr>
<tr>
<td>50.5026</td>
<td>0.4992</td>
<td></td>
</tr>
<tr>
<td>46.3850</td>
<td>3.9686</td>
<td>Green</td>
</tr>
<tr>
<td>43.0314</td>
<td>0.2704</td>
<td></td>
</tr>
</tbody>
</table>

The investor’s gambling wheel represents the three outcomes by associating a color with each probability. Only the green values provide an option payoff, and the estimate resulting from the Monte Carlo testing is:

\[ V = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} V_i(T) = \frac{e^{-0.06}}{140} \left( 31 \times 0 + 70 \times 0 + 39 \times 3.9686 \right) = 1.0412 \]

**Solution 7**

**A Chapter 19, Monte Carlo Valuation in the Binomial Model**

The risk-neutral probability of up and down movements is:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.06-0.05) \times 0.5} - 0.92770}{1.08877 - 0.92770} = 0.4800 \quad 1 - p^* = 0.5200 \]

The risk-neutral probabilities of two ups, an up and a down, and two downs are:

- uu: \( 0.4800 \times 0.4800 = 0.2304 \)
- ud or du: \( 2 \times 0.4800 \times 0.5200 = 0.4992 \)
- dd: \( 0.5200 \times 0.5200 = 0.2704 \)

The tree of stock prices, option payoffs, the associated probabilities, and the associated colors is:

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</tr>
</thead>
<tbody>
<tr>
<td>59.2710</td>
<td>12.2710</td>
<td>Red</td>
</tr>
<tr>
<td>54.4385</td>
<td>0.2304</td>
<td></td>
</tr>
<tr>
<td>50.0000</td>
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</tr>
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<td>50.5026</td>
<td>0.4992</td>
<td></td>
</tr>
<tr>
<td>46.3850</td>
<td>0.2704</td>
<td>Green</td>
</tr>
<tr>
<td>43.0314</td>
<td>0.0000</td>
<td></td>
</tr>
</tbody>
</table>
The investor’s gambling wheel represents the three outcomes by associating a color with each probability. Only the red and blue values provide option payoffs, and the estimate resulting from the Monte Carlo testing is:

\[
\bar{V} = e^{-r(T-t)} \sum_{i=1}^{n} V_i(T) = e^{-0.06} \left( 31 \times 12.2710 + 70 \times 3.5026 + 39 \times 0 \right) = 4.2082
\]

Solution 8

A Chapter 19, Sum of 12 Uniformly Distributed Random Variables

The sum of the uniform random variables is 5.863:

\[
0.126 + 0.205 + 0.080 + 0.303 + 0.992 + 0.481 + 0.162 \\
+ 0.786 + 0.279 + 0.703 + 0.752 + 0.994 = 5.863
\]

The estimate for the standard normal draw is therefore:

\[
\hat{Z} = \sum_{i=1}^{12} u_i - 6 = 5.863 - 6 = -0.137
\]

The Black-Scholes framework implies that the stock price follows a lognormal distribution, so:

\[
S_T = S_0 e^{(\alpha-\delta-0.5\sigma^2)(T-t)+\sigma z \sqrt{T-t}} = 100 e^{(0.08-0.03-0.5 \times 0.3^2) \times (2-0) + 0.3 \times (-0.137) \sqrt{2-0}} \\
= 95.30
\]

Solution 9

A Chapter 19, Sum of 12 Uniformly Distributed Random Variables

The sum of the uniform random variables is 5.863:

\[
0.126 + 0.205 + 0.080 + 0.303 + 0.992 + 0.481 + 0.162 \\
+ 0.786 + 0.279 + 0.703 + 0.752 + 0.994 = 5.863
\]

The estimate for the standard normal draw is therefore:

\[
\hat{Z} = \sum_{i=1}^{12} u_i - 6 = 5.863 - 6 = -0.137
\]

For the risk-neutral distribution of the stock, we replace \( \alpha \) with \( r \). The Black-Scholes framework implies that the stock price follows a lognormal distribution, so:

\[
S_T = S_0 e^{(r-\delta-0.5\sigma^2)(T-t)+\sigma z \sqrt{T-t}} = 100 e^{(0.05-0.03-0.5 \times 0.3^2) \times (2-0) + 0.3 \times (-0.137) \sqrt{2-0}} \\
= 89.75
\]
Solution 10

E  Chapter 19, Inverse Cumulative Normal Distribution

The steps to obtaining a standard normal random variable are:

1. Obtain an observation $\hat{x}$ of a random variable $x$. This observation is 4.6.

2. Find the probability that $x$ is less than $\hat{x}$:
   \[ F(\hat{x}) = \text{Prob}(x < \hat{x}) \]
   For a uniform distribution, we have:
   \[ F(x) = \frac{x - a}{b - a} \text{ for } U(a, b) \]
   Therefore, the cumulative probability of 4.6 is:
   \[ F(4.6) = \frac{4.6 - 3}{5 - 3} = 0.8 \]

3. Find the value of $\hat{z}$ that is the $F(\hat{x})$ quantile of the normal distribution:
   \[ N(\hat{z}) = F(\hat{x}) \]
   \[ N(\hat{z}) = 0.80000 \]
   \[ \hat{z} = N^{-1}[0.80000] = 0.84162 \]

Thus 0.84162 is the observation from the standard normal distribution.

For the risk-neutral distribution of the stock, we replace $\alpha$ with $r$. The Black-Scholes framework implies that the stock price follows a lognormal distribution, so:

\[
S_T = S_0 e^{(r - \delta - 0.5\sigma^2)(T - t) + \sigma \sqrt{T - t}} = 100 e^{(0.05 - 0.03 - 0.5 \times 0.3^2) 	imes (2 - 0) + 0.3 \times (0.84162) \sqrt{2 - 0}}
\]
\[= 135.9435 \]

Solution 11

B  Chapter 19, Inverse Cumulative Normal Distribution

The steps to obtaining a standard normal random variable are:

1. Obtain an observation $\hat{x}$ of a random variable $x$. This observation is 0.21072.

2. Find the probability that $x$ is less than $\hat{x}$:
   \[ F(\hat{x}) = \text{Prob}(x < \hat{x}) \]
   For the exponential distribution in this question, we have:
   \[ F(x) = 1 - e^{-0.5x} \]
   Therefore, the cumulative probability of 0.21072 is:
   \[ F(0.21072) = 1 - e^{-0.5 \times 0.21072} = 0.10000 \]
3. Find the value of $\hat{z}$ that is the $F(\hat{x})$ quantile of the normal distribution:

$$N(\hat{z}) = F(\hat{x})$$

$$N(\hat{z}) = 0.10000$$

$$N^{-1}(0.10000) = -1.28155$$

$$\hat{z} = -1.28155$$

Thus $-1.28155$ is the observation from the standard normal distribution.

For the risk-neutral distribution of the stock, we replace $\alpha$ with $r$. The Black-Scholes framework implies that the stock price follows a lognormal distribution, so:

$$S_T = S_t e^{(r-\delta-0.5\sigma^2)(T-t)+\sigma\sqrt{T-t}} = 100e^{(0.05-0.03-0.5\times0.3^2)\times(2-0)+0.3\times(-1.28155)\sqrt{2-0}}$$

$$= 55.2271$$

**Solution 12**

**E** Chapter 19, Sequence of Stock Prices

We convert the draws from the uniform distribution into draws from the standard normal distribution:

$$F(0.15) = N(\hat{z}_1) \Rightarrow N^{-1}(0.15000) = -1.03643 \Rightarrow \hat{z}_1 = -1.03643$$

$$F(0.65) = N(\hat{z}_2) \Rightarrow N^{-1}(0.65000) = 0.38532 \Rightarrow \hat{z}_2 = 0.38532$$

We use these draws to find the new stock prices:

$$S_T = S_t e^{(r-\delta-0.5\sigma^2)nh+\sigma(z_1+z_2+\cdots+z_n)\sqrt{h}}$$

$$= 100e^{(0.11-0.03-0.5\times0.3^2)\times2\times0.5+0.3(-1.03643+0.38532)\sqrt{0.5}} = 90.2018$$

**Solution 13**

**A** Chapter 19, Geometric Average Strike Call Option

We convert the draws from the uniform distribution into draws from the standard normal distribution:

$$F(0.12) = N(\hat{z}_1) \Rightarrow N^{-1}(0.12000) = -1.17499 \Rightarrow \hat{z}_1 = -1.17499$$

$$F(0.87) = N(\hat{z}_2) \Rightarrow N^{-1}(0.87000) = 1.12639 \Rightarrow \hat{z}_2 = 1.12639$$

$$F(0.50) = N(\hat{z}_3) \Rightarrow N^{-1}(0.50000) = 0.00000 \Rightarrow \hat{z}_3 = 0.00000$$
We use these draws to find the new stock prices:

\[ S_1 = S_0 e^{(r - \delta - 0.5\sigma^2) \frac{1}{3} + \sigma z_1 \sqrt{\frac{1}{3}}} = 100 e^{(0.11 - 0.03 - 0.5 \times 0.3^2) \frac{1}{3} + 0.30 \times (-1.17499)} \sqrt{\frac{1}{3}} = 82.54327 \]

\[ S_2 = S_1 e^{(r - \delta - 0.5\sigma^2) \frac{1}{3} + \sigma z_2 \sqrt{\frac{1}{3}}} = 82.54327 e^{(0.11 - 0.03 - 0.5 \times 0.3^2) \frac{1}{3} + 0.30 \times (1.12639)} \sqrt{\frac{1}{3}} = 101.50274 \]

\[ S_3 = S_2 e^{(r - \delta - 0.5\sigma^2) \frac{1}{3} + \sigma z_3 \sqrt{\frac{1}{3}}} = 101.50274 e^{(0.11 - 0.03 - 0.5 \times 0.3^2) \frac{1}{3} + 0.30 \times (0.00000)} \sqrt{\frac{1}{3}} = 102.69387 \]

The geometric average of the stock prices is:

\[ \bar{S} = (82.54327 \times 101.50274 \times 102.69387)^{\frac{1}{3}} = 95.11185 \]

The payoff of the geometric average strike Asian call is:

\[ \text{Average strike Asian call option payoff} = \max(S_T - \bar{S}, 0) = 102.69387 - 95.11185 = 7.5820 \]

**Solution 14**

C Chapter 19, Asian Call Options

The fact that one trial out of 10,000 resulted in a price of $15.00 is of no use.

As the sampling frequency increases, the price of the Asian option declines. Since the sampling frequency of Option 4 is greater than the sampling frequency of Option 2, it must have a lower price than Option 2:

Option 4 Price ≤ 12.15

Since the sampling frequency of Option 4 is less than the sampling frequency of Option 3, it must have a higher price than Option 3:

Option 4 Price ≥ 9.71

Furthermore, an arithmetic average is always greater than a geometric average, so the price of Option 4 must be higher than the price of Option 1:

Option 4 Price ≥ 9.77

The only choice that satisfies these conditions is $10.30.

**Solution 15**

A Chapter 19, Standard Deviation of Monte Carlo Estimate

The standard deviation of an individual option price is:

\[ \sigma_V = 13.15 \]
There are $5 \times 500 = 2,500$ draws, so the standard deviation of the Monte Carlo estimate is:

$$\sigma_{\bar{V}(n)} = \frac{\sigma_{\bar{V}}}{\sqrt{n}} = \frac{13.15}{\sqrt{2,500}} = 0.263$$

### Solution 16

#### Chapter 19, Standard Deviation of Monte Carlo Estimate

The formula for the standard deviation of the Monte Carlo price estimate is:

$$\sigma_{\bar{V}(n)} = \frac{\sigma_{\bar{V}}}{\sqrt{n}}$$

We have:

$$\sigma_{\bar{V}(500)} = \frac{\sigma_{\bar{V}}}{\sqrt{500}}$$

$$\sigma_{\bar{V}(4,000)} = \frac{\sigma_{\bar{V}}}{\sqrt{4,000}} = \frac{\sigma_{\bar{V}}}{\sqrt{8 \times 500}} = \frac{\sigma_{\bar{V}(500)}}{\sqrt{8}}$$

If we were given the standard deviation of an individual option price, then we would make use of it. But since we are not given the standard deviation, we begin by estimating the standard deviation of the estimate obtained using 500 draws. We have 5 observations of $\bar{V}(500)$ available to us, so we can estimate $\sigma_{\bar{V}(500)}$:

Using the TI-30X IIS, the procedure is:

1. [2nd][STAT] (Select 1-VAR) [ENTER]
2. [DATA]
3. X1= 9.44 [Enter]
4. X2= 10.80 [Enter]
5. X3= 9.78 [Enter]
6. X4= 8.95 [Enter]
7. X5= 9.80 [Enter]

   (Hit the down arrow twice)

8. [STATVAR] (Arrow over to Sx)

   The result is: 0.678586767

To exit the statistics mode: [2nd] [EXITSTAT] [ENTER]

The estimate for $\sigma_{\bar{V}(500)}$ is therefore 0.678586767.
Alternatively, using the BA II Plus calculator, the procedure is:

[2nd][DATA] [2nd][CLR WORK]
X01= 9.440 [ENTER] ↓↓ (Hit the down arrow twice)
X02= 10.80 [ENTER] ↓↓
X03= 9.78 [ENTER] ↓↓
X04= 8.95 [ENTER] ↓↓
X05= 9.80 [ENTER]

[2nd][STAT] ↓↓↓

The result is: 0.67858677

To exit the statistics mode: [2nd][QUIT]

The resulting solution is:

\[ \sigma_v(4,000) = \frac{\sigma_v(500)}{\sqrt{8}} = \frac{0.67858677}{\sqrt{8}} = 0.2399 \]

**Solution 17**

E Chapter 19, Standard Deviation of Monte Carlo Estimate

The standard deviation of the Monte Carlo estimate is the standard deviation of the individual price estimates divided by the square root of the number of paths:

\[ \sigma_v(n) = \frac{\sigma_v}{\sqrt{n}} = \frac{\sigma_C}{\sqrt{n}} = \frac{11.40}{\sqrt{n}} \]

We can solve for the value of \( n \) that leads to a standard deviation that is 1% of the price:

\[ \frac{11.40}{\sqrt{n}} = 0.01 \quad \Rightarrow \quad n = 24,657 \]

**Solution 18**

C Chapter 19, Monte Carlo Valuation of a European Put

We convert the draws from the uniform distribution into draws from the standard normal distribution:

\[
\begin{align*}
F(0.90) &= N(\hat{z}_1) \quad \Rightarrow \quad N^{-1}(0.90000) = 1.28155 \quad \Rightarrow \quad \hat{z}_1 = 1.28155 \\
F(0.74) &= N(\hat{z}_2) \quad \Rightarrow \quad N^{-1}(0.74000) = 0.64335 \quad \Rightarrow \quad \hat{z}_2 = 0.64335 \\
F(0.21) &= N(\hat{z}_3) \quad \Rightarrow \quad N^{-1}(0.21000) = -0.80642 \quad \Rightarrow \quad \hat{z}_3 = -0.80642 \\
F(0.48) &= N(\hat{z}_4) \quad \Rightarrow \quad N^{-1}(0.48000) = -0.05015 \quad \Rightarrow \quad \hat{z}_4 = -0.05015
\end{align*}
\]
When using the Monte Carlo method to value an option, we use the risk-neutral distribution of the stock. Therefore, the four stock prices are found using the following formula:

\[ S_T = S_t e^{(r - \delta - 0.5\sigma^2)(T-t) + \sigma \sqrt{T-t}z} \]

where: \( z \sim N(0,1) \)

The four stock prices are and the associated payoffs to the put option are:

1. \( S_T^1 = 100e^{(0.07-0.02-0.5\times0.25^2)(1)+0.25\times1.28155\sqrt{T}} = 140.37363 \quad V_1(1) = 0 \)
2. \( S_T^2 = 100e^{(0.07-0.02-0.5\times0.25^2)(1)+0.25\times0.64335\sqrt{T}} = 119.67236 \quad V_2(1) = 0 \)
3. \( S_T^3 = 100e^{(0.07-0.02-0.5\times0.25^2)(1)+0.25\times(-0.80642)\sqrt{T}} = 83.28889 \quad V_3(1) = 18.7111 \)
4. \( S_T^4 = 100e^{(0.07-0.02-0.5\times0.25^2)(1)+0.25\times(-0.05015)\sqrt{T}} = 100.62318 \quad V_3(1) = 1.3768 \)

The Monte Carlo estimate is:

\[ \bar{V} = e^{-r(T-t)} \frac{\sum_{i=1}^{n} V_i(T)}{n} = e^{-0.07(1-0)} \frac{0 + 18.7111 + 1.3768}{4} = 4.6825 \]

**Solution 19**

**D** Chapter 19, Monte Carlo Valuation of an Asian Put

The arithmetic averages of the stock prices are:

<table>
<thead>
<tr>
<th></th>
<th>Arithmetic Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.133</td>
</tr>
<tr>
<td>2</td>
<td>60.767</td>
</tr>
<tr>
<td>3</td>
<td>61.847</td>
</tr>
<tr>
<td>4</td>
<td>53.440</td>
</tr>
</tbody>
</table>

The payoff of an average strike Asian put option is the average stock price minus the final stock price, when that amount is greater than zero:

\[ \text{Average strike Asian put option payoff} = \text{Max} [\bar{S} - S_T, 0] \]

These payoffs are shown in the rightmost column below:

<table>
<thead>
<tr>
<th></th>
<th>Arithmetic Average</th>
<th>Final Stock Price</th>
<th>Option Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.133</td>
<td>32.43</td>
<td>6.703</td>
</tr>
<tr>
<td>2</td>
<td>60.767</td>
<td>47.26</td>
<td>13.507</td>
</tr>
<tr>
<td>3</td>
<td>61.847</td>
<td>70.11</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>53.440</td>
<td>52.44</td>
<td>1.000</td>
</tr>
</tbody>
</table>
The Monte Carlo estimate is:

\[
\hat{V} = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} V_i(T) = \frac{e^{-0.10 \times (1-0)}}{4} [6.703 + 13.507 + 0.000 + 1.000] = 4.80
\]

**Solution 20**

**D** Chapter 19, Control Variate Valuation

The arithmetic and geometric averages of the stock prices are:

<table>
<thead>
<tr>
<th></th>
<th>Arithmetic</th>
<th>Geometric</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.133</td>
<td>37.423</td>
</tr>
<tr>
<td>2</td>
<td>60.767</td>
<td>59.675</td>
</tr>
<tr>
<td>3</td>
<td>61.847</td>
<td>61.353</td>
</tr>
<tr>
<td>4</td>
<td>53.440</td>
<td>53.425</td>
</tr>
</tbody>
</table>

The payoff of an average strike Asian put option is the average stock price minus the final stock price, when that amount is greater than zero:

\[
\text{Average strike Asian put option payoff} = \text{Max} [\bar{S} - S_T, 0]
\]

These payoffs are shown in the rightmost columns below:

<table>
<thead>
<tr>
<th></th>
<th>Arithmetic</th>
<th>Geometric</th>
<th>Final Stock Price</th>
<th>Arithmetic Option Payoff</th>
<th>Geometric Option Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.133</td>
<td>37.423</td>
<td>32.43</td>
<td>6.703</td>
<td>4.993</td>
</tr>
<tr>
<td>2</td>
<td>60.767</td>
<td>59.675</td>
<td>47.26</td>
<td>13.507</td>
<td>12.415</td>
</tr>
<tr>
<td>3</td>
<td>61.847</td>
<td>61.353</td>
<td>70.11</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>53.440</td>
<td>53.425</td>
<td>52.44</td>
<td>1.000</td>
<td>0.985</td>
</tr>
</tbody>
</table>

The Monte Carlo estimate for the arithmetic average strike Asian put is:

\[
\hat{A} = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} V_i(T) = \frac{e^{-0.10 \times (1-0)}}{4} [6.703 + 13.507 + 0.000 + 1.000] = 4.7979
\]

The Monte Carlo estimate for the geometric average strike Asian put is:

\[
\hat{G} = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} V_i(T) = \frac{e^{-0.10 \times (1-0)}}{4} [4.993 + 12.415 + 0.000 + 0.985] = 4.1605
\]

The true price for the geometric option is $2.02, so the control variate price of the arithmetic average option is:

\[
A^* = \hat{A} + (G - \hat{G}) = 4.7979 + (2.02 - 4.1605) = 2.6574
\]
Solution 21

E Chapter 19, Control Variate Valuation

The arithmetic and geometric averages of the stock prices are:

<table>
<thead>
<tr>
<th></th>
<th>Arithmetic Average</th>
<th>Geometric Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.133</td>
<td>37.423</td>
</tr>
<tr>
<td>2</td>
<td>60.767</td>
<td>59.675</td>
</tr>
<tr>
<td>3</td>
<td>61.847</td>
<td>61.353</td>
</tr>
<tr>
<td>4</td>
<td>53.440</td>
<td>53.425</td>
</tr>
</tbody>
</table>

The payoff of an average strike Asian put option is the average stock price minus the final stock price, when that amount is greater than zero:

\[ \text{Average strike Asian put option payoff} = \max\left[ 0, S - S_T \right] \]

These payoffs are shown in the rightmost columns below:

<table>
<thead>
<tr>
<th></th>
<th>Arithmetic Average</th>
<th>Geometric Average</th>
<th>Final Stock Price</th>
<th>Arithmetic Option Y_i e^{0.10}</th>
<th>Geometric Option X_i e^{0.10}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.133</td>
<td>37.423</td>
<td>32.43</td>
<td>6.703</td>
<td>4.993</td>
</tr>
<tr>
<td>2</td>
<td>60.767</td>
<td>59.675</td>
<td>47.26</td>
<td>13.507</td>
<td>12.415</td>
</tr>
<tr>
<td>3</td>
<td>61.847</td>
<td>61.353</td>
<td>70.11</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>53.440</td>
<td>53.425</td>
<td>52.44</td>
<td>1.000</td>
<td>0.985</td>
</tr>
</tbody>
</table>

The values in the two rightmost columns are the payoffs, which means that they are the discounted Monte Carlo prices, \( Y_i \) and \( X_i \), times \( e^{rT} \).

The estimate for \( \beta \) is:

\[
\beta = \frac{\sum_{i=1}^{n} (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} = \frac{\left( e^{rT} \right)^2 \sum_{i=1}^{n} (Y_i - \bar{Y})(X_i - \bar{X})}{\left( e^{rT} \right)^2 \sum_{i=1}^{n} (X_i - \bar{X})^2}
\]

We get the same estimate for \( \beta \) regardless of whether we use the time 0 prices or the time 1 payoffs (as shown in the rightmost expression above). To save time, we use the time 1 payoffs from the table above.

We perform a regression using the sixth column in the table above as the x-values and the fifth column as the y-values. The resulting slope coefficient is:

\[
\beta = 1.0959
\]

During the exam, it is more efficient to let the calculator perform the regression and determine the slope coefficient.
Using the TI-30XS Multiview calculator, we first clear the data by pressing [data] [data] ↓↓↓ until Clear ALL is shown, and then press [enter].

Fill out the table as shown below:

<table>
<thead>
<tr>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.993</td>
<td>6.703</td>
<td>6.703</td>
</tr>
<tr>
<td>12.415</td>
<td>13.507</td>
<td>13.507</td>
</tr>
<tr>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.985</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Next, press:

[2nd] [stat] 2 (i.e., select 2-Var Stats)
Select L1 for x-data and press [enter].
Select L2 for y-data and press [enter].
Select CALC and then [enter]

Exit the data table by pressing [2nd] [quit].

Obtain access to the statistics by pressing [2nd] [stat] 3

Note the following statistics:

\[ \bar{x} = 4.59825 \]
\[ \bar{y} = 5.3025 \]
\[ a = 1.0959 \] (This is \( \beta \))

**Alternatively, the TI-30X IIS or the BA II Plus can be used to obtain the values above. To use one of those calculators, follow the steps outlined at the end of this solution and then return to this point to continue the solution.**

The Monte Carlo estimate for the arithmetic average strike Asian put is:

\[ \bar{A} = \bar{Y} = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} V_i(T) = e^{-0.10 \times (1-0)} \left[ 5.3025 \right] = 4.80 \]

The Monte Carlo estimate for the geometric average strike Asian put is:

\[ \bar{G} = \bar{X} = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} V_i(T) = e^{-0.10 \times (1-0)} \left[ 4.59825 \right] = 4.16 \]

The true price for the geometric option is $2.02, so the control variate price of the arithmetic average option is:

\[ A^* = \bar{A} + \beta (G - \bar{G}) = 4.80 + 1.0959(2.02 - 4.16) = 2.45 \]
Alternative Calculators

Using the TI-30X IIS, the procedure is:

\[ \text{[2nd][STAT]} \quad \text{(Select 2-VAR)} \quad \text{[ENTER]} \]

[DATA]

\[ \text{X1}= 4.993 \quad \downarrow \quad \text{(Hit the down arrow once)} \]

\[ \text{Y1}= 6.703 \quad \downarrow \]

\[ \text{X2}= 12.415 \quad \downarrow \]

\[ \text{Y2}= 13.507 \quad \downarrow \]

\[ \text{X3}= 0.000 \quad \downarrow \]

\[ \text{Y3}= 0.000 \quad \downarrow \]

\[ \text{X4}= 0.985 \quad \downarrow \]

\[ \text{Y4}= 1.000 \quad \text{[ENTER]} \]

[STATVAR]

Press the right arrow and note the following statistics:

\[ \bar{x} = 4.59825 \]

\[ \bar{y} = 5.3025 \]

\[ a = 1.0959 \quad \text{(This is } \beta \text{)} \]

To exit the statistics mode:  \[ \text{[2nd][EXITSTAT]} \quad \text{[ENTER]} \]

Using the BA II Plus calculator, the procedure is:

\[ \text{[2nd][DATA]} \quad \text{[2nd][CLR WORK]} \]

\[ \text{X01}= 4.993 \quad \text{[ENTER]} \quad \downarrow \quad \text{(Hit the down arrow once)} \]

\[ \text{Y01}= 6.703 \quad \text{[ENTER]} \quad \downarrow \]

\[ \text{X02}= 12.415 \quad \text{[ENTER]} \quad \downarrow \]

\[ \text{Y02}= 13.507 \quad \text{[ENTER]} \quad \downarrow \]

\[ \text{X03}= 0.000 \quad \text{[ENTER]} \quad \downarrow \]

\[ \text{Y03}= 0.000 \quad \text{[ENTER]} \quad \downarrow \]

\[ \text{X04}= 0.985 \quad \text{[ENTER]} \quad \downarrow \]

\[ \text{Y04}= 1.000 \quad \text{[ENTER]} \]

[2nd][STAT]
Press the down arrow and note the following statistics:

\[ \bar{x} = 4.59825 \]
\[ \bar{y} = 5.3025 \]
\[ b = 1.0959 \quad \text{(This is } \beta) \]

To exit the statistics mode: [2nd][QUIT]

Solution 22

Chapter 19, Variance and the Control Variate Method

The covariance of the two naïve price estimates is:

\[ \text{Cov}[\bar{A}, \bar{G}] = \rho_{\bar{A}, \bar{G}} \sqrt{\text{Var}[\bar{A}] \times \text{Var}[\bar{G}]} = 0.8 \times \sqrt{3 \times 5} = 3.0984 \]

The variance of the control variate estimate for the price of option G is:

\[ \text{Var}[G^*] = \text{Var}[\bar{G}] + \text{Var}[\bar{A}] - 2 \text{Cov}[\bar{A}, \bar{G}] = 3 + 5 - 2 \times 3.0984 = 1.8032 \]

The reduction in the variance is:

\[ \text{Var}[\bar{G}] - \text{Var}[G^*] = 3 - 1.8032 = 1.1968 \]

Solution 23

Chapter 19, Variance and the Control Variate Method

The variance of the control variate estimate is:

\[ \text{Var}[A^*] = \text{Var}[\bar{A}](1 - \rho^2_{A,B}) = 2(1 - 0.75^2) = 0.875 \]

Solution 24

Chapter 19, Antithetic Variate Method

The original estimate is based on the final stock prices and their corresponding payoffs:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( S_{0.5}^i )</th>
<th>Option Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35.50</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>41.70</td>
<td>1.70</td>
</tr>
<tr>
<td>3</td>
<td>37.95</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>29.41</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The original Monte Carlo estimate for the value of the call option is:

\[ \bar{V} = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} V_i(T) = \frac{e^{-0.08 \times (0.5 - 0.0)}}{4} [0 + 1.70 + 0 + 0] = 0.4083 \]
The antithetic variate method utilizes each draw from the standard normal distribution twice. To infer the draws from the normal distribution, we use the following formula:

$$S_T = S_te^{(r-\delta-0.5\sigma^2)(T-t)+\sigma z\sqrt{T-t}}$$

For the first stock price of 35.50, we have:

$$35.50 = 40e^{(0.08-0.00-0.5\times0.35^2)(0.5-0.0)+0.35z_1\sqrt{0.5-0.0}}$$

$$35.50 = 40e^{0.009375+0.35z_1\sqrt{0.5}}$$

$$z_1 = -0.52011$$

For the second stock price of 41.70, we have:

$$41.70 = 40e^{(0.08-0.00-0.5\times0.35^2)(0.5-0.0)+0.35z_2\sqrt{0.5-0.0}}$$

$$41.70 = 40e^{0.009375+0.35z_2\sqrt{0.5}}$$

$$z_2 = 0.13030$$

For the third stock price of 37.95, we have:

$$37.95 = 40e^{(0.08-0.00-0.5\times0.35^2)(0.5-0.0)+0.35z_3\sqrt{0.5-0.0}}$$

$$37.95 = 40e^{0.009375+0.35z_3\sqrt{0.5}}$$

$$z_3 = -0.25046$$

For the fourth stock price of 29.41, we have:

$$29.41 = 40e^{(0.08-0.00-0.5\times0.35^2)(0.5-0.0)+0.35z_4\sqrt{0.5-0.0}}$$

$$29.41 = 40e^{0.009375+0.35z_4\sqrt{0.5}}$$

$$z_4 = -1.28055$$

We take the opposite of each of these values and generate new stock prices. The first new stock price is therefore:

$$40e^{(0.08-0.00-0.5\times0.35^2)(0.5-0.0)+0.35(0.52011)\sqrt{0.5-0.0}} = 45.92$$
The other 3 new prices are calculated in a similar manner. The new prices and their associated payoffs appear in the bottom 4 rows of the table below:

<table>
<thead>
<tr>
<th>i</th>
<th>( z_i )</th>
<th>( S_{0.5}^i )</th>
<th>Option Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.52</td>
<td>35.50</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>0.13</td>
<td>41.70</td>
<td>1.70</td>
</tr>
<tr>
<td>3</td>
<td>-0.25</td>
<td>37.95</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>-1.28</td>
<td>29.41</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>0.52</td>
<td>45.92</td>
<td>5.92</td>
</tr>
<tr>
<td>6</td>
<td>-0.13</td>
<td>39.10</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>0.25</td>
<td>42.96</td>
<td>2.96</td>
</tr>
<tr>
<td>8</td>
<td>1.28</td>
<td>55.43</td>
<td>15.43</td>
</tr>
</tbody>
</table>

The new antithetic variate estimate for the price of the option is:

\[
\hat{V} = e^{-r(T-t)} \frac{1}{n} \sum_{i=1}^{n} V_i(T) = e^{-0.08 \times (0.5 - 0.0)} \left[ 0 + 1.70 + 0 + 0 + 5.92 + 0 + 2.96 + 15.43 \right] \\
= 3.124382
\]

The antithetic variate price estimate exceeds the original price estimate by the following amount:

\[ 3.124382 - 0.4083 = 2.7160 \]

**Solution 25**

C Chapter 19, Control Variate Method

Let’s refer to the cash-or-nothing call option as Option Y. We have:

\[
\bar{Y} = 0.5228 \\
\sigma_Y = 0.0046
\]

Let’s refer to the regular call option as Option X. We have:

\[
X = 13.36 \\
\bar{X} = 13.4815 \\
\sigma_X = 0.1916
\]

We can use the control variate price to solve for \( \beta \):

\[
Y^* = \bar{Y} + \beta (X - \bar{X}) \\
0.5209 = 0.5228 + \beta (13.36 - 13.4815) \\
\beta = 0.015638
\]
We can use $\beta$ to find the covariance of the naïve price estimates:

$$\beta = \frac{Cov[\bar{X}, \bar{Y}]}{Var[\bar{X}]}$$

$$0.015638 = \frac{Cov[\bar{X}, \bar{Y}]}{0.1916^2}$$

$$Cov[\bar{X}, \bar{Y}] = 0.0005741$$

The variance of the control variate estimate is:

$$Var[Y^*] = Var[\bar{Y}] + \beta^2 Var[\bar{X}] - 2\beta Cov[\bar{X}, \bar{Y}]$$

$$= 0.0046^2 + 0.015638^2 \times 0.1916^2 - 2 \times 0.015638 \times 0.0005741$$

$$= 0.0000121827$$

The standard deviation of the control estimate is the square root of the variance:

$$\sqrt{0.0000121827} = 0.00349$$

**Solution 26**

**D** Chapter 19, Stratified Sampling

The formula for stratified sampling is:

$$\hat{u}_i = \frac{u_i + i - 1}{k}$$

for $i \leq k$.

Since there are 3 intervals, we have $k = 3$. When $i$ exceeds $k$, we start over with the second $i$ in the numerator:

$$\hat{u}_1 = \frac{u_1 + 1 - 1}{3} = \frac{0.156}{3} = 0.052$$

$$\hat{u}_2 = \frac{u_2 + 2 - 1}{3} = \frac{0.695 + 1}{3} = 0.565$$

$$\hat{u}_3 = \frac{u_3 + 3 - 1}{3} = \frac{0.788 + 2}{3} = 0.929$$

$$\hat{u}_4 = \frac{u_4 + 1 - 1}{3} = \frac{0.188}{3} = 0.063$$

$$\hat{u}_5 = \frac{u_5 + 2 - 1}{3} = \frac{0.751 + 1}{3} = 0.584$$

$$\hat{u}_6 = \frac{u_6 + 3 - 1}{3} = \frac{0.132 + 2}{3} = 0.711$$

The sum of these numbers is:

$$0.052 + 0.565 + 0.929 + 0.063 + 0.584 + 0.711 = 2.903$$
Solution 27

B Chapter 19, Stratified Sampling

We begin by stratifying the uniform random variables:

\[
\begin{align*}
\hat{u}_1 &= \frac{u_1 + 1 - 1}{4} = \frac{0.152}{4} = 0.038 \\
\hat{u}_2 &= \frac{u_2 + 2 - 1}{4} = \frac{0.696 + 1}{4} = 0.424 \\
\hat{u}_3 &= \frac{u_3 + 3 - 1}{4} = \frac{0.788 + 2}{4} = 0.697 \\
\hat{u}_4 &= \frac{u_4 + 4 - 1}{4} = \frac{0.188 + 3}{4} = 0.797 
\end{align*}
\]

We convert these draws into draws from the standard normal distribution:

\[
\begin{align*}
F(0.038) &= N(\hat{z}_1) \quad \Rightarrow \quad N^{-1}(0.03800) = -1.77438 \quad \Rightarrow \quad \hat{z}_1 = -1.77438 \\
F(0.424) &= N(\hat{z}_2) \quad \Rightarrow \quad N^{-1}(0.42400) = -0.19167 \quad \Rightarrow \quad \hat{z}_2 = -0.19167 \\
F(0.697) &= N(\hat{z}_3) \quad \Rightarrow \quad N^{-1}(0.69700) = 0.51579 \quad \Rightarrow \quad \hat{z}_3 = 0.51579 \\
F(0.797) &= N(\hat{z}_4) \quad \Rightarrow \quad N^{-1}(0.79700) = 0.83095 \quad \Rightarrow \quad \hat{z}_4 = 0.83095 
\end{align*}
\]

We use these draws to find the stock prices along the path:

\[
\begin{align*}
S_1 &= S_0 e^{(r-\delta-0.5\sigma^2)\frac{1}{4}+\sigma \hat{z}_1 \sqrt{\frac{1}{4}}} = 40 e^{(0.08-0.00-0.5\times0.3^2)\frac{1}{4}+0.30\times(-1.77438)\sqrt{\frac{1}{4}}} = 30.92214 \\
S_2 &= S_1 e^{(r-\delta-0.5\sigma^2)\frac{1}{4}+\sigma \hat{z}_2 \sqrt{\frac{1}{4}}} = 30.92214 e^{(0.08-0.00-0.5\times0.3^2)\frac{1}{4}+0.30\times(-0.19167)\sqrt{\frac{1}{4}}} = 30.30983 \\
S_3 &= S_2 e^{(r-\delta-0.5\sigma^2)\frac{1}{4}+\sigma \hat{z}_3 \sqrt{\frac{1}{4}}} = 30.30983 e^{(0.08-0.00-0.5\times0.3^2)\frac{1}{4}+0.30\times(0.51579)\sqrt{\frac{1}{4}}} = 33.03576 \\
S_4 &= S_3 e^{(r-\delta-0.5\sigma^2)\frac{1}{4}+\sigma \hat{z}_4 \sqrt{\frac{1}{4}}} = 33.03576 e^{(0.08-0.00-0.5\times0.3^2)\frac{1}{4}+0.30\times(0.83095)\sqrt{\frac{1}{4}}} = 37.7499 
\end{align*}
\]

The arithmetic average of the stock prices is:

\[
\bar{S} = \frac{30.92214 + 30.30983 + 33.03576 + 37.7499}{4} = 33.0044
\]

The payoff of the arithmetic average strike Asian call is:

\[
\text{Average strike Asian call option payoff} = \text{Max}[S_T - \bar{S}, 0] = 37.7499 - 33.0044 = 4.7455
\]
Solution 28

**E Chapter 19, Normal Random Variables as Quantiles**

We convert the draws from the uniform distribution into draws from the standard normal distribution:

- \( F(0.209) = N(\hat{z}_1) \) \( \Rightarrow \) \( N^{-1}(0.20900) = -0.80990 \) \( \Rightarrow \) \( \hat{z}_1 = -0.80990 \)
- \( F(0.881) = N(\hat{z}_2) \) \( \Rightarrow \) \( N^{-1}(0.88100) = 1.18000 \) \( \Rightarrow \) \( \hat{z}_2 = 1.18000 \)
- \( F(0.025) = N(\hat{z}_3) \) \( \Rightarrow \) \( N^{-1}(0.02500) = -1.95996 \) \( \Rightarrow \) \( \hat{z}_3 = -1.95996 \)

We use these draws to find the new stock prices:

\[
S_T = S_0 e^{(\alpha - \delta - 0.5\sigma^2)(T-t) + \sigma z \sqrt{T-t}}
\]

1. \( S_3 = 50e^{(0.12 - 0.00 - 0.5 \times 0.30^2)3 + 0.30 \times (-0.80990)\sqrt{3}} = 41.10734 \)
2. \( S_3 = 50e^{(0.12 - 0.00 - 0.5 \times 0.30^2)3 + 0.30 \times (1.18000)\sqrt{3}} = 115.60382 \)
3. \( S_3 = 50e^{(0.12 - 0.00 - 0.5 \times 0.30^2)3 + 0.30 \times (-1.95996)\sqrt{3}} = 22.61465 \)

The average of the stock prices is:

\[
\frac{41.10734 + 115.60382 + 22.61465}{3} = 59.7753
\]

Solution 29

**E Chapter 19, Stratified Sampling Method**

The stratified sampling method assigns the first and sixth uniform \((0, 1)\) random numbers to the segment \((0.00, 0.20)\), the second and seventh uniform \((0, 1)\) random variables to the segment \((0.20, 0.40)\), the third and eighth uniform \((0, 1)\) random variables to the segment \((0.40, 0.60)\), the fourth and the ninth uniform \((0, 1)\) random variables to the segment \((0.60, 0.80)\), and the fifth and the tenth uniform \((0,1)\) random variables to the segment \((0.80, 1.00)\).

The lowest simulated normal random variables will come from the segment \((0.00, 0.20)\). The lower of the two values in this segment is the sixth one: 0.347, so we use it to find the corresponding standard normal random variable:

\[
\begin{align*}
\hat{u}_6 &= 0.347 \\
\hat{V}_6 &= \frac{0.347 + (1-1)}{5} = 0.06940 \\
Z_6 &= N^{-1}(0.06940) = -1.48027
\end{align*}
\]
The highest simulated normal random variable will come from the segment (0.80, 1.00). The higher of the two values in this segment is the fifth one: 0.782, so we use it to find the corresponding standard normal random variable:

\[ u_5 = 0.782 \]
\[ \hat{u}_5 = V_5 = \frac{0.782 + (5 - 1)}{5} = 0.95640 \]
\[ Z_5 = N^{-1}(0.95640) = 1.71036 \]

The difference between the largest and smallest simulated normal random variates is:

\[ 1.71036 - (-1.48027) = 3.19063 \]

**Solution 30**

C  Chapter 19, Control Variate Method

Let the variable \( Y \) be associated with the $52-strike option and the variable \( X \) be associated with the $50-strike option. In the regression analysis, \( Y \) will be the dependent variable and \( X \) will be the independent variable.

The payoffs depend on the simulated stock prices:

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>Payoff With Strike = 50 ( X_i e^{0.025} )</th>
<th>Payoff With Strike = 52 ( Y_i e^{0.025} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>43.30</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>47.30</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>52.00</td>
<td>2.00</td>
<td>0.00</td>
</tr>
<tr>
<td>54.00</td>
<td>4.00</td>
<td>2.00</td>
</tr>
<tr>
<td>58.90</td>
<td>8.90</td>
<td>6.90</td>
</tr>
</tbody>
</table>

The values in the two rightmost columns are the payoffs, which means that they are the discounted Monte Carlo prices, \( X_i \) and \( Y_i \), times \( e^{0.10 \times 0.25} \).

The estimate for \( \beta \) is:

\[ \beta = \frac{\sum_{i=1}^{n} (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} = \frac{\left(e^{rT}\right)^2 \sum_{i=1}^{n} (Y_i - \bar{Y})(X_i - \bar{X})}{\left(e^{rT}\right)^2 \sum_{i=1}^{n} (X_i - \bar{X})^2} \]

We get the same estimate for \( \beta \) regardless of whether we use the time 0 prices or the time 0.25 payoffs (as shown in the rightmost expression above). To save time, we use the time 0.25 payoffs from the table above.
We perform a regression using the second column in the table above as the \(x\)-values and the third column as the \(y\)-values. The resulting slope coefficient is:

\[
\beta = 0.78251
\]

During the exam, it is more efficient to let the calculator perform the regression and determine the slope coefficient.

Using the TI-30XS Multiview calculator, we first clear the data by pressing [data] [data] ↓↓↓ until Clear ALL is shown, and then press [enter].

Fill out the table as shown below:

<table>
<thead>
<tr>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>---</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>4.00</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>8.90</td>
<td>6.90</td>
<td></td>
</tr>
</tbody>
</table>

Next, press:

\[2^{nd}\] [stat] 2 (i.e., select 2-Var Stats)
Select L1 for \(x\)-data and press [enter].
Select L2 for \(y\)-data and press [enter].
Select CALC and then [enter]

Scroll down to: \(a = 0.7825135017\).

Exit the statistics mode by pressing \(2^{nd}\) [quit].

Since the $50-strike call is our control, we have:

\[
\bar{X} = \frac{0 + 0 + 2.00 + 4.00 + 8.90}{5} \times e^{-0.025} = 2.9064
\]
\[
\bar{Y} = \frac{0 + 0 + 2.00 + 6.90}{5} \times e^{-0.025} = 1.7361
\]

The estimate for the price of the $52-strike option is:

\[
C^*(52) = Y^* = \bar{Y} + \beta(X - \bar{X}) = 1.7361 + 0.78251(2.67 - 2.9064) = 1.5510
\]

**Solution 31**

**B** Chapter 19, Control Variate Method

Let the variable \(Y\) be associated with the $52-strike option and the variable \(X\) be associated with the $50-strike option. In the regression analysis, \(Y\) will be the dependent variable and \(X\) will be the independent variable.
The payoffs depend on the simulated stock prices:

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>Payoff With Strike = 50</th>
<th>Payoff With Strike = 52</th>
</tr>
</thead>
<tbody>
<tr>
<td>43.30</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>47.30</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
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<td>2.00</td>
</tr>
<tr>
<td>58.90</td>
<td>8.90</td>
<td>6.90</td>
</tr>
</tbody>
</table>

The values in the two rightmost columns are the payoffs, which means that they are the discounted Monte Carlo prices, $X_i$ and $Y_i$, times $e^{0.10 \times 0.25}$.

Using the TI-30XS Multiview calculator, we first clear the data by pressing [data] [data] ↓↓↓ until Clear ALL is shown, and then pressing [enter].

Fill out the table as shown below:

<table>
<thead>
<tr>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>4.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>8.90</td>
<td>6.90</td>
<td>6.90</td>
</tr>
</tbody>
</table>

Next, press:

- [2nd] [stat] 2 (i.e., select 2-Var Stats)
- Select L1 for x-data and press [enter].
- Select L2 for y-data and press [enter].
- Select CALC and then [enter]

Scroll down to obtain the estimates for the standard deviation of the 52-strike option’s payoffs and the correlation coefficient:

$Sy = 2.99032$

$r = 0.968649$

Exit the statistics mode by pressing [2nd] [quit].
To obtain the standard deviation of the individual values of the 52-strike option, we adjust for the fact that the payoffs occur at time 0.25. Then we find the standard deviation of the naïve estimate:

\[
\sigma_Y = S_0 e^{-0.10 \times 0.25} = 2.99032 \times e^{-0.025} = 2.91649
\]

\[
\sigma_{\overline{Y}} = \frac{\sigma_Y}{\sqrt{n}} = \frac{2.91649}{\sqrt{5}} = 1.30429
\]

The variance of the estimate is then:

\[
Var[Y^*] = Var[\overline{Y}] \left(1 - \rho_{X,\overline{Y}}^2\right) = 1.30429^2 \left(1 - 0.968649^2\right) = 0.10499
\]

The standard deviation of the estimate is:

\[
\sqrt{0.10499} = 0.32403
\]

**Solution 32**

Chapter 19, Variance of Control Variate Estimate

The formula from the ActuarialBrew.com Review Notes that has \(X\) as the control variate and \(Y^*\) as the control variate estimate is:

\[
Var[Y^*] = Var[\overline{Y}] \left(1 - \rho_{X,\overline{Y}}^2\right)
\]

In this question, however, the control variate is denoted by \(Y\) and the control variate estimate is denoted by \(X^*\), so we switch the roles of \(X\) and \(Y\):

\[
Var[X^*] = Var[\overline{X}] \left(1 - \rho_{X,\overline{X}}^2\right) = 5^2 \left(1 - 0.4^2\right) = 21
\]
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Chapter 20 – Solutions

Solution 1
C  Chapter 20, Sharpe Ratio
Stock X and Stock Q are perfectly correlated since each one is driven by the same stochastic process, so they must have the same Sharpe ratio:

\[
\frac{\alpha_X - r}{\sigma_X} = \frac{\alpha_Q - r}{\sigma_Q}
\]

\[
\frac{0.05 - r}{0.11} = \frac{0.08 - r}{0.31}
\]

\[
(0.31)(0.05) - 0.31r = (0.11)(0.08) - 0.11r
\]

\[
0.20r = (0.31)(0.05) - (0.11)(0.08)
\]

\[
r = 0.0335
\]

Solution 2
E  Chapter 20, Diffusion Process
A stochastic process is a diffusion process if its future values do not depend on the path taken to reach the current value. All of the properties listed in the question have this property.

Solution 3
D  Chapters 18 and 20, Confidence Intervals
\(Y(t)\) follows geometric Brownian motion with the following parameters:

\[
\alpha_Y - \delta_Y = 0.66 \quad \sigma_Y = -0.6
\]

The upper \(z\)-value for the 90% confidence interval is found below:

\[
P\left(z > z^U\right) = \frac{p}{2} \Rightarrow P\left(z > z^U\right) = \frac{1 - 0.90}{2} \Rightarrow P\left(z > z^U\right) = 0.05000 \Rightarrow z^U = 1.64485
\]

The upper limit of the confidence interval for \(Y(4)\) is:

\[
Y(0)e^{(\alpha_Y - \delta_Y - 0.5\sigma^2_Y)T + \sigma_Y\sqrt{T}} = 27e^{(0.66 - 0.5\times0.6^2)(4) + 0.6\times1.64485\times\sqrt{4}} = 1,325.6483
\]

The corresponding upper limit of the confidence interval for \(S(4)\) is the cubic root of the upper limit for \(Y(4)\):

\[
U = (1,325.6483)^{\frac{1}{3}} = 10.9852
\]
Solution 4

E  Chapter 20, Correlated Itô Processes

The differential equations and their expectations are:
\[ dZ(t) = dQ(t) \quad \text{and} \quad dZ'(t) = \sqrt{0.75}dQ(t) + 0.5dX(t) \]
\[ E[dZ(t)] = 0 \quad \text{and} \quad E[dZ'(t)] = \sqrt{0.75} \times 0 + 0.5 \times 0 = 0 \]

The covariance is:
\[ \text{Cov}(dZ(t), dZ(t)) = \text{Cov}(dZ'(t), dZ'(t)) = \text{Cov}(dQ(t), dQ(t)) = 0.75 \]
\[ \text{Cov}(dZ(t), dQ(t)) = \text{Cov}(dZ(t), dX(t)) = 0.5 \]
\[ \text{Cov}(dQ(t), dX(t)) = \sqrt{0.75} \cdot 0.5 \times 0.8660 \]
\[ = 0.3660 \]

From the multiplication rules, we have:
\[ (dQ(t))^2 = dt \]

Since \( X(t) \) and \( Q(t) \) are independent pure Brownian motions, we have:
\[ E[dX(t)dQ(t)] = E[dX(t)]E[dQ(t)] = 0 \times 0 = 0 \]

Therefore, the covariance is:
\[ \sqrt{0.75}E[(dQ(t))^2] + 0.5E[dX(t)dQ(t)] = \sqrt{0.75}E[dt] + 0.5 \times 0 = 0.8660dt \]

Solution 5

C  Chapter 20, Correlated Itô Processes

Since \( Z'(t) \), \( Q(t) \), and \( X(t) \) are pure Brownian motions, we have:
\[ \text{Var}[Z'(t)] = t \]
\[ \text{Var}[Q(t)] = t \]
\[ \text{Var}[X(t)] = t \]
We can now solve for $a$:

$$Z'(t) = 0.8Q(t) + aX(t)$$

$$\text{Var}[Z'(t)] = 0.64\text{Var}[Q(t)] + a^2\text{Var}[X(t)]$$

$$t = 0.64t + a^2t$$

$$1 = 0.64 + a^2$$

$$a^2 = 0.36$$

$$a = 0.6$$

**Solution 6**

D Chapter 20, Correlated Itô Processes

Since $Z'(t)$ exhibits pure Brownian motion, its variance is $t$:

$$\text{Var}[Z'(t)] = t$$

$$\text{Var} \left[ \sqrt{(a^2 - 1)}W_1(t) + 0.8W_2(t) \right] = t$$

$$(a^2 - 1)t + (0.8)^2 t = t$$

$$(a^2 - 1) + (0.8)^2 = 1$$

$$(a^2 - 1) = 1 - (0.8)^2$$

$$a = 1.166$$

**Solution 7**

B Chapter 20, Multiplication Rules

The geometric Brownian motion process is described below:

$$dX(t) = 0.05X(t)dt + 0.08X(t)dZ(t)$$

We then square $dX(t)$:

$$[dX(t)]^2 = [0.05X(t)dt + 0.08X(t)dZ(t)]^2$$

$$= 0.0025[X(t)]^2 (dt)^2 + 0.008[X(t)]^2 dtdZ(t) + 0.0064[X(t)]^2 [dZ(t)]^2$$

Now we use the multiplication rules.

The rule that $(dt)^2 = 0$ means that the first term is 0.

The rule that $dt \times dZ = 0$ means that the second term is 0.

The rule that $(dZ)^2 = dt$ applies to the third term.

Applying the rules, we have:

$$[dX(t)]^2 = 0.0064[X(t)]^2 dt$$
Solution 8

D Chapter 20, Sharpe Ratio and Arbitrage

The Sharpe ratio of Asset 1 is greater than that of Asset 2:

\[
\frac{0.08 - 0.04}{0.23} > \frac{0.07 - 0.04}{0.25} \Rightarrow 0.1739 > 0.12
\]

Since both assets follow geometric Brownian motion, a strategy that involves purchasing \( \frac{1}{\sigma_1 S_1} \) shares of Asset 1 results in an arbitrage profit of:

\[
\frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} dt = \left[ \frac{0.08 - 0.04}{0.23} - \frac{0.07 - 0.04}{0.25} \right] dt = 0.053913 dt
\]

Therefore a strategy that involves purchasing 1 share of Asset 1 results in an arbitrage profit of:

\[
\sigma_1 S_1 \times 0.053913 dt = (0.23)(20)(0.053913) dt = 0.248 dt
\]

Solution 9

E Chapter 20, Sharpe Ratio and Arbitrage

The Sharpe ratio of Stock Q is greater than that of Stock X:

\[
\frac{0.07 - 0.03}{0.12} > \frac{0.05 - 0.03}{0.11} \Rightarrow 0.3333 > 0.1818
\]

Since both assets follow geometric Brownian motion, a strategy that involves purchasing \( \frac{1}{\sigma_Q S_Q} \) shares of Stock Q and selling \( \frac{1}{\sigma_X S_X} \) shares of Stock X results an arbitrage profit of:

\[
\frac{\alpha_Q - r}{\sigma_Q} - \frac{\alpha_X - r}{\sigma_X} dt = \left[ \frac{0.07 - 0.03}{0.12} - \frac{0.05 - 0.03}{0.11} \right] dt = 0.1515 dt
\]

Therefore a strategy that involves selling 1 share of Stock X results in an arbitrage profit of:

\[
\sigma_X S_X \times 0.1515 dt = (0.11)(182)(0.1515) dt = 3.033 dt
\]

Solution 10

A Chapter 20, Prepaid Forward Price of $1

We don’t need the information provided in statement (i) to answer this question.

The usual formula for a prepaid forward is:

\[
P_{0,T}^P(S) = S(0)e^{-\delta T}
\]
Let’s find the prepaid forward price of $1, in euros. From the perspective of a euro-denominated investor, the initial asset price is:

\[
\frac{1}{x(0)} = \frac{1}{1.40}
\]

From the perspective of a euro-denominated investor, the dividend yield of $1 is 5%.

The prepaid forward price of one dollar, in euros, is therefore:

\[
\frac{1}{x(0)} e^{-0.05 \times 3} = \frac{1}{1.40} e^{-0.05 \times 3} = 0.61479
\]

The prepaid forward price of $1,000, in euros, is:

\[0.61479 \times 1,000 = 614.79\]

**Solution 11**

**B** Chapter 20, Sharpe Ratio

Stock C has a risk-free return of \( \alpha_C \). Therefore, the continuously compounded risk-free rate of return is \( \alpha_C \). So, the Sharpe ratio of Asset A is:

\[
\frac{3\alpha_C - \alpha_C}{6\alpha_C} = \frac{3 - 1}{6} = 0.333
\]

Asset B is perfectly correlated with Asset A since each are driven by the same stochastic process, so it has the same Sharpe ratio as Asset A.

**Solution 12**

**C** Chapter 20, Sharpe Ratio and Arbitrage

The Sharpe ratio of Stock Q is greater than that of Stock X:

\[
\frac{0.09 - 0.06}{0.25} > \frac{0.08 - 0.06}{0.20}
\]

\[0.12 > 0.10\]

Since both assets follow geometric Brownian motion, a strategy of purchasing \( \frac{1}{\sigma_Q S_Q} \) shares of Stock Q and selling \( \frac{1}{\sigma_X S_X} \) shares of Stock X results an arbitrage profit of:

\[
\left[ \frac{\alpha_Q - r}{\sigma_Q} - \frac{\alpha_X - r}{\sigma_X} \right] dt = \left[ \frac{0.09 - 0.06}{0.25} - \frac{0.08 - 0.06}{0.20} \right] dt = 0.02 dt
\]

The strategy also calls for lending $1:

\[\left( \frac{1}{\sigma_X} - \frac{1}{\sigma_Q} \right) = \left( \frac{1}{0.20} - \frac{1}{0.25} \right) = 1\]
The investor is allowed to lend $200, so the arbitrage can be scaled up by a factor of:
\[
\frac{200}{1} = 200
\]
Therefore, the arbitrage profit is:
\[
200 \times 0.02dt = 4dt
\]

**Solution 13**

**A**  Chapter 20, Sharpe Ratio and Arbitrage

The Sharpe ratios for Stock X and Stock Q must be equal or arbitrage would be possible:
\[
\frac{0.08 - 0.06}{0.20} = \frac{\alpha_Q - 0.06}{-0.25}
\]
\[-0.25(0.08 - 0.06) = 0.20(\alpha_Q - 0.06)
\]
\[\alpha_Q = 0.035\]

*Notice that \( \alpha_Q \) is less than the risk-free rate of return. From Asset X we can see that assets that are positively related to \( Z \) have a positive risk premium. This implies that assets such as Asset Q that are negatively related to \( Z \) must have a negative risk premium.*

**Solution 14**

**C**  Chapter 20, Sharpe Ratio and CAPM

We use the Sharpe ratio of Stock X to determine the Sharpe ratio of the market:
\[
0.0625 = 0.4 \times \frac{\alpha_M - r}{\sigma_M}
\]
\[\frac{\alpha_M - r}{\sigma_M} = 0.15625\]

We can now determine the Sharpe ratio of Stock Y:
\[
\rho_{Y,M} \frac{\alpha_M - r}{\sigma_M} = 0.75 \times 0.15625 = 0.1171875
\]

The risk premium of Stock Y can now be determined:
\[
\frac{\alpha_Y - r}{\sigma_Y} = 0.1171875
\]
\[\alpha_Y - r = (0.20)(0.1171875) = 0.0234375\]
Solution 15

The risk premium for Stock A is:
\[ \alpha_A - r = \beta_A \left( \alpha_M - r \right) \]
Substituting this expression into the formula for the Sharpe ratio of Stock A, we can determine the risk premium for the market:

\[ \frac{\alpha_A - r}{\sigma_A} = \frac{\beta_A \left( \alpha_M - r \right)}{\sigma_A} \]
\[ 0.15 = \frac{0.7 \left( \alpha_M - r \right)}{0.28} \]
\[ \left( \alpha_M - r \right) = 0.06 \]

The Sharpe ratio for Stock B is:
\[ \frac{\alpha_B - r}{\sigma_B} = \frac{\beta_B \left( \alpha_M - r \right)}{\sigma_B} \]
\[ 0.5 \left( 0.06 \right) = 0.12 \]

Solution 16

The risk premium for Stock A is:
\[ \alpha_A - r = \beta_A \left( \alpha_M - r \right) \]
Substituting this expression into the formula for the Sharpe ratio of Stock A, we can determine the risk premium for the market:

\[ \frac{\alpha_A - r}{\sigma_A} = \frac{\beta_A \left( \alpha_M - r \right)}{\sigma_A} \]
\[ 0.06 = \frac{0.1 \left( \alpha_M - r \right)}{0.20} \]
\[ \left( \alpha_M - r \right) = 0.12 \]

The Sharpe ratio for Stock B is:
\[ \frac{\alpha_B - r}{\sigma_B} = \frac{\beta_B \left( \alpha_M - r \right)}{\sigma_B} \]
\[ 0.5 \left( 0.12 \right) = 0.12 \]
Solution 17

D Chapter 20, Sharpe Ratio and Arbitrage

The Sharpe ratios for Stock X and Stock Q must be equal or arbitrage would be possible:

\[
\frac{0.08 - r}{0.20} = \frac{0.04 - r}{-0.25} \\
-0.25(0.08 - r) = 0.20(0.04 - r) \\
0.25r + 0.20r = 0.02 + 0.008 \\
r = 0.0622
\]

Solution 18

A Chapter 20, Geometric Brownian Motion Equivalencies

When the price follows geometric Brownian motion, the natural log of the price follows arithmetic Brownian motion:

\[
dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t) \quad \Leftrightarrow \quad d[\ln S(t)] = \left(\alpha - 0.5\sigma^2\right)dt + \sigma dZ
\]

We have:

\[
C = S^{0.5} \\
\ln C = 0.5\ln S \\
d[\ln C] = 0.5d[\ln S] = 0.5\left[(0.08 - 0.5 \times 0.18^2)dt + 0.18dZ\right] = 0.0319dt + 0.09dZ \\
dC = (0.0319 + 0.5 \times 0.09^2)Cdt + 0.09CdZ = 0.03595Cdt + 0.09CdZ
\]

Alternative Solution

Since the stock is governed by geometric Brownian motion, we can use the following formula:

\[
\frac{d(S^\alpha)}{S^\alpha} = \left[a(\alpha - \delta) + 0.5a(a-1)\sigma^2\right]dt + a\sigma dZ \\
d(S^\alpha) = \left[a(\alpha - \delta) + 0.5a(a-1)\sigma^2\right]S^\alpha dt + a\sigma S^\alpha dZ
\]

We have:

\[
a = 0.5 \quad \alpha - \delta = 0.08 \quad \sigma = 0.18
\]

Putting these values into the formula gives us:

\[
d\left(S^{0.5}\right) = \left(0.5(0.08) + 0.5(0.5)(-0.5)(0.18)^2\right)S^{0.5}dt + (0.5)(0.18)S^{0.5}dZ \\
= (0.04 - 0.00405)\sqrt{S}dt + 0.09\sqrt{S}dZ = 0.03595\sqrt{S}dt + 0.09\sqrt{S}dZ \\
= 0.03595Cdt + 0.09CdZ
\]
Solution 19

E Chapter 20, Geometric Brownian Motion Equivalencies

When the price follows geometric Brownian motion, the natural log of the price follows arithmetic Brownian motion:

\[ dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t) \quad \Leftrightarrow \quad d[\ln S(t)] = (\alpha - 0.5\sigma^2)dt + \sigma dZ \]

We have:

\[ y = S^{0.5} \]
\[ \ln y = 0.5 \ln S \]
\[ d[\ln y] = 0.5d[\ln S] = 0.5[(0.07 - 0.5 \times 0.20^2)dt + 0.20dZ] = 0.025dt + 0.10dZ \]
\[ dy = (0.025 + 0.5 \times 0.10^2)ydt + 0.10ydz = 0.03ydt + 0.10ydz \]

Solution 20

B Chapter 20, Sharpe Ratio

Since the derivative is perfectly correlated with the stock, it has the same Sharpe ratio as the stock:

\[ \frac{\alpha - r}{\sigma} = \frac{0.07 - 0.02}{0.20} = \frac{0.05}{0.20} = 0.25 \]

Solution 21

C Chapter 20, Valuing a Claim on \( S^a \)

Since the stock is governed by geometric Brownian motion, we can use the following formula:

\[ d(S^a) = \left(a(\alpha - \delta) + 0.5a(a-1)\sigma^2\right)S^adt + a\sigma S^adZ \]

We have:

\[ a = 0.5 \quad \alpha = 0.07 \quad \delta = 0 \quad \sigma = 0.20 \]

Putting these values into the formula gives us a new geometric Brownian motion that describes the price of the derivative:

\[ dG = d\left(S^{0.5}\right) = \left(0.5(0.07 - 0) + 0.5(0.5)(-0.5)(0.20)^2\right)S^{0.5}dt + (0.5)(0.20)S^{0.5}dZ \]
\[ = (0.035 - 0.005)\sqrt{S}dt + 0.10\sqrt{S}dZ \]
\[ = 0.03\sqrt{S}dt + 0.10\sqrt{S}dZ \]
\[ = 0.03Gdt + 0.10GdZ \]

The drift term is the first term:

\[ 0.03Gdt \]
Solution 22

E Chapter 20, Delta and $S^a$

The dividend yield of $[S(2)]^2$ is:

$$\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2 = 0.05 - 2(0.05 - 0) - 0.5 \times 2 \times 1 \times 0.2^2 = -0.09$$

The market-maker sold a prepaid forward of the $S^2$ asset. The prepaid forward price is:

$$F_{t,T}^P \left[ S(T)^a \right] = e^{-(\delta^*)(T-t)}S(t)^a = e^{0.09 \times 2}S^2 = e^{0.18}S^2$$

The partial derivative of the prepaid forward price with respect to the stock price is:

$$\frac{\partial \left( e^{0.18}S^2 \right)}{\partial S} = 2e^{0.18}S = 2e^{0.18} \times 3 = 7.1833$$

The delta of the contingent claim is 7.1833. Since the market-maker sold the contingent claim, the delta of the market-maker’s un-hedged position is $-7.1833$. To bring the position’s delta to zero, the market-maker purchases 7.1833 shares of stock.

Solution 23

E Chapter 20, Power Option

The dividend yield of $[S(1)]^2$ is:

$$\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2 = 0.05 - 2(0.05 - 0) - 0.5 \times 2 \times 1 \times 0.2^2 = -0.09$$

The usual expression for put-call parity is:

$$C + Ke^{-rT} = F_{0,T}^P(S) + P$$

In this case, the underlying asset is $S^2$ and the strike asset is $K^2$:

$$C + K^2e^{-rT} = F_{0,T}^P(S^2) + P$$

$$C - P = F_{0,T}^P(S^2) - K^2e^{-rT}$$

$$C - P = e^{-(\delta^*)T}S^2 - K^2e^{-rT}$$

To find the delta of the position consisting of a long call and a short put, we take the partial derivative of both sides with respect to $S$:

$$\frac{\partial (C - P)}{\partial S} = \frac{\partial \left( e^{-(\delta^*)T} \times S^2 - K^2e^{-rT} \right)}{\partial S} = 2S \times e^{-(\delta^*)T} + 0 = 2 \times 3 \times e^{0.09 \times 2} = 7.1833$$

Since the delta of the un-hedged position is 7.1833, the market-maker brings the delta to zero by selling 7.1833 shares of stock.
Solution 24

A Chapter 20, Valuing a Claim on $S^a$

Since the stock is governed by geometric Brownian motion, we can use the following formula:

$$d(S^a) = \left( a(\alpha - \delta) + 0.5a(a - 1)\sigma^2 \right) S^a dt + a\sigma S^a dZ$$

We have:

$$a = 0.5 \quad \alpha = 0.09 \quad \delta = 0.02 \quad \sigma = 0.20$$

Putting these values into the formula gives us:

$$d\left(S^{0.5}\right) = \left( 0.5(0.09 - 0.02) + 0.5(0.5)(-0.5)(0.20)^2 \right) S^{0.5} dt + (0.5)(0.20)S^{0.5}dZ$$

$$= \left( 0.035 - 0.005 \right) \sqrt{S} dt + 0.10 \sqrt{S} dZ$$

$$= 0.03 \sqrt{S} dt + 0.10 \sqrt{S} dZ$$

Since the expected change in $Z(t)$ is zero, the expected change in the function is:

$$0.03 \sqrt{S} dt = 0.03 \sqrt{60} dt = 0.232 dt$$

Solution 25

A Chapter 20, Risk-Neutral Process

The risk-neutral price process is:

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t)$$

We observe that:

$$\mu = r - \delta \quad \text{and} \quad \sigma = 0.2$$

The dividend yield of $S^3$ is:

$$\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2 = r - 3(r - \delta) - 0.5 \times 3 \times 2 \times 0.2^2 = r - 3(r - \delta) - 0.12$$

The formula for the forward price of $S^3$ is:

$$F_{t,T} \left[ S(T)^a \right] = e^{(r - \delta^*)(T - t)} S(t)^a$$

Setting this expression equal to the forward price provided in statement (ii), we have:

$$e^{[r - r + 3(r - \delta) + 0.12](T - t)} S(t)^3 = S(t)^3 e^{0.36(T - t)}$$

$$3(r - \delta) + 0.12 = 0.36$$

$$r - \delta = 0.08$$

$$\mu = 0.08$$
Solution 26

E Chapter 20, Itô’s Lemma

We have:

\[ V = \ln S^2 \]

\[ V_S = \frac{1}{S^2} \times 2S = 2S^{-1} \quad V_{SS} = -2S^{-2} \quad V_t = 0 \]

From Itô’s Lemma:

\[ dV = d(\ln S) = V_S dS + \frac{1}{2} V_{SS} (dS)^2 + V_t dt \]

\[ = 2S^{-1}dS - \frac{1}{2} \times 2S^{-2} (dS)^2 + 0dt \]

\[ = 2S^{-1}(0.09dt + 0.25dZ) - S^{-2}(0.25)^2 dt \]

\[ = S^{-1}(0.18dt + 0.50dZ) - 0.0625S^{-2} dt \]

\[ = \left[ 0.18S^{-1} - 0.0625S^{-2} \right] dt + 0.50S^{-1}dZ \]

Solution 27

B Chapter 20, Itô’s Lemma

We have:

\[ V = \ln S^2 \]

\[ V_S = \frac{1}{S^2} \times 2S = 2S^{-1} \quad V_{SS} = -2S^{-2} \quad V_t = 0 \]

From Itô’s Lemma and the multiplication rules:

\[ d[\ln S(t)] = V_S dS + \frac{1}{2} V_{SS} (dS)^2 + V_t dt \]

\[ = 2S^{-1}dS - \frac{1}{2} \times 2S^{-2} (dS)^2 + 0 \]

\[ = 2S^{-1} \times 0.4[0.12 - S]dt + 2S^{-1} \times 0.33dZ - S^{-2}(0.33)^2 dt \]

\[ = \left[ \frac{0.096}{S} - 0.8 - \frac{0.1089}{S^2} \right] dt + \frac{0.66}{S} dZ \]

\[ = \left[ 0.8 + \frac{0.096}{S} - \frac{0.1089}{S^2} \right] dt + \frac{0.66}{S} dZ \]
Solution 28

D Chapter 20, Geometric Brownian Motion Equivalencies

A lognormal stock price implies that changes in the stock price follow geometric Brownian motion, and vice versa:

\[ S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)} \iff dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t) \]

We can use the equation provided in the question to find values for \( \alpha - \delta \) and \( \sigma \):

\[
\begin{align*}
S(t) &= 20e^{0.035t + 0.3Z(t)} \\
\sigma &= 0.3 \\
0.035 &= \alpha - \delta - 0.5\sigma^2 \\
0.035 &= \alpha - 0.5(0.3^2) \\
\alpha - \delta &= 0.035 + 0.5(0.3^2) = 0.08
\end{align*}
\]

The solution is:

\[ dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t) = 0.08S(t)dt + 0.3S(t)dZ(t) \]

Solution 29

B Chapter 20, Geometric Brownian Motion Equivalencies

A lognormal stock price implies that changes in the stock price follow geometric Brownian motion, and vice versa:

\[ S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)} \iff dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t) \]

We can use the equation provided in the question to find values for \( \alpha - \delta \) and \( \sigma \):

\[
\begin{align*}
S(t) &= e^{0.20 + 0.05t}e^{0.4Z(t)} \\
S(0) &= e^{0.20} \\
\sigma &= 0.4 \\
0.05 &= \alpha - \delta - 0.5\sigma^2 \\
0.05 &= \alpha - 0.5(0.4^2) \\
\alpha - \delta &= 0.05 + 0.5(0.4^2) = 0.13
\end{align*}
\]

The solution is:

\[ dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ = 0.13S(t)dt + 0.4S(t)dZ(t) \]
Solution 30

D  Chapter 20, Geometric BM Equivalencies and the Sharpe Ratio

A lognormal stock price implies that changes in the stock price follow geometric Brownian motion, and vice versa:

\[ X(t) = X(0)e^{(\alpha_X - \delta_X - 0.5\sigma_X^2)t + \sigma_X Z(t)} \iff dX(t) = (\alpha_X - \delta_X)X(t)dt + \sigma_X X(t)dZ \]

We can use the equation provided in the question to find values for \( \alpha_X \) and \( \sigma_X \) in the price process for Stock X:

\[ X(t) = 15e^{0.14t + 0.07t^2}e^{0.42Z(t)} \Rightarrow X(t) = 15e^{0.14}e^{0.07t + 0.42Z(t)} \]

\[ X(0) = 15e^{0.14} \]

\[ \sigma_X = 0.42 \]

\[ 0.07 = \alpha_X - \delta_X - 0.5\sigma_X^2 \]

\[ 0.07 = \alpha_X - 0 - 0.5(0.42^2) \]

\[ \alpha_X = 0.07 + 0.5(0.42^2) = 0.1582 \]

The price process for Stock X is therefore:

\[ dX(t) = 0.1582X(t)dt + 0.42X(t)dZ(t) \]

From the price process for Stock Q, we observe that:

\[ \alpha_Q = 0.08 \]

\[ \sigma_Q = 0.13 \]

Stock X and Stock Q are perfectly correlated since they are driven by the same stochastic process, so they must have the same Sharpe ratio:

\[ \frac{0.1582 - r}{0.42} = \frac{0.08 - r}{0.13} \]

\[ 0.13(0.1582) - 0.13r = 0.42(0.08) - 0.42r \]

\[ 0.29r = 0.42(0.08) - 0.13(0.1582) \]

\[ r = 0.04494 \]

Solution 31

A  Chapter 20, Itô's Lemma

We have:

\[ V = S^2 \]

\[ V_S = 2S \]

\[ V_{SS} = 2 \]

\[ V_t = 0 \]
From Itô’s Lemma and the multiplication rules:

\[
\frac{d[S(t)]^2}{dt} = V_S dS + \frac{1}{2} V_{SS} (dS)^2 + V_t dt = 2S dS + \frac{1}{2} \times 2(dS)^2 + 0
\]

\[
= 2S[0.09 dt + 0.25dZ] + 0.25^2 dt = [0.18S + 0.0625]dt + 0.5sdZ
\]

**Solution 32**

E Chapter 20, Geometric Brownian Motion Equivalencies

When the price follows geometric Brownian motion, the natural log of the price follows arithmetic Brownian motion:

\[
ds(t) = \alpha S(t) dt + \sigma S(t) dZ(t) \iff d[\ln S(t)] = (\alpha - 0.5\sigma^2)dt + \sigma dZ
\]

We have:

\[
d[\ln S^2] = 2d[\ln S] = 2\left[(0.09 - 0.5 \times 0.25^2)dt + 0.25dZ\right] = 0.1175dt + 0.5dZ
\]

\[
ds^2 = (0.1175 + 0.5 \times 0.5^2)S^2 dt + 0.5S^2 dZ = 0.2425S^2 dt + 0.5S^2 dZ
\]

**Solution 33**

E Chapter 20, Correlated Itô Processes

From the geometric Brownian motion equivalencies, we have:

\[
ds(t) = (\alpha - \delta)S(t) dt + \sigma S(t) dZ(t) \iff \ln S(t) = \ln S(0) + (\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)
\]

Furthermore, we can find the difference between the log and its expected value:

\[
\ln S(t) = \ln S(0) + (\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t) \implies \ln S(t) - E[\ln S(t)] = \sigma Z(t)
\]

The covariance of \(\ln Q(t)\) and \(\ln X(t)\) is:

\[
Cov[\ln Q(t), \ln X(t)] = E[\ln Q(t) - E[\ln Q(t)]][\ln X(t) - E[\ln X(t)]]
\]

\[
= E[-0.3Z'(t) \times 0.2Z(t)]
\]

\[
= -0.06E[Z'(t)Z(t)] = -0.06 \times 0.7 t
\]

\[
= -0.042 t
\]

The correlation coefficient is:

\[
\rho = \frac{Cov(\ln(Q_t), \ln(X_t))}{\sqrt{\sigma_Q^2} \times \sqrt{\sigma_X^2}} = \frac{-0.042 t}{0.06t} = -0.7t
\]

Alternatively, we could observe that the correlation coefficient has to be negative since \(Q(t)\) moves inversely to \(X(t)\). Since the two pure Brownian motions that drive \(Q(t)\) and \(X(t)\) have a correlation of 0.7, the correlation of \(\ln Q(t)\) and \(\ln X(t)\) must be \(-0.7\).
We can value this option as a call option that has 2 shares of Stock Q as the underlying asset and 1 share of Stock X as the strike asset. We can use the Black-Scholes formula for exchange options. We begin by finding the volatility parameter. In the Black-Scholes formula, the volatility parameters are the standard deviation of the returns, and therefore they are positive.

Consequently, we use the absolute value of Stock Q’s volatility parameter, and \(-0.3\) becomes 0.3 in the expression below:

\[
\sigma = \sqrt{\sigma_Q^2 + \sigma_X^2 - 2\rho\sigma_Q\sigma_X} = \sqrt{0.3^2 + 0.2^2 - 2 \times (-0.7) \times 0.3 \times 0.2} = 0.4626
\]

The next step is to find \(d_1\) and \(d_2\) and \(N(d_1)\) and \(N(d_2)\):

\[
d_1 = \frac{\ln \left( \frac{2Q(0)e^{-\delta Qt}}{X(0)e^{-\delta Xt}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{2 \times 3e^{-0.0 \times 1}}{5e^{-0.02 \times 1}} \right) + \frac{0.4626^2 \times 1}{2}}{0.4626 \times \sqrt{1}} = 0.66866
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.66866 - 0.4626\sqrt{1} = 0.20606
\]

\(N(0.66866) = 0.74814\)

\(N(0.20606) = 0.58163\)

The price of the exchange call option is:

\[
\text{ExchangeCallPrice} = 2Q(0)e^{-\delta Qt} N(d_1) - X(0)e^{-\delta Xt} N(d_2)
\]

\[
= 2 \times 3e^{-0.0 \times 1} \times 0.74814 - 5e^{-0.02 \times 1} \times 0.58163 = 1.638275
\]

**Solution 34**

C Chapter 20, Correlated Itô Processes

The covariance of \(Z(t)\) and \(Z'(t)\) is:

\[
\text{Cov}[Z(t), Z'(t)] = E[(Z(t) - E[Z(t)])(Z'(t) - E[Z'(t)])] = E[Z(t) \times Z'(t)] = 0.7t
\]

The expected value of \(X(t)Q(t)\) is:

\[
E[X(t) \times Q(t)] = E\left[5e^{0.05 - 0.5(0.2)^2}t + 0.2Z(t) \times 3e^{0.08 - 0.5(-0.3)^2}t - 0.3Z'(t) \right]
\]

\[
= 15e^{0.065t} E\left[e^{0.2Z(t) - 0.3Z'(t)} \right]
\]

We can determine the expected value and the variance of the exponent in the expected value above:

\[
E[0.2Z(t) - 0.3Z'(t)] = 0
\]

\[
\text{Var}[0.2Z(t) - 0.3Z'(t)] = 0.2^2 t + (-0.3)^2 t + 2(0.2)(-0.3)(0.7t) = 0.046t
\]
We can now finish the calculation for the expected value of $X(t)Q(t)$:

$$E[X(t) \times Q(t)] = 15e^{0.065t}E \left[ e^{0.2Z(t) - 0.3Z'(t)} \right] = 15e^{0.065t}e^{0 + 0.5 \times 0.046t} = 15e^{0.088t}$$

For $t = 2$, we have:

$$E[X(2) \times Q(2)] = 15e^{0.088 \times 2} = 17.8866$$

**Solution 35**

**D** Chapter 20, Ornstein-Uhlenbeck Process

The general form of the Ornstein-Uhlenbeck process is:

$$dX(t) = \lambda \times [\alpha - X(t)]dt + \sigma dZ(t)$$

With $\alpha = 0$, the Ornstein-Uhlenbeck process can be written as:

$$dX(t) = \lambda \times [0 - X(t)]dt + \sigma dZ(t)$$

$$\frac{dX(t)}{X(t)} = -\lambda dt + \frac{\sigma}{X(t)}dZ(t)$$

Answer D is in this form with:

$$\lambda = 0.4$$
$$\sigma = 0.20$$

**Solution 36**

**E** Chapter 20, Risk-Neutral Process

The risk-neutral price process is of the form:

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t)$$

Therefore, we have:

$$r - \delta = 0.05$$
$$0.08 - \delta = 0.05$$
$$\delta = 0.03$$

The true price process is of the form:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)$$

Therefore, we have:

$$\alpha - \delta = 0.09$$
$$\alpha = 0.09 + 0.03 = 0.12$$
Solution 37

C  Chapter 20, Risk-Neutral Process

The true price process is of the form:

\[ \frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \]

Therefore, we have:

\[ \alpha - \delta = 0.13 \]
\[ \delta = 0.15 - 0.13 = 0.02 \]

The risk-neutral price process is of the form:

\[ \frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t) \]

Therefore, we have:

\[ r - \delta = 0.08 \]
\[ r = 0.08 + 0.02 = 0.10 \]

Solution 38

B  Chapter 20, Risk-Neutral Process and Sharpe Ratio

The risk-neutral price process is of the form:

\[ \frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t) \]

Since the dividend rate is 0:

\[ r - \delta = 0.07 \]
\[ r = 0.07 \]

The volatility is:

\[ \sigma = 0.30 \]

The Sharpe ratio is:

\[ \frac{\alpha - r}{\sigma} = \frac{0.13 - 0.07}{0.30} = 0.20 \]

Solution 39

E  Chapter 20, Itô’s Lemma

Let’s call the expression in the first given statement \( V(t) \):

\[ V(t) = \int_{0}^{t} Z(s)dZ(s) = \alpha [Z(t)]^2 - \beta t \]
By definition, we have:

\[ dV(t) = d\left[ \int_0^t Z(s) dZ(s) \right] = Z(t) dZ(t) \]

We can find another expression for \( dV(t) \) using Itô's Lemma and \( V(t) = \alpha[Z(t)]^2 - \beta t \). The partial derivatives are:

\[
\begin{align*}
V_Z &= 2\alpha Z \\
V_{ZZ} &= 2\alpha \\
V_t &= -\beta
\end{align*}
\]

From Itô's Lemma, we have:

\[
\begin{align*}
dV(t) &= V_Z dZ(t) + 0.5 V_{ZZ} (dZ(t))^2 + V_t dt \\
&= 2\alpha Z(t) dZ(t) + 0.5(2\alpha) dt - \beta dt \\
&= 2\alpha Z(t) dZ(t) + (\alpha - \beta) dt
\end{align*}
\]

Since we previously saw that \( dV(t) = Z(t) dZ(t) \), we can find the values of \( \alpha \) and \( \beta \) by noting that the coefficient to \( Z(t) dZ(t) \) is 1 and that the coefficient to \( dt \) is zero:

\[
\begin{align*}
2\alpha &= 1 & \Rightarrow & & \alpha &= 0.5 \\
\alpha - \beta &= 0 & \Rightarrow & & \beta &= 0.5
\end{align*}
\]

The value of \( \alpha + \beta \) is:

\[ \alpha + \beta = 0.5 + 0.5 = 1 \]

**Solution 40**

**B** Chapter 20, Risk-Neutral Process

In the Black-Scholes framework:

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta) dt + \sigma dZ(t)
\]

We can use statements (ii) and (iii) in the question to find \( \alpha \) and \( \sigma \):

\[
\begin{align*}
\alpha - \delta &= 0.08 & \Rightarrow & & \alpha - 0.04 &= 0.08 & \Rightarrow & & \alpha &= 0.12 \\
\sigma &= 0.30
\end{align*}
\]

The relationship between \( Z(T) \) and \( \tilde{Z}(T) \) is described by:

\[
\tilde{Z}(T) = Z(T) + \frac{\alpha - \gamma}{\sigma} T
\]
Under the risk-neutral probability measure $\tilde{Z}(T)$ is a pure Brownian motion, so:

$$E^* \left[ \tilde{Z}(T) \right] = E^* \left[ Z(T) + \frac{\alpha - r}{\sigma} T \right]$$

$$0 = E^* \left[ Z(T) \right] + \frac{\alpha - r}{\sigma} T$$

$$E^* \left[ Z(T) \right] = \frac{r - \alpha}{\sigma} T$$

Since the expectation of $Z(0.25)$ is $-0.05$, we have:

$$E^* \left[ Z(0.25) \right] = \frac{r - \alpha}{\sigma} \times 0.25$$

$$-0.05 = \frac{r - 0.12}{0.30} \times 0.25$$

$$r = 0.06$$

**Solution 41**

**A** Chapter 20, Geometric Brownian Motion Equivalencies

This question can be answered using Itô’s Lemma, but the method below is quicker.

Since we are given the risk-free rate in both the U.S. and Great Britain, we have:

$$r - r^* = 0.06 - 0.09 = -0.03$$

The forward price follows geometric Brownian motion:

$$G(t) = S(t)e^{-0.03(T-t)} = S(0)e^{[0.05 - 0.5(0.30)^2]t + 0.3Z(t)} e^{-0.03(T-t)}$$

$$= S(0)e^{-0.03T} e^{[0.05 - 0.5(0.30)^2]t + 0.3Z(t)} e^{0.03t}$$

$$= G(0)e^{[0.08 - 0.5(0.30)^2]t + 0.3Z(t)}$$

Since the forward price follows geometric Brownian motion, we can use the following equivalency:

$$dG(t) = (\hat{\alpha} - \hat{\sigma})G(t) dt + \hat{\sigma}G(t)dZ(t) \quad \Leftrightarrow \quad G(t) = G(0)e^{(\hat{\alpha} - \hat{\sigma} - 0.5\hat{\sigma}^2)t + \hat{\sigma}[Z(t)]}$$

We use the $^*$ symbol above to avoid confusion with the parameters for $S(t)$.

Since we have the rightmost expression above, we can express the left side as:

$$dG(t) = 0.08G(t)dt + 0.3G(t)dZ(t) = G(t) \left[ 0.08dt + 0.3dZ(t) \right]$$

**Alternative Solution (Using Itô’s Lemma)**

The expression for $G(t)$ is therefore:

$$G(t) = S(t)e^{-0.03(T-t)}$$
The partial derivatives are:
\[ G_S = e^{-0.03(T-t)} \]
\[ G_{SS} = 0 \]
\[ G_t = \frac{\partial}{\partial t} \left[ S(t)e^{-0.03T}e^{0.03t} \right] = S(t)e^{-0.03T}(0.03)e^{0.03t} = S(t)(0.03)e^{-0.03(T-t)} \]

From Itô’s Lemma, we have:
\[
dG(t) = G_S dS(t) + \frac{1}{2} G_{SS} (dS(t))^2 + G_t dt \\
= e^{-0.03(T-t)}dS(t) + \frac{1}{2}(0)(dS(t))^2 + S(t)(0.03)e^{-0.03(T-t)} dt \\
= e^{-0.03(T-t)}dS(t) + G(t)(0.03)dt \\
= e^{-0.03(T-t)}S(t)[0.05dt + 0.3dZ(t)] + G(t)(0.03)dt \\
= G(t)[0.05dt + 0.3dZ(t)] + G(t)(0.03)dt \\
= G(t)[0.08dt + 0.3dZ(t)]
\]

**Solution 42**

When the price follows geometric Brownian motion, the natural log of the price follows arithmetic Brownian motion:
\[
dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t) \Leftrightarrow \quad d[\ln S(t)] = (\alpha - 0.5\sigma^2)dt + \sigma dZ \\
\]

Therefore:
\[
\frac{dY(t)}{Y(t)} = Gdt + HdZ(t) \Leftrightarrow \quad d[\ln Y(t)] = \left( G - 0.5H^2 \right) dt + HdZ \\
\]

The arithmetic Brownian motion provided in the question for \( d[\ln Y(t)] \) allows us to find an expression for \( H \) and \( G \):
\[
d[\ln Y(t)] = 0.07dt + \sigma dZ(t) \quad \text{and} \quad d[\ln Y(t)] = \left( G - 0.5H^2 \right) dt + HdZ \\
\]
\[
H = \sigma \\
G - 0.5H^2 = 0.07 \\
G = 0.07 + 0.5H^2
\]
Since \( X \) and \( Y \) have the same source of randomness, \( dZ(t) \), they must have the same Sharpe ratio:
\[
\frac{0.09 - 0.05}{0.16} = \frac{G - 0.05}{H}
\]
\[
0.25 = \frac{G - 0.05}{H}
\]
\[
0.25 = \frac{0.07 + 0.5H^2 - 0.05}{H}
\]
\[
0.25H = 0.02 + 0.5H^2
\]
\[
0.5H^2 - 0.25H + 0.02 = 0
\]

We use the quadratic formula to solve for \( H \):
\[
H = \frac{0.25 \pm \sqrt{(-0.25)^2 - 4(0.5)(0.02)}}{2(0.5)} = 0.1 \text{ or } 0.4
\]

Since we are given that \( \sigma < 0.30 \) and we know that \( H = \sigma \), it must be the case that:
\[
H = 0.1
\]

We can now find the value of \( G \):
\[
0.25 = \frac{G - 0.05}{H} = \frac{G - 0.05}{0.1}
\]
\[
G = 0.075
\]

**Solution 43**

**A  Chapter 20, Black-Scholes Framework**

Statement (i) is true, because when the Black-Scholes framework applies, the stock prices are lognormally distributed:
\[
\ln S(t + h) - \ln S(t) \sim N \left( (\alpha - 0.5\sigma^2)h, \sigma^2 h \right)
\]

If we are given \( S(t) \), this implies:
\[
\ln S(t + h) - \ln S(t) \sim N \left( \ln S(t) + (\alpha - 0.5\sigma^2)h, \sigma^2 h \right)
\]
\[
\text{Var} \left[ \ln(S(t + h)|S(t)) \right] = \sigma^2 h
\]

Statement (ii) is false, because when the Black-Scholes framework applies, the stock prices follow geometric Brownian motion:
\[
\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t)
\]
The variance is:

\[
\text{Var}\left[ \frac{dS(t)}{S(t)} \right] = \text{Var}\left[ \alpha dt + \sigma dZ(t) | Z(t) \right] = 0 + \sigma^2 \text{Var}\left[ dZ(t) | Z(t) \right] = \sigma^2 dt
\]

Statement (iii) is false, because when the Black-Scholes framework applies, the stock prices follow geometric Brownian motion:

\[
dS(t) = \alpha S(t) dt + \sigma S(t) dZ(t)
\]

Therefore, we have:

\[
\text{Var}\left[ S(t + dt) | S(t) \right] = 0 + \sigma^2 \text{Var}\left[ dZ(t) \right] = \sigma^2 dt
\]

Solution 44

C  Chapter 20, Properties of Brownian Motion

Statement I is true. The quadratic variation of the pure Brownian motion process from time 0 to time \( T \) is equal to \( T \).

Statement II is true. Since the quadratic variation is equal to \( T \), it is finite.

Statement III is false. The total variation is infinite.

Solution 45

E  Chapter 20, Probability

The stock price follows geometric Brownian motion with:

\[
\alpha = 0.10 - 0.02 = 0.08
\]

\[
\sigma = 0.34
\]

The stock prices are lognormally distributed:

\[
\ln S(t) \sim N\left( \ln S(0) + (\alpha - 0.5 \sigma^2) t, \, \sigma^2 t \right)
\]

\[
\ln S(0.25) \sim N\left( \ln 30 + (0.08 - 0.5 \times 0.34^2)(0.25), \, 0.34^2 \times 0.25 \right)
\]

\[
\ln S(0.25) \sim N(3.4067, \, 0.0289)
\]

The probability that \( S(0.25) \) is less than $32 is:

\[
\text{Prob} \left[ S(0.25) < 32 \right] = \text{Prob} \left[ \ln S(0.25) < \ln 32 \right]
\]

\[
= \text{Prob} \left[ \frac{\ln S(0.25) - 3.4067}{\sqrt{0.0289}} < \frac{\ln 32 - 3.4067}{\sqrt{0.0289}} \right]
\]

\[
= \text{Prob} \left[ Z < 0.34699 \right] = 0.63570
\]
Solution 46

C Chapter 20, Valuing a Claim on $S^a$

The stock’s price follows geometric Brownian motion, and the claim pays $S(5)^a$, where $a = -1$. Therefore, the forward price is:

$$F_{0,T} \left[ S(T)^a \right] = e^{(r - \delta)(T)} S(0)^a = e^{-a(\sigma^2 + 0.5a(a-1))} S(0)^a$$

$$= e^{-1(0.07 - 0.02) + 0.5(-1)(-2)0.25^2} \times 2^{-1} = e^{0.0125 \times 0.5} = 0.5322$$

Solution 47

B Chapter 20, Valuing a Claim on $S^a$

The drift term of the Itô process can be expressed in terms of the expected return on the stock and the dividend yield of the stock:

$$0.08 dt = (\alpha - \delta) dt$$

$$0.08 = \alpha - 0.02$$

$$\alpha = 0.10$$

The expected return on the stock is 10%.

The stock’s price follows geometric Brownian motion, and the claim pays $S(T)^a$, where $a = -1$. The expected return on the claim is therefore:

$$\gamma = a(\alpha - \delta) + r = -1(0.10 - 0.07) + 0.07 = 0.04$$

The ratio of the expected return on the claim to the expected return on the stock is:

$$\frac{\gamma}{\alpha} = \frac{0.04}{0.10} = 0.40$$

Solution 48

B Chapter 20, Valuing a Claim on $S^a$

The stock price follows geometric Brownian motion, and the claim pays $S(T)^a$, where $a = 0.5$. The prepaid forward price is therefore:

$$F_{0,T}^P \left[ S(T)^a \right] = e^{(\delta^2)(T)} S(0)^a = e^{-a(r - \delta - 0.5a(a-1))} S(0)^a$$

$$= e^{-0.08 - 0.5(0.08 - 0.03) - 0.5(0.5)(-0.5)0.3^2} \times \frac{1}{\sqrt{100}} = 8.7590$$
Solution 49

A  Chapter 20, Valuing a Claim on $S^a$

The stock price follows geometric Brownian motion, and the claim pays $S(T)^a$, where $a = 2$. The dividend rate on the claim is therefore:

$$\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2 = 0.04 - 2(0.04 - 0) - 0.5(2)(2 - 1)(0.2)^2 = -0.08$$

Solution 50

D  Chapter 20, Valuing a Claim on $S^a$

The stock price follows geometric Brownian motion, and the claim pays $S(T)^a$, where $a = 2$. The forward price on the claim is therefore:

$$F_{0,T} \left[ S(T)^a \right] = e^{(r - \delta^*)(T)}S(0)^a = e^{a(r - \delta) + 0.5a(a - 1)\sigma^2}T S(0)^a$$
$$= \left[ S(0)e^{(r - \delta)T} \right]^a e^{0.5a(a - 1)\sigma^2 T} = (10.305)^2 e^{0.5(2)(2 - 1)(0.25)a(0.5)}$$
$$= 109.56$$

Solution 51

D  Chapter 20, Valuing a Claim on $S^a$

The forward price of an asset is its current price accumulated at the risk-free rate minus the asset’s dividend rate:

$$F_{t,T} \left[ S(T)^2 \right] = S(t)^2 e^{(r - \delta^*)(T-t)}$$

The forward price is equal to the current price of the derivative:

$$F_{t,T} \left[ S(T)^2 \right] = S(t)^2$$
$$S(t)^2 e^{(r - \delta^*)(T-t)} = S(t)^2$$
$$e^{(r - \delta^*)(T-t)} = 1$$
$$(r - \delta^*)(T-t) = 0$$
$$\delta^* = r$$
$$\delta^* = 0.10$$

Solution 52

E  Chapter 20, Valuing a Claim on $S^a$

The forward price of the derivative is:

$$F_{t,T} \left[ S(T)^a \right] = S(0)^a e^{(r - \delta^*)(T)} = S(t)^a e^{a(r - \delta) + 0.5a(a - 1)\sigma^2}(T-t)$$
The forward price is equal to the current price of the derivative:

\[ F_{t,T} \left[ S(T)^2 \right] = S(t)^2 \]

\[ S(t)^2 e^{\left[ a(r - \delta) + 0.5a(a-1)\sigma^2 \right](T-t)} = S(t)^2 \]

\[ S(t)^2 e^{\left[ 2(0.10-\delta) + 0.5(2)(2-1)(0.40)^2 \right](T-t)} = S(t)^2 \]

\[ e^{[0.20-2\delta+0.16](T-t)} = 1 \]

\[ (0.36 - 2\delta)(T - t) = 0 \]
\[ 0.36 - 2\delta = 0 \]
\[ \delta = 0.18 \]

Solution 53

A Chapter 20, Prepaid Forward Price of \( S^a \)

We make use of the following expression for the prepaid forward price of the derivative:

\[ F_{t,T}^P \left[ S(T)^a \right] = e^{-(\delta^a)(T-t)} S(t)^a \]

We can substitute this expression for the prepaid forward price in the equation provided in the question:

\[ F_{t,T}^P \left[ S(T)^x \right] = S(t)^x \]

\[ e^{-(\delta^a)(T-t)} S(t)^x = S(t)^x \]

\[ e^{\left[ -r + x(r - \delta) + 0.5x(x-1)\sigma^2 \right](T-t)} = 1 \]

\[ [-r + x(r - \delta) + 0.5x(x-1)\sigma^2](T - t) = 0 \]

\[ -r + x(r - \delta) + 0.5x(x-1)\sigma^2 = 0 \]

\[ -0.16 + x(0.16 - 0.00) + 0.5x^2(0.10)^2 - 0.5x(0.10)^2 = 0 \]

\[ 0.005x^2 + 0.155x - 0.16 = 0 \]

\[ 5x^2 + 155x - 160 = 0 \]

\[ x = \frac{-155 \pm \sqrt{155^2 - 4(5)(-160)}}{2(5)} \]

\[ x = 1 \quad \text{or} \quad x = -32 \]

We were told in the question that 1 is a solution. The other solution is \(-32\).
Solution 54

C  Chapter 20, Geometric Brownian Motion Equivalencies

When a process follows geometric Brownian motion, the natural log of the process follows arithmetic Brownian motion:

\[ dS(t) = \alpha S(t)dt + \beta S(t)dZ(t) \quad \iff \quad d[\ln S(t)] = (\alpha - 0.5\beta^2)dt + \beta dZ \]

We have:

\[
\begin{align*}
Y &= S^2 \\
\ln Y &= 2\ln S \\
d\ln Y &= 2d\ln S = 2[(\alpha - 0.5\beta^2)dt + \beta dZ] = 2(\alpha - 0.5\beta^2)dt + 2\beta dZ \\
\frac{dY}{Y} &= [2(\alpha - 0.5\beta^2) + 0.5 \times (2\beta)^2]dt + 2\beta dZ = (2\alpha + \beta^2)dt + 2\beta dZ
\end{align*}
\]

This implies that:

\[ \mu = 2\alpha + \beta^2 \quad \text{and} \quad \sigma = 2\beta \]

Consequently:

\[ \frac{\mu}{\sigma} = \frac{2\alpha + \beta^2}{2\beta} \]

Solution 55

A  Chapter 20, Geometric Brownian Motion Equivalencies

When a process follows geometric Brownian motion, the natural log of the process follows arithmetic Brownian motion:

\[ dX(t) = \mu X(t)dt + \sigma X(t)dZ(t) \quad \iff \quad d[\ln X(t)] = (\mu - 0.5\sigma^2)dt + \sigma dZ \]

We have:

\[
\begin{align*}
Y &= X^3 \\
\ln Y &= 3\ln X \\
d\ln Y &= 3d[\ln X] = 3[(\mu - 0.5\sigma^2)dt + \sigma dZ] = 3(\mu - 0.5\sigma^2)dt + 3\sigma dZ \\
\frac{dY}{Y} &= [3(\mu - 0.5\sigma^2) + 0.5 \times (3\sigma)^2]dt + 3\sigma dZ = (3\mu + 3\sigma^2)dt + 3\sigma dZ
\end{align*}
\]

Therefore:

\[ \frac{\alpha}{\beta} = \frac{3\mu + 3\sigma^2}{3\sigma} = \frac{\mu + \sigma^2}{\sigma} \]
Solution 56

E Chapter 20, Risk-Neutral Process

The risk-neutral price process for $S(t)$ is:

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t)$$

We are given that $\delta = 0.03$ and:

$$\frac{dS(t)}{S(t)} = 0.04dt + 0.12d\tilde{Z}(t)$$

We can solve for $r$:

$$r - \delta = 0.04$$
$$r - 0.03 = 0.04$$
$$r = 0.07$$

The expected return is twice the risk-free rate of return:

$$\alpha = 2 \times r = 2 \times 0.07 = 0.14$$

The expected value of the stock at the end of two years is:

$$E[S(t + 2) | S(t)] = S(t)e^{(\alpha - \delta)^2} = 100e^{(0.14 - 0.03)^2} = 124.61$$

Solution 57

A Chapter 20, Stochastic Differential Equations

The $t^2$ within the integral can be pulled out of the integral:

$$X(t) = 6t^3 + t \int_0^t d \tilde{Z}(s) = 6t^3 + t \int_0^t s \tilde{d}Z(s)$$

The first term of the differential is easily obtained:

$$dX(t) = 18t^2 dt + d \left[ t^3 \int_0^t s \tilde{d}Z(s) \right]$$

For the second term, we use the product rule:

$$d \left[ t^3 \int_0^t \tilde{d}Z(s) \right] = 3t^2 dt \left[ \int_0^t \tilde{d}Z(s) \right] + t^3 [td\tilde{Z}(t)] = 3t^2 dt \left[ \int_0^t \tilde{d}Z(s) \right] + t^4 d\tilde{Z}(t)$$
We can now write \( dX(t) \) as:

\[
dX(t) = 18t^2 \, dt + d \left[ t^3 \int_0^t s \, dZ(s) \right] = 18t^2 \, dt + 3t^2 \, dt \int_0^t s \, dZ(s) + t^4 \, dZ(t)
\]

\[
= \frac{3}{t} \left\{ 6t^3 + t^3 \int_0^t s \, dZ(s) \right\} \, dt + t^4 \, dZ(t) = \frac{3}{t} X(t) \, dt + t^4 \, dZ(t) = \frac{3X(t) \, dt + t^5 \, dZ(t)}{t}
\]

**Solution 58**

**D** Chapter 20, Geometric Brownian Motion and Mutual Funds

The instantaneous percentage increase of the mutual fund is the weighted average of the return on the stock (including its dividend yield) and the return on the risk-free asset, minus the dividend yield of the mutual fund:

\[
d\left( \frac{dW(t)}{W(t)} \right) = \frac{b}{S(t)} \left( dS(t) + \delta \, dt \right) + (1 - b) \, r \, dt - \delta_{\text{Mutual Fund}} \times dt
\]

\[
= 0.70 \left[ 0.07 dt + 0.20 dZ(t) + 0.05 dt \right] + 0.30(0.06) dt - 0.03 dt
\]

\[
= 0.084 dt + 0.14 dZ(t) + 0.018 dt - 0.03 dt
\]

\[
= 0.072 dt + 0.14 dZ(t)
\]

This can be rewritten as:

\[
dW(t) = 0.072W(t) dt + 0.14 W(t) dZ(t)
\]

The expression above matches Choice (D).

**Solution 59**

**E** Chapter 20, Probability

Using the geometric Brownian motion equivalencies, we can express the Brownian motion for \( Y(t) \) in the form at right below:

\[
d \left( \ln \left[ Y(t) \right] \right) = \left( 0.235 - 0.5 \beta^2 \right) dt + \beta dZ(t) \quad \Leftrightarrow \quad \frac{dY(t)}{Y(t)} = 0.235 dt + \beta dZ(t)
\]
Both stocks have only \( Z(t) \) as their source of randomness. Therefore, they have the same Sharpe ratio:

\[
\frac{0.4\beta - 0.085}{0.2} = \frac{0.235 - 0.085}{\beta}
\]

\[0.4\beta^2 - 0.085\beta = 0.03\]

\[0.4\beta^2 - 0.085\beta - 0.03 = 0\]

\[
\beta = \frac{0.085 \pm \sqrt{(-0.085)^2 - 4(0.4)(-0.085)}}{2 \times 0.4}
\]

\[\beta = -0.1875 \quad \text{or} \quad \beta = 0.40\]

Since we are given that \( \beta > 0 \), we have \( \beta = 0.40 \). We can now express the processes for \( X(t) \) and \( Y(t) \) as:

\[
dX(t) = 0.16X(t)dt + 0.2X(t)dZ(t)
\]

\[
dY(t) = 0.235Y(t)dt + 0.4Y(t)dZ(t)
\]

Again making use of the geometric Brownian motion equivalencies, we can express the ratio of \( X(t) \) to \( Y(t) \):

\[
\frac{X(t)}{Y(t)} = \frac{10e^{(0.16 - 0.5 \times 0.2^2)t + 0.2Z(t)}}{12e^{(0.235 - 0.5 \times 0.4^2)t + 0.4Z(t)}} = 0.8333e^{-0.015t - 0.2Z(t)}
\]

We now recognize that \( \frac{X(t)}{Y(t)} \) follows geometric Brownian motion with the following parameters:

\[
\sigma = -0.20
\]

\[
\alpha - \delta = -0.015 + 0.5 \times (-0.20)^2 = 0.005
\]

The probability that \( X(2) \) is greater than \( Y(2) \) is can be found by calculating \( N(\hat{d}_2) \) with \( K = 1 \). Although the volatility parameter in the Brownian motion is \( -0.20 \), the volatility used in the Black-Scholes formula is defined as the standard deviation of the return. Therefore, we use 0.20 in the formula for \( \hat{d}_2 \):

\[
P[X(2) > Y(2)] = P\left[\frac{X(2)}{Y(2)} > 1\right] = N(\hat{d}_2) = N\left[\frac{\ln \left(\frac{S_t}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}}\right]
\]

\[
= N\left[\frac{\ln 0.8333 + (0.005 - 0.5 \times 0.20^2)2}{0.20\sqrt{2}}\right] = N(-0.75067)
\]

\[= 0.22643\]
Solution 60

D Chapter 20, Probability

The stock price follows geometric Brownian motion with:

\[ \alpha - \delta = 0.08 \]
\[ \sigma = 0.32 \]

The stock prices are lognormally distributed:

\[ \ln S(t + h) - \ln S(t) \sim N \left( \alpha - \delta - 0.5\sigma^2 h, \sigma^2 h \right) \]
\[ \ln S(t + h) - \ln S(t) \sim N \left( \ln S(t) + (\alpha - \delta - 0.5\sigma^2) h, \sigma^2 h \right) \]
\[ \ln S(6) \sim N \left( \ln [S(2)] + (0.08 - 0.5 \times 0.32^2)(4), 0.32^2 \times 4 \right) \]
\[ \ln S(6) \sim N(3.11093, 0.40960) \]

Since the dividends are reinvested, at time 6 the value of the investment is:

\[ e^{0.04(6-2)} S(6) = e^{0.16} S(6) \]

The probability that \( e^{0.16} S(6) \) is greater than or equal to $25 is:

\[ \text{Prob} \left[ e^{0.16} S(6) \geq 25 \right] = \text{Prob} \left[ S(6) \geq e^{-0.16} 25 \right] = \text{Prob} \left[ S(6) \geq 21.30359 \right] \]
\[ = \text{Prob} \left[ \ln S(6) \geq \ln(21.30359) \right] = \text{Prob} \left[ \ln S(6) \geq 3.05888 \right] \]
\[ = \text{Prob} \left[ \frac{\ln S(6) - 3.11093}{0.40960} \geq \frac{3.05888 - 3.11093}{0.40960} \right] \]
\[ = \text{Prob} \left[ Z \geq -0.081338 \right] = 1 - \text{Prob} \left[ Z < -0.081338 \right] \]
\[ = 1 - 0.46677 \]
\[ = 0.53323 \]

Solution 61

C Chapter 20, Correlated Itô Processes

Both processes exhibit geometric Brownian motion:

For \( X(t) \), we have:

\[ \alpha_X - \delta_X = 0.08 \quad \sigma_X = 0.4 \]

For \( Y(t) \), we have:

\[ \alpha_Y - \delta_Y - 0.5\sigma_Y^2 = 0.08 \quad \sigma_Y = 0.4 \]
X(t) is lognormally distributed:
\[
\ln X(0.25) - \ln X(0) \sim N\left(\left(\alpha_X - \delta_X - 0.5\sigma_X^2\right)0.25, 0.25\sigma_X^2\right)
\]
\[
\ln X(0.25) - \ln(1) \sim N\left(\left(0.08 - 0.5(0.4^2)\right)0.25, 0.25 \times 0.4^2\right)
\]
\[
\ln X(0.25) \sim N\left(0.00000, 0.040000\right)
\]

Y(t) is also lognormally distributed:
\[
\ln Y(0.25) - \ln Y(0) \sim N\left(\left(\alpha_Y - \delta_Y - 0.5\sigma_Y^2\right)0.25, 0.25\sigma_Y^2\right)
\]
\[
\ln Y(0.25) - \ln(0.75) \sim N\left((0.08)0.25, 0.25 \times 0.4^2\right)
\]
\[
\ln Y(0.25) \sim N\left(-0.26768, 0.040000\right)
\]

The covariance of the logs is:
\[
Cov[\ln X(t), \ln Y(t)] = \sigma_X \sigma_Y \rho t = (0.4)(0.4)(0.4)(0.25) = 0.01600
\]

The expected value of \(\ln Y(0.25) - \ln X(0.25)\) is:
\[
E[\ln Y(0.25) - \ln X(0.25)] = -0.26768 - 0 = -0.26768
\]

The variance of \(\ln Y(0.25) - \ln X(0.25)\) is:
\[
Var[\ln Y(0.25) - \ln X(0.25)] = Var[\ln Y(0.25)] + Var[\ln X(0.25)] - 2Cov[\ln Y(0.25), \ln X(0.25)]
\]
\[
= 0.04000 + 0.04000 - 2(0.01600) = 0.04800
\]

We are now ready to calculate the probability that \(X(0.25)\) is greater than \(Y(0.25)\):
\[
P\left[X(0.25) > Y(0.25)\right] = P\left[\ln X(0.25) > \ln Y(0.25)\right]
\]
\[
= P\left[\ln Y(0.25) - \ln X(0.25) < 0\right]
\]
\[
= P\left[\frac{\ln Y(0.25) - \ln X(0.25) - (-0.26768)}{\sqrt{0.04800}} < \frac{0 - (-0.26768)}{\sqrt{0.04800}}\right]
\]
\[
= P[Z < 1.22180]
\]
\[
= 0.88911
\]

Solution 62

A Chapter 20, Ornstein-Uhlenbeck Process

This question is quite easy if we have memorized the differential and integral forms of the Ornstein-Uhlenbeck process. For the sake of thoroughness, we show how to derive the correct answer below.
The integral equation contains 3 terms. The first two terms do not contain random variables, so their differentials are easy to find. The third term will be more difficult:

\[
dX(t) = -\lambda X(0)e^{-\lambda t} dt + \lambda \alpha e^{-\lambda t} dt + d \left[ \int_0^t \sigma e^{-\lambda(t-s)} dZ(s) \right]
\]

The third term has a function of \( t \) in the integral. We can pull the \( t \)-dependent portion out of the integral, so that we are finding the differential of a product. We then use the following version of the product rule to find the differential:

\[
d[U(t)V(t)] = dU(t)V(t) + U(t)dV(t)
\]

The differential of the third term is:

\[
d \left[ \sigma \int_0^t e^{-\lambda(t-s)} dZ(s) \right] = \sigma d \left[ \int_0^t e^{\lambda s} dZ(s) \right]
\]

\[
= \sigma \left[ e^{-\lambda t} \int_0^t e^{\lambda s} dZ(s) + e^{-\lambda t} \sigma \int_0^t e^{\lambda s} dZ(s) \right]
\]

\[
= -\lambda \sigma e^{-\lambda t} \int_0^t e^{\lambda s} dZ(s) dt + e^{-\lambda t} e^\lambda dZ(t)
\]

\[
= -\lambda \sigma \left[ e^{-\lambda(t-s)} \int_0^t e^{\lambda s} dZ(s) \right] dt + \sigma dZ(t)
\]

Putting the three differentials together, we have:

\[
dX(t) = -\lambda X(0)e^{-\lambda t} dt + \lambda \alpha e^{-\lambda t} dt - \lambda \sigma \left( \int_0^t e^{-\lambda(t-s)} dZ(s) \right) dt + \sigma dZ(t)
\]

\[
= -\lambda \left[ X(0)e^{-\lambda t} - \alpha e^{-\lambda t} + \sigma \left( \int_0^t e^{-\lambda(t-s)} dZ(s) \right) \right] dt + \sigma dZ(t)
\]

\[
= -\lambda \left[ X(0)e^{-\lambda t} + \alpha - \alpha e^{-\lambda t} + \sigma \left( \int_0^t e^{-\lambda(t-s)} dZ(s) \right) - \alpha \right] dt + \sigma dZ(t)
\]

\[
= -\lambda \left[ X(0)e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \sigma \left( \int_0^t e^{-\lambda(t-s)} dZ(s) \right) - \alpha \right] dt + \sigma dZ(t)
\]

\[
= -\lambda [X(t) - \alpha] dt + \sigma dZ(t)
\]

\[
= \lambda [\alpha - X(t)] dt + \sigma dZ(t)
\]
Solution 63

**E Chapter 20, Ornstein-Uhlenbeck Process**

The trick to this question is to recognize that the process is an Ornstein-Uhlenbeck Process.

The differential form and its solution of an Ornstein-Uhlenbeck are:

\[
dX(t) = \lambda \times [\alpha - X(t)]dt + \sigma dZ(t)
\]

\[
X(t) = X(0)e^{-\lambda t} + \alpha \left(1 - e^{-\lambda t}\right) + \sigma \int_{0}^{t} e^{-\lambda(t-s)}dZ(s)
\]

The process provided in the question can be re-written so that we can identify its parameters:

\[
dX(t) = 0.048dt + 0.3dZ(t) - 0.4X(t)dt
\]

\[
= [0.048 - 0.4X(t)]dt + 0.3dZ(t)
\]

\[
= 0.4(0.12 - X(t))dt + 0.3dZ(t)
\]

From the final version above, we see that the process is an Ornstein-Uhlenbeck process with:

- \(\lambda = 0.4\)
- \(\alpha = 0.12\)
- \(\sigma = 0.3\)

Therefore, there is a solution of the form:

\[
X(t) = X(0)e^{-\lambda t} + \alpha \left(1 - e^{-\lambda t}\right) + \sigma \int_{0}^{t} e^{-\lambda(t-s)}dZ(s)
\]

\[
= X(0)e^{-0.4t} + 0.12 \left(1 - e^{-0.4t}\right) + 0.3\int_{0}^{t} e^{-0.4(t-s)}dZ(s)
\]

None of the answer choices are in this exact form, but a quick inspection shows that only choice (E) can provide \(e^{-0.4(t-s)}\) within the integral.

Depending on the time available during the exam, we might want to select choice (E) as the answer at this point and move on to the next question. For completeness though, we continue with the solution below.
Let’s see if choice (E) can be put in the form shown above:

\[
X(t) = e^{-0.4t} \left[ 0.12 \left( e^{0.4t} + 1 \right) + \int_0^t 0.3e^{0.4s} dZ(s) \right]
\]

\[
= 0.12 \left( 1 + e^{-0.4t} \right) + 0.3 \int_0^t e^{-0.4(t-s)} dZ(s)
\]

\[
= 0.12 \left( 1 + e^{-0.4t} - e^{-0.4t} + e^{-0.4t} \right) + 0.3 \int_0^t e^{-0.4(t-s)} dZ(s)
\]

\[
= 2(0.12)e^{-0.4t} + 0.12 \left( 1 - e^{-0.4t} \right) + 0.3 \int_0^t e^{-0.4(t-s)} dZ(s)
\]

\[
= 0.24e^{-0.4t} + 0.12 \left( 1 - e^{-0.4t} \right) + 0.3 \int_0^t e^{-0.4(t-s)} dZ(s)
\]

Thus we see that choice (E) results in the correct form with \( X(0) = 0.24 \).

**Solution 64**

**E Chapter 20, Correlation Coefficient**

Although we are not given the values of \( \alpha \) and \( \sigma \), the arithmetic Brownian motion is described by:

\[
X(T) - X(0) = \alpha T + \sigma Z(T)
\]

The correlation coefficient is:

\[
\rho = \frac{Cov[X(8), X(17)]}{\sqrt{Var[X(8)] \cdot Var[X(17)]}}
\]

The variances in the denominator are:

\[
Var[X(8)] = 8\sigma^2
\]

\[
Var[X(17)] = 17\sigma^2
\]

Now let’s find the covariance:

\[
Cov[X(8), X(17)]
\]

\[
= E \left[ ((X(8) - E[X(8)])(X(17) - E[X(17)])) \right]
\]

\[
= E \left[ (X(8) - X(0) - 8\alpha)(X(17) - X(0) - 17\alpha) \right]
\]

\[
= E \left[ (\sigma Z(8))(\sigma Z(17)) \right]
\]

\[
= E [Z(8)Z(17)] \sigma^2
\]

\[
= Min[8,17] \sigma^2
\]

\[
= 8\sigma^2
\]
The correlation coefficient is:

\[
\rho = \frac{\text{Cov}[X(8), X(17)]}{\sqrt{\text{Var}[X(8)] \times \text{Var}[X(17)]}} = \frac{8\sigma^2}{\sqrt{8\sigma^2 \times 17\sigma^2}} = \frac{\sqrt{8}}{\sqrt{17}} = 0.6860
\]

**Solution 65**

**C** Chapter 20, Valuing a Claim on \( S^a \)

Since the payoff consists of \( S(2) \) shares of stock at the end of 2 years, the value of this payoff at time 2 is:

\[
S(2) \times S(2) = [S(2)]^2
\]

The value of the derivative is therefore the prepaid forward price of \( S(T)^a \), where \( a = 2 \) and \( T = 2 \). The prepaid forward price is:

\[
F_{0,T}^P \left[ S(T)^a \right] = e^{-\left(\delta^*(T)\right)T} S(0)^a = e^{-\left[r-a(r-\delta)-0.5a(a-1)\sigma^2\right]T} S(0)^a
\]

\[
= e^{-0.04 + 2(0.04-0.05)+0.5(2)(2-1)0.3^2} (12)^2 = 152.9045
\]

**Solution 66**

**D** Chapter 20, Power Option

Since \( S(t) \) is a geometric Brownian motion, so is \( [S(t)]^a \):

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \quad \Rightarrow \quad \frac{d(S^a)}{S^a} = \left[ a(\alpha - \delta) + 0.5a(a-1)\sigma^2 \right]dt + a\sigma dZ(t)
\]

The coefficient of \( dZ(t) \) is the asset’s volatility. For \( [S(t)]^3 \), let’s call this volatility \( \sigma^* \):

\[
\sigma^* = a\sigma = 3 \times 0.3 = 0.9
\]

The dividend yield on \( [S(t)]^3 \) is:

\[
\delta^* = r - a(r-\delta) - 0.5a(a-1)\sigma^2 = 0.15 - 3(0.15 - 0.24) - 0.5(3)(3-1)(0.3)^2 = 0.15
\]

The initial price of \( [S(t)]^3 \) is:

\[
[S(0)]^3 = 10^3 = 1,000
\]

We can use the Black-Scholes formula to find the value of a 1-year European call option with \( [S(t)]^3 \) as the underlying asset and $900 as the strike price. This option has a payoff of \( [S(1)]^3 - 900 \) when \( [S(1)]^3 > 900 \) and zero otherwise.
The first step is to calculate $d_1$ and $d_2$:

$$d_1 = \frac{\ln(S^3 / K) + \left[ r - (\delta^*) + 0.5(\sigma^*)^2 \right] T}{(\sigma^*)\sqrt{T}}$$

$$= \frac{\ln(1,000/900) + (0.15 - 0.15 + 0.5 \times 0.9^2) \times 1}{0.9\sqrt{1}} = 0.56707$$

$$d_2 = d_1 - (\sigma^*)\sqrt{T} = 0.56707 - 0.9\sqrt{1} = -0.33293$$

We have:

$$N(d_1) = N(0.56707) = 0.71467$$

$$N(d_2) = N(-0.33293) = 0.36959$$

The value of the European power call option is:

$$C_{Eur} = S^3 e^{-(\delta^*)T} N(d_1) - Ke^{-rT} N(d_2)$$

$$= 1,000e^{-0.15 \times 1} \times 0.71467 - 900e^{-0.15 \times 1} \times 0.36959 = 328.8240$$

**Solution 67**

**E** Chapter 20, Sharpe Ratio

The Sharpe ratio of Asset 1 must be equal to that of Asset 2:

$$\frac{0.08 - 0.04}{0.25} = \frac{0.02 - 0.04}{\sigma}$$

$$\sigma = -0.125$$

Over the next instant, the returns on Stock 1 and Stock 2 are:

Return on $S_1 = 0.08S_1 dt + 0.250S_1 dZ = 0.08(75) dt + 0.250(75) dZ = 6.0 dt + 18.75 dZ$

Return on $S_2 = 0.02S_2 dt - 0.125S_2 dZ = 0.02(25) dt - 0.125(25) dZ = 0.5 dt - 3.125 dZ$

Since the investor purchases 1 share of Stock 1, the amount of Stock 2 that must be purchased to remove the random term (i.e., the $dZ$ term) is:

$$N = \frac{18.75}{3.125} = 6$$

When the return on Stock 1 is added to the return on 6 shares of Stock 2, there is no random term:

Return on $S_1 = 6.0 dt + 18.75 dZ$

Return on 6 shares of $S_2 = 3.0 dt - 18.75 dZ$

$$\frac{9.0 dt}{9.0 dt}$$
Solution 68

**E** Chapter 20, Sharpe Ratio

The Sharpe ratio of Asset 1 must be equal to that of Asset 2:

\[
\frac{0.08 - 0.04}{0.25} = \frac{0.02 - 0.04}{\sigma}
\]

\[
\sigma = -0.125
\]

Over the next instant, the returns on Stock 1 and Stock 2 are:

Return on \( S_1 \):
\[
0.08S_1 dt + 0.250S_1 dZ = 0.08(75)dt + 0.250(75)dZ = 6.0dt + 18.75dZ
\]

Return on \( S_2 \):
\[
0.02S_2 dt - 0.125S_2 dZ = 0.02(25)dt - 0.125(25)dZ = 0.5dt - 3.125dZ
\]

Since the investor purchases 1 share of Stock 1, the amount of Stock 2 that must be purchased to remove the random term (i.e., the \( dZ \) term) is:

\[
N = \frac{18.75}{3.125} = 6
\]

The cost of purchasing 1 share of Stock 1 and 6 shares of Stock 2 is:

\[
1 \times 75 + 6 \times 25 = 225
\]

Since the portfolio is a zero-investment portfolio, $225 is borrowed.

Solution 69

**B** Chapter 20, Itô’s Lemma

The expression for \( G(t) \) is:

\[
G = t^2 S
\]

The partial derivatives are:

\[
G_S = t^2
\]

\[
G_{SS} = 0
\]

\[
G_t = 2tS
\]

From Itô’s Lemma, we have:

\[
dG(t) = G_S dS(t) + \frac{1}{2} G_{SS} [dS(t)]^2 + G_t dt
\]

\[
= t^2 dS(t) + 0.5 \times 0 \times [dS(t)]^2 + 2tSdt
\]

\[
= t^2[0.5Sdt + 0.4SdZ] + 2tSdt
\]

\[
= t^2S[0.5dt + 0.4dZ] + t^2S \frac{2}{t} dt
\]
Since $G = t^2 S$, let’s divide both sides by $t^2 S$ and reorganize the expression:

$$
\frac{dG(t)}{G(t)} = \left[0.5dt + 0.4dZ\right] + \frac{2}{t} dt
$$

$$
= \left(0.5 + \frac{2}{t}\right) dt + 0.4dZ
$$

$$
= \left(0.5t + 2\right) dt + 0.4dZ
$$

**Solution 70**

**B Chapter 20, Itô’s Lemma**

The expression for $G(t)$ is:

$$
G = S^t
$$

The partial derivatives are:

$$
G_S = tS^{t-1}
$$

$$
G_{SS} = t(t - 1)S^{t-2}
$$

$$
G_t = S^t \ln S
$$

From Itô’s Lemma, we have:

$$
dG(t) = G_S dS(t) + \frac{1}{2} G_{SS} [dS(t)]^2 + G_t dt
$$

$$
= tS^{t-1} dS(t) + 0.5t(t - 1)S^{t-2} [dS(t)]^2 + S^t \ln S dt
$$

$$
= tS^{t-1} \left[-0.5S dt + SdZ\right] + 0.5t(t - 1)S^{t-2} S^2 dt + S^t \ln S dt
$$

$$
= -0.5tS^t dt + tS^t dZ + 0.5t(t - 1)S^t dt + S^t \ln S dt
$$

Since $G = S^t$, let’s divide both sides by $S^t$ and reorganize the expression:

$$
\frac{dG(t)}{G(t)} = -0.5t dt + tdZ + 0.5t(t - 1) dt + \ln S dt
$$

$$
= -0.5t dt + 0.5t^2 dt - 0.5t dt + \ln S dt + tdZ
$$

$$
= \left(0.5t^2 - t + \ln S\right) dt + tdZ
$$

We can find an expression for the natural log of $S$:

$$
G = S^t
$$

$$
\ln G = t \ln S
$$

$$
\ln S = \frac{\ln G}{t}
$$
Substituting this in for the natural log of \( S \), we have:

\[
\frac{dG(t)}{G(t)} = \left(0.5t^2 - t + \frac{\ln G(t)}{t}\right)dt + tdZ
\]

**Solution 71**

**E Chapter 20, Geometric Brownian Motion and Mutual Funds**

The instantaneous percentage increase of the mutual fund is the weighted average of the return on the stock (including its dividend yield) and the return on the risk-free asset:

\[
dW(t) = b \frac{dS(t)}{S(t)} + \delta dt + (1 - b) r dt
\]

\[
= b [ (\alpha - \delta) dt + \sigma dZ(t) + \delta dt ] + (1 - b) r dt
\]

\[
= [ b \alpha + (1 - b) r ] dt + b \sigma dZ(t)
\]

The expression above does not match Choice (A), because it does not include the \( \delta dt \) term. Therefore, we can rule out Choice (A).

The expression above does not match Choice (B) either since the final term in Choice (B) does not include the factor \( b \). Therefore, we can rule out Choice (B).

As written above, we see that \( W(t) \) is a geometric Brownian motion. Therefore, the price can be expressed as:

\[
W(t) = W(0)e^{[ b \alpha + (1 - b) r - 0.5 b^2 \sigma^2 ]t + b \sigma Z(t)}
\]

Although the expression above is close to Choice (C), it is slightly different, so we can rule out Choice (C). Therefore, we turn to Choices (D) and (E).

Since Choices (D) and (E) contain \([ S(t) / S(0) ]^b \), let’s find the value of this expression:

\[
S(t) = S(0)e^{(\alpha - \delta - 0.5 \sigma^2) t + \sigma Z(t)}
\]

\[
\frac{S(t)}{S(0)}^b = e^{b(\alpha - \delta - 0.5 \sigma^2) t + b \sigma Z(t)}
\]

\[
\frac{S(t)}{S(0)}^b = e^{(b \alpha - b \delta - 0.5 b \sigma^2) t + b \sigma Z(t)}
\]
We can now describe the process of the mutual fund with:

\[
W(t) = W(0)e^{b\alpha + (1-b)r-0.5b^2\sigma^2} + b\sigma Z(t)
\]

\[
= W(0)e^{b\alpha - b\delta - 0.5b\sigma^2} + b\sigma Z(t) e^{b\delta + (1-b)r-0.5b^2\sigma^2 + 0.5b^2\sigma^2} t
\]

\[
= W(0)\left[\frac{S(t)}{S(0)}\right] e^{b\delta + (1-b)r-0.5b^2\sigma^2 + 0.5b^2\sigma^2} t
\]

\[
= W(0)\left[\frac{S(t)}{S(0)}\right] e^{[b\delta + (1-b)(r+0.5b^2\sigma^2)]t}
\]

This matches Choice (E).

**Solution 72**

B Chapter 20, Geometric Brownian Motion and Mutual Funds

The instantaneous percentage increase of the mutual fund is the weighted average of the return on the stock (including its dividend yield) and the return on the risk-free asset:

\[
\frac{dW(t)}{W(t)} = b\left[\frac{dS(t)}{S(t)} + \delta dt\right] + (1-b)rdt
\]

\[
= 1.5\left[(0.10-\delta)dt + 0.20dZ(t) + \delta dt\right] - 0.5(0.04)dt
\]

\[
= 0.15dt + 0.30dZ(t) - 0.02dt
\]

\[
= 0.13dt + 0.3dZ(t)
\]

The expression above matches Choice (B).

**Solution 73**

D Chapter 20, Geometric Brownian Motion and Mutual Funds

The stock’s price follows geometric Brownian motion, so:

\[
S(t+h) = S(t)e^{(\alpha - \delta - 0.5\sigma^2)h + \sigma[Z(t+h) - Z(t)]}
\]

\[
S(4) = S(0)e^{[0.07 - 0.5(0.20)^2]4 + 0.20Z(4)}
\]

\[
100 = 110e^{0.20 + 0.20Z(4)}
\]

\[
Z(4) = -1.4766
\]
We can use the value of the stock in the mutual fund at time 4 to determine the value of \( b \):

\[
100b\% \times 80 = 96 \\
100b\% = 1.20 \\
b = 1.20
\]

The instantaneous percentage increase of the mutual fund is the weighted average of the return on the stock (including its dividend yield) and the return on the risk-free asset:

\[
\frac{dW(t)}{W(t)} = b \left[ \frac{dS(t)}{S(t)} + \delta dt \right] + (1 - b)rdt
\]

\[
= 1.2 \left[ 0.07 dt + 0.20dZ(t) + 0.03dt \right] - 0.2(0.04)dt \\
= 1.2 \left[ 0.10 dt + 0.20dZ(t) \right] - 0.2(0.04)dt \\
= 0.12dt + 0.24dZ(t) - 0.008dt \\
= 0.112dt + 0.24dZ(t)
\]

As written above, we see that \( W(t) \) is a geometric Brownian motion. Therefore, the price can be expressed as:

\[
W(t) = W(0)e^{0.112-(0.5)(0.24)^2}t + 0.24Z(t) \\
W(t) = W(0)e^{0.0832t + 0.24Z(t)} \\
W(4) = W(0)e^{0.0832(4) + 0.24Z(4)} \\
80 = W(0)e^{0.0832(4) + 0.24(-1.4766)} \\
W(0) = 81.7445
\]

**Solution 74**

**A**  Chapter 20, Multiplication Rules

As \( n \) goes to infinity, we can replace the summation with an integral:

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \left[ Z[jh] - Z[(j-1)h] \right]^4 = \int_0^T [dZ(t)]^4
\]

We can now use the multiplication rules to evaluate the integral:

\[
\int_0^T [dZ(t)]^4 = \int_0^T [dZ(t)]^2 \times [dZ(t)]^2 = \int_0^T dt \times dt = \int_0^T 0 = 0
\]

The expected value and variance of zero are both zero:

\( N(0,0) \)
Solution 75

E Chapter 20, Multiplication Rules

As \( n \) goes to infinity, we can replace the summation with an integral:

\[
X = \lim_{n \to \infty} \sum_{j=1}^{n} \left[ Z[jh] - Z[(j-1)h] + \{Z[jh] - Z[(j-1)h]\}^2 + \{Z[jh] - Z[(j-1)h]\}^3 \right]
\]

\[
= \int_{0}^{T} \left[ dZ(t) + [dZ(t)]^2 + [dZ(t)]^3 \right] = \int_{0}^{T} [dZ(t)] + \int_{0}^{T} [dZ(t)]^2 + \int_{0}^{T} [dZ(t)]^3
\]

We can use the multiplication rules to evaluate the second and third integrals:

\[
X = \int_{0}^{T} dZ(t) + \int_{0}^{T} dt + \int_{0}^{T} [dt \times dZ(t)] = Z(T) + T + 0 = Z(T) + T
\]

The expected value of \( X \) is:

\[
E[X] = E\left[Z(T) + T\right] = 0 + T = T
\]

The variance of \( X \) is:

\[
Var[X] = Var\left[Z(T) + T\right] = T + 0 = T
\]

Since the expected value and the variance are both equal to \( T \), the distribution is:

\( N(T, T) \)

Solution 76

E Chapter 20, Product Rule for Stochastic Differential Equations

To simplify the notation, we use \( r \) for \( r(t) \) and \( r_0 \) for \( r(0) \).

The first two terms of \( r(t) \) do not contain random variables, so their differentials are easy to find. The third term will be more difficult:

\[
r = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma \int_{0}^{t} e^{a(s-t)} \sqrt{r(s)} dZ(s)
\]

\[
dr = -ar_0 e^{-at} dt + ab e^{-at} dt + \sigma \left[ \int_{0}^{t} e^{a(s-t)} \sqrt{r(s)} dZ(s) \right]
\]

The third term has a function of \( t \) in the integral. We can pull the \( t \)-dependent portion out of the integral, so that we are finding the differential of a product. We then use the following version of the product rule to find the differential:

\[
d[U(t)V(t)] = dU(t)V(t) + U(t)dV(t)
\]
The differential of the third term is:

\[
d \left[ \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)} dZ(s) \right] = d \left[ \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)} dZ(s) \right] \\
= -a \sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)} dZ(s) + \sigma e^{-at} e^a \sqrt{r(t)} dZ(t) \\
= -a \sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)} dZ(s) + \sigma \sqrt{r(t)} dZ(t)
\]

Putting all three terms together, we have:

\[
dr = -ar_0 e^{-at} dt + ab e^{-at} dt - a \sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)} dZ(s) + \sigma \sqrt{r} dZ(t)
\]

The portion in the brackets above is equal to the expression for \( r(t) \) provided in the question:

\[
dr = -ardt + abdt + \sigma \sqrt{rdZ(t)}
\]

Putting the functional relationship back in, we see that this differential equation matches Choice E:

\[
dr = a[b - r(t)] dt + \sigma \sqrt{r(t)} dZ(t)
\]

The equation above is the Cox-Ingersoll-Ross Model for the short rate, which appears in the Chapter 24 Review Note.

Solution 77

D Chapter 20, Valuing a Claim on \( S^a \)

The derivative is of the form \( S^a \), and the derivative has a dividend yield of zero:

\[ \delta^* = 0 \]
We can use the formula for the dividend yield of the claim to solve for $a$:

$$\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2$$

$$0 = 0.07 - a(0.07 - 0.00) - 0.5a(a - 1)\sigma^2$$

$$0 = 0.07 - 0.07a - 0.5\sigma^2a^2 + 0.5\sigma^2a$$

$$0 = -0.5\sigma^2a^2 + (0.5\sigma^2 - 0.07)a + 0.07$$

We make use of the quadratic formula:

$$a = \frac{0.07 - 0.5\sigma^2 \pm \sqrt{0.25\sigma^4 - 0.07\sigma^2 + 0.0049 - 4(-0.5\sigma^2)(0.07)}}{-\sigma^2}$$

$$a = \frac{0.5\sigma^2 - 0.07 \pm \sqrt{0.25\sigma^4 + 0.07\sigma^2 + 0.0049}}{\sigma^2}$$

$$a = \frac{0.5\sigma^2 - 0.07 \pm (0.5\sigma^2 + 0.07)}{\sigma^2}$$

$$a = -\frac{0.14}{\sigma^2} \quad \text{or} \quad a = 1$$

The question specifies that $a$ is negative, so $a$ cannot be equal to 1:

$$a = -\frac{0.14}{\sigma^2}$$

$$-k = \frac{-0.14}{\sigma^2}$$

$$k = 0.14$$

Solution 78

D Chapter 20, Valuing a Claim on $S^a$

The derivative is of the form $S^a$, and the derivative has a dividend yield of equal the dividend yield of the stock:

$$\delta^* = \delta = 0.03$$

We can use the formula for the dividend yield of the claim to solve for $a$:

$$\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2$$

$$0.03 = 0.09 - a(0.09 - 0.03) - 0.5a(a - 1)\sigma^2$$

$$0 = 0.06 - 0.06a - 0.5\sigma^2a^2 + 0.5\sigma^2a$$

$$0 = -0.5\sigma^2a^2 + (0.5\sigma^2 - 0.06)a + 0.06$$
We make use of the quadratic formula:

\[ a \frac{0.5\sigma^2 + a^2}{0.5\sigma^2} + (0.5\sigma^2 - 0.06) = 0 \]

\[ a = \frac{-0.5\sigma^2 + 0.06 \pm \sqrt{[0.25\sigma^4 - 0.06\sigma^2 + 0.0036] - 4(-0.5\sigma^2)(0.06)}}{-\sigma^2} \]

\[ a = \frac{-0.5\sigma^2 + 0.06 \pm \sqrt{0.25\sigma^4 - 0.06\sigma^2 + 0.0036 + 0.12\sigma^2}}{-\sigma^2} \]

\[ a = \frac{-0.5\sigma^2 + 0.06 \pm \sqrt{0.25\sigma^4 + 0.06\sigma^2 + 0.0036}}{-\sigma^2} \]

\[ a = \frac{-0.5\sigma^2 + 0.06 \pm (0.5\sigma^2 + 0.06)}{-\sigma^2} \]

\[ a = -0.12 \frac{0.12}{\sigma^2} \quad \text{or} \quad a = 1 \]

The question specifies that \( a \) is negative, so \( a \) cannot be equal to 1:

\[ a = -0.12 \frac{0.12}{\sigma^2} \]

\[ -k = -0.12 \]

\[ k = 0.12 \]

**Solution 79**

**E**

Chapter 20, Geometric Brownian Motion Equivalencies

We make use of the following equivalency:

\[ \frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \quad \iff \quad \ln \left[ \frac{S(t + h)}{S(t)} \right] \sim N \left( (\alpha - \delta - 0.5\sigma^2)h, \sigma^2 h \right) \]

\[ \frac{dS(t)}{S(t)} = 0.04 dt + 0.3dZ(t) \quad \iff \quad \ln \left[ \frac{S(t + h)}{S(t)} \right] \sim N \left( -0.005h, 0.09h \right) \]

To avoid a cluttered appearance, we use \( S_t \) to represent \( S(t) \) in the solution below.
We can rewrite the expression for $G$, so that it is the product of independent random variables:

$$G = [S_1 \times S_2 \times S_3]^{1/3} = \left( \frac{S_1}{S_0} \right) \left( \frac{S_2}{S_1} \right) \left( \frac{S_3}{S_2} \right) = \left[ \frac{S_0}{S_0} \right]^{1/3} = S_0 \times \left( \frac{S_1}{S_0} \right) \left( \frac{S_2}{S_1} \right) \left( \frac{S_3}{S_2} \right)$$

Taking the natural log, we have:

$$\ln G = \ln(S_0) + \ln \left( \frac{S_1}{S_0} \right) + \frac{2}{3} \ln \left( \frac{S_2}{S_1} \right) + \frac{1}{3} \ln \left( \frac{S_3}{S_2} \right)$$

The variance is:

$$\text{Var}[\ln G] = 0 + 0.09 + \left( \frac{2}{3} \right)^2 (0.09) + \left( \frac{1}{3} \right)^2 (0.09) = 0.14$$

Solution 80

We make use of the following equivalency:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta) dt + \sigma dZ(t) \quad \Rightarrow \quad \ln \left( \frac{S(t+h)}{S(t)} \right) \sim N (\alpha - \delta - 0.5\sigma^2) h, \quad \sigma^2 h$$

$$\frac{dS(t)}{S(t)} = 0.04 dt + 0.3 dZ(t) \quad \Rightarrow \quad \ln \left( \frac{S(t+h)}{S(t)} \right) \sim N (-0.005 h, \quad 0.09 h)$$

To avoid a cluttered appearance, we use $S_t$ to represent $S(t)$ in the solution below.

We can rewrite the expression for $G$, so that it is the product of independent random variables:

$$G = [S_1 \times S_2 \times S_3]^{1/3} = \left( \frac{S_1}{S_0} \right) \left( \frac{S_2}{S_1} \right) \left( \frac{S_3}{S_2} \right) = \left[ \frac{S_0}{S_0} \right]^{1/3} = S_0 \times \left( \frac{S_1}{S_0} \right) \left( \frac{S_2}{S_1} \right) \left( \frac{S_3}{S_2} \right)$$

Taking the natural log, we have:

$$\ln G = \ln(S_0) + \ln \left( \frac{S_1}{S_0} \right) + \frac{2}{3} \ln \left( \frac{S_2}{S_1} \right) + \frac{1}{3} \ln \left( \frac{S_3}{S_2} \right)$$
The expected value is:

\[ E[\ln G] = \ln(2) + (-0.005) + \frac{2}{3} \times (-0.005) + \frac{1}{3} \times (-0.005) = 0.6831 \]

**Solution 81**

Chapter 20, Geometric Brownian Motion Equivalencies

We make use of the following equivalency:

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta) dt + \sigma dZ(t) \quad \Leftrightarrow \quad \ln \left( \frac{S(t + h)}{S(t)} \right) \sim N \left( (\alpha - \delta - 0.5\sigma^2)h, \sigma^2 h \right)
\]

\[
\frac{dS(t)}{S(t)} = 0.04 dt + 0.3 dZ(t) \quad \Leftrightarrow \quad \ln \left( \frac{S(t + h)}{S(t)} \right) \sim N \left( -0.005h, 0.09h \right)
\]

To avoid a cluttered appearance, we use \( S_t \) to represent \( S(t) \) in the solution below.

We can rewrite the expression for \( G \), so that it is the product of independent random variables:

\[
G = [S_1 \times S_2 \times S_3]^{1/3} = \left( S_0 \frac{S_1}{S_0} \right) \left( S_0 \frac{S_1}{S_1} \right) \left( S_0 \frac{S_1}{S_0} \frac{S_2}{S_1} \right) \left( S_0 \frac{S_1}{S_0} \frac{S_2}{S_1} \frac{S_3}{S_2} \right)^{1/3}
\]

\[
= \left( S_0 \frac{S_1}{S_0} \right)^3 \times \left( S_0 \frac{S_2}{S_1} \right)^2 \times \left( S_0 \frac{S_3}{S_2} \right)^{1/3} = S_0 \times \left( \frac{S_1}{S_0} \right)^2 \times \left( \frac{S_2}{S_1} \right)^2 \times \left( \frac{S_3}{S_2} \right)^{1/3}
\]

Taking the natural log, we have:

\[
\ln G = \ln(S_0) + \ln \left( \frac{S_1}{S_0} \right) + \frac{2}{3} \ln \left( \frac{S_2}{S_1} \right) + \frac{1}{3} \ln \left( \frac{S_3}{S_2} \right)
\]

The expected value and variance are:

\[
E[\ln G] = \ln(2) + (-0.005) + \frac{2}{3} \times (-0.005) + \frac{1}{3} \times (-0.005) = 0.6831
\]

\[
Var[\ln G] = 0 + 0.09 + \left( \frac{2}{3} \right)^2 (0.09) + \left( \frac{1}{3} \right)^2 (0.09) = 0.14
\]

Therefore, the distribution of \( \ln[G] \) is normal with the following parameters:

\( \ln G \sim N(0.6831, 0.14) \)

The expected value of \( G \) is:

\[
E[G] = e^{0.6831 \times 0.5(0.14)} = 2.1237
\]
Solution 82

E  Chapter 20, Geometric Brownian Motion Equivalencies

We make use of the following equivalency, substituting $x$ and $y$ for $S$ as needed:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \iff d[\ln S(t)] = \left(\alpha - \delta - 0.5\sigma^2\right)dt + \sigma dZ(t)$$

Since $y(t)$ is the exchange rate denominated in euros per dollar, we have:

$$y(t) = \frac{1}{x(t)} = [x(t)]^{-1}$$

$$\ln y(t) = -\ln x(t)$$

$$d[\ln y(t)] = -d[\ln x(t)] = -\left[(0.03 - 0.5 \times 0.1^2)dt + 0.10dZ(t)\right]$$

$$d[\ln y(t)] = -0.025dt - 0.10dZ(t)$$

$$\frac{dy(t)}{y(t)} = \left[-0.025 + 0.5 \times (-0.10)^2\right]dt - 0.10dZ(t) = -0.02dt - 0.10dZ(t)$$

Solution 83

B  Chapter 20, Geometric Brownian Motion Equivalencies

We make use of the following equivalency, substituting $x$ and $y$ for $S$ as needed:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \iff d[\ln S(t)] = \left(\alpha - \delta - 0.5\sigma^2\right)dt + \sigma dZ(t)$$

Since $x(t)$ is the exchange rate denominated in dollars per euro, we have:

$$x(t) = \frac{1}{y(t)} = [y(t)]^{-1}$$

$$\ln x(t) = -\ln y(t)$$

$$d[\ln x(t)] = -d[\ln y(t)] = -\left[(r_e - r + \sigma^2 - 0.5 \times (-\sigma)^2)dt - \sigma dZ(t)\right]$$

$$d[\ln x(t)] = \left(r - r_e - 0.5\sigma^2\right)dt + \sigma dZ(t)$$

$$\frac{dx(t)}{x(t)} = \left[r - r_e - 0.5\sigma^2 + 0.5\sigma^2\right]dt + \sigma dZ(t) = (r - r_e)dt + \sigma dZ(t)$$

Solution 84

D  Chapter 20, Sharpe Ratio

The Sharpe ratio of Stock 1 must be equal to that of Stock 2:

$$\frac{0.10 - 0.08}{0.05} = \frac{0.06 - 0.08}{k}$$

$$k = -0.05$$
Over the next instant, the returns on Stock 1 and Stock 2 are:

Return on \( S_1 \) = 0.10\( S_1 \)\( dt \) + 0.05\( S_1 \)\( dZ \) = 0.10(80)\( dt \) + 0.05(80)\( dZ \) = 8.0\( dt \) + 4.0\( dZ \)

Return on \( S_2 \) = 0.06\( S_2 \)\( dt \) - 0.05\( S_2 \)\( dZ \) = 0.06(40)\( dt \) - 0.05(40)\( dZ \) = 2.4\( dt \) - 2.0\( dZ \)

Since the investor purchases 1 share of Stock 1, the amount of Stock 2 that must be purchased to remove the random term (i.e., the \( dZ \) term) is:

\[
\frac{4}{2} = 2
\]

When the return on Stock 1 is added to the return on 2 shares of Stock 2, there is no random term:

Return on \( S_1 \) = 8.0\( dt \) + 4.0\( dZ \)

Return on 2 shares of \( S_2 \) = \( \frac{2 \times (2.4\ dt - 2.0\ dZ)}{12.8\ dt} \)

\[\text{Solution 85} \]

\( B \) Chapter 20, Sharpe Ratio

We provide 2 solutions to this question. The first solution does not require us to determine \( k \).

Stock 1 has an expected return of 12% and Stock 2 has an expected return of 8.4%. Since the portfolio earns the risk-free rate of 10%, we have:

\[
0.12x\% + 0.084(1-x\%) = 0.10 \\
0.12x\% + 0.084 - 0.084x\% = 0.10 \\
0.036x\% = 0.016 \\
x\% = 0.4444
\]

Alternative Solution

The Sharpe ratio of Stock 1 must be equal to that of Stock 2:

\[
\frac{0.12 - 0.10}{0.05} = \frac{0.084 - 0.10}{k} \\
k = -0.04
\]

The resulting differential equations are:

\[dS_1(t) = 0.12S_1(t)dt + 0.05S_1(t)dZ(t)\]

\[dS_2(t) = 0.084S_2(t)dt - 0.04S_2(t)dZ(t)\]

The quantities of Stock 1 and Stock 2 that are purchased are:

\[Q_1 = \frac{1,000 \times x\%}{S_1(t)} \quad Q_2 = \frac{1,000 \times (1-x\%)}{S_2(t)}\]
The partial differential equation describing the return on the portfolio is:

\[ Q_1dS_1(t) + Q_2dS_2(t) = Q_1 \left[ 0.12S_1(t)dt + 0.05S_1(t)dZ(t) \right] + Q_1 \left[ 0.084S_2(t)dt - 0.04S_2(t)dZ(t) \right] \]

\[ = \left[ 0.12S_1(t)Q_1 + 0.084S_2(t)Q_2 \right]dt + \left[ 0.05S_1(t)Q_1 - 0.04S_2(t)Q_2 \right]dZ(t) \]

Since the portfolio earns the risk-free rate, the coefficient to the random term \( dZ(t) \) must be zero:

\[ 0.05S_1(t)Q_1 - 0.04S_2(t)Q_2 = 0 \]
\[ 0.05S_1(t)Q_1 = 0.04S_2(t)Q_2 \]
\[ 0.05S_1(t) \frac{1000 \times x\%}{S_1(t)} = 0.04S_2(t) \frac{1000 \times (1 - x\%)}{S_2(t)} \]
\[ 0.05x\% = 0.04(1 - x\%) \]
\[ 0.09x\% = 0.04 \]
\[ x\% = 0.4444 \]

**Solution 86**

**E Chapter 20, Geometric Brownian Motion**

From the process for the stock price, we observe that:

\[ \sigma = 0.30 \]

The volatility is the annualized standard deviation of the natural log of the prepaid forward price:

\[ \sqrt{\frac{\text{Var}\left[ \ln \left( F_{t,T}^P(S) \right) \right]}{t}} = \sigma \]
\[ \sqrt{\frac{\text{Var}\left[ \ln \left( F_{t,3}^P(S) \right) \right]}{t}} = 0.30 \]
\[ \text{Var}\left[ \ln \left( F_{t,3}^P(S) \right) \right] = 0.09t \]

**Alternative Solution**

The prepaid forward price is:

\[ F_{t,3}^P(S) = e^{-\delta(3-t)}S(t) = e^{-0.05(3-t)}S(0)e^{(0.09-0.5 \times 0.30^2)t + 0.30[Z(t) - Z(0)]} \]

The natural log of the prepaid forward price is:

\[ \ln \left[ F_{t,3}^P(S) \right] = -0.05(3-t) + \ln S(0) + (0.09 - 0.5 \times 0.30^2)t + 0.30[Z(t) - Z(0)] \]
The first three terms above are not random, so their variances are zero:

$$Var\left\{\ln F_{t,3}(S)\right\} = 0 + 0 + 0 + Var\left\{0.30[Z(t) - Z(0)]\right\} = 0.30^2 t = 0.09t$$

**Solution 87**

C Chapter 20, Geometric Brownian Motion

From the process for the stock price, we observe that:

$$\sigma = 0.30$$

The volatility is the annualized standard deviation of the natural log of the forward price:

$$\sqrt{\frac{Var[\ln F_{t,T}]}{t}} = \sigma$$

The question uses $t$ instead of $T$ to represent the maturity of the contract, but we can make the appropriate substitutions:

$$\sqrt{\frac{Var[\ln F_{3,t}]}{3}} = 0.30$$

$$Var[\ln F_{3,t}] = (0.30)^2 \times 3 = 0.27$$

**Alternative Solution**

The forward price is:

$$F_{3,t}(S) = e^{(r-\delta)(t-3)} S(3) = e^{-0.01(t-3)} S(0)e^{(0.09-0.5\times0.30^2)3+0.30[Z(3)-Z(0)]}$$

The natural log of the prepaid forward price is:

$$\ln F_{3,t}(S) = -0.01(t-3) + \ln S(0) + (0.09 - 0.5 \times 0.30^2)3 + 0.30[Z(3) - Z(0)]$$

The first three terms above are not random, so their variances are zero:

$$Var\left\{\ln F_{3,t}(S)\right\} = 0 + 0 + 0 + Var\left\{0.30[Z(3) - Z(0)]\right\} = 0.30^2 \times 3 = 0.27$$

**Solution 88**

B Chapter 20, Geometric Brownian Motion Equivalencies

The forward price at time $t$ is:

$$F(t) = F_{t,T} = S(t)e^{(r-\delta)(T-t)} = S(t)e^{(0.06-0.03)(T-t)} = S(t)e^{0.03(T-t)}$$
We can determine the natural log of the forward price and the differential of the natural log. Then we convert the differential of the natural log of the forward price into the differential of the forward price:

\[
F(t) = S(t)e^{0.03(T-t)} \\
\ln F(t) = \ln S(t) + 0.03T - 0.03t \\
d[\ln F(t)] = d[\ln S(t)] + 0 - 0.03dt = (0.16 - 0.5 \times 0.2^2)dt + 0.2dZ(t) - 0.03dt \\
\implies dF(t) = 0.13F(t)dt + 0.2F(t)dZ(t) = F(t)[0.13dt + 0.2dZ(t)]
\]

**Alternative Solution**

The partial derivatives are:

\[
\begin{align*}
F_S &= e^{0.03(T-t)} \\
F_{SS} &= 0 \\
F_t &= \frac{\partial}{\partial t} [S(t)e^{0.03T}e^{-0.03t}] = S(t)e^{0.03T}(-0.03)e^{-0.03t} = S(t)(-0.03)e^{0.03(T-t)}
\end{align*}
\]

From Itô’s Lemma, we have:

\[
dF(t) = F_S dS(t) + \frac{1}{2} F_{SS} [dS(t)]^2 + F_t dt
\]

\[
= e^{0.03(T-t)} dS(t) + \frac{1}{2} (0)[dS(t)]^2 + S(t)(-0.03)e^{0.03(T-t)}dt
\]

\[
= e^{0.03(T-t)} S(t) + F(t)(-0.03)dt \\
= e^{0.03(T-t)} S(t)[0.16dt + 0.2dZ(t)] + F(t)(-0.03)dt \\
= F(t)[0.13dt + 0.2dZ(t)] + F(t)(-0.03)dt \\
= F(t)[0.13dt + 0.2dZ(t)]
\]

**Solution 89**

**E** Chapter 20, Geometric Brownian Motion Equivalencies

The prepaid forward price at time \( t \) is:

\[
G(t) = F_{t,T}^P (S) = S(t)e^{-\delta(T-t)} = S(t)e^{-0.03(T-t)}
\]
We can determine the natural log of the prepaid forward price and the differential of the natural log. Then we convert the differential of the natural log of the prepaid forward price into the differential of the prepaid forward price:

\[
G(t) = S(t)e^{-0.03(T-t)}
\]

\[
\ln G(t) = \ln S(t) - 0.03T + 0.03t
\]

\[
d[\ln G(t)] = d[\ln S(t)] - 0 + 0.03dt = (0.16 - 0.5 \times 0.2^2)dt + 0.2dZ(t) + 0.03dt
\]

\[
dG(t) = 0.19G(t)dt + 0.2G(t)dZ(t) = G(t)[0.19dt + 0.2dZ(t)]
\]

**Solution 90**

D Chapter 20, Multiplication Rules

Statement (i) is true, because when the Black-Scholes framework applies, the natural log of stock prices follows arithmetic Brownian motion.

Statement (ii) is false, because when the Black-Scholes framework applies, stock prices are lognormally distributed as described below:

\[
\ln S(t + h) - \ln S(t) \sim N((\alpha - \delta - 0.5\sigma^2)h, \sigma^2h)
\]

Since the stock prices are lognormally distributed, we have:

\[
E[X(t + h) - X(t)] = E[\ln S(t + h) - \ln S(t)] = (\alpha - \delta - 0.5\sigma^2)h
\]

Now consider Statement (iii). Small increments in a Brownian motion are represented by the differential as the time increment becomes increasingly small. For example, in the textbook’s discussion of quadratic variation, \(Z\left(\frac{jT}{n}\right) - Z\left(\frac{(j-1)T}{n}\right)\) in the summation becomes \(dZ(t)\) in the integral form:

\[
\text{Quadratic Variation} = \lim_{n \to \infty} \sum_{j=1}^{n} \left[Z\left(\frac{jT}{n}\right) - Z\left(\frac{(j-1)T}{n}\right)\right]^2 = \int_0^T [dZ(t)]^2 = T
\]
Likewise, \( X\left( \frac{jT}{n} \right) - X\left( \frac{(j-1)T}{n} \right) \) in the summation can be replaced by \( dX(t) \) in the integral form. Below we use the multiplication rules to simplify the solution:

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \left( X\left( \frac{jT}{n} \right) - X\left( \frac{(j-1)T}{n} \right) \right)^2 = \int_{0}^{T} [dX(t)]^2 = \int_{0}^{T} \left[ (\alpha - \delta - 0.5\sigma^2)dt + \sigma dZ(t) \right]^2 \\
= \int_{0}^{T} \left[ (\alpha - \delta - 0.5\sigma^2)^2 (dt)^2 + 2(\alpha - \delta - 0.5\sigma^2)dt\sigma dZ(t) + \sigma^2 [dZ(t)]^2 \right] \\
= \int_{0}^{T} \left[ (\alpha - \delta - 0.5\sigma^2)^2 \times 0 + 2(\alpha - \delta - 0.5\sigma^2)\sigma \times 0 + \sigma^2 dt \right] \\
= \int_{0}^{T} \left[ 0 + 0 + \sigma^2 dt \right] = \sigma^2 \int_{0}^{T} dt = \sigma^2 T
\]

Therefore, Statement (iii) is true.

**Solution 91**

**D** Chapter 20, Drift and Itô’s Lemma

The first equation and its partial derivatives are:

\[
U(t) = 3Z(t) - 5 \\
U_Z = 3 \\
U_{ZZ} = 0 \\
U_t = 0
\]

This results in:

\[
dU(Z,t) = U_ZdZ + \frac{1}{2} U_{ZZ} (dZ)^2 + U_t dt = 3dZ + 0 + 0 = 3dZ
\]

Since there is no \( dt \) term, the drift is zero for \( dU \).

The second equation and its the partial derivatives are:

\[
V(t) = 2[Z(t)]^3 - 3t \\
V_Z = 6Z^2 \\
V_{ZZ} = 12Z \\
V_t = -3
\]

This results in:

\[
dV(Z,t) = V_ZdZ + \frac{1}{2} V_{ZZ} (dZ)^2 + V_t dt = 6[Z(t)]^2 dZ + \frac{1}{2} \times 12Z(t)(dZ)^2 - 3dt \\
= 6[Z(t)]^2 dZ + 6Z(t)dt - 3dt = 6[Z(t)]^2 dZ + [6Z(t) - 3]dt
\]

Since the \( dt \) term is nonzero, the drift is nonzero for \( dV \).
For the third equation, we are given:

\[ W(t) = t^3 Z(t) - 3 \int_0^t s^2 Z(s) ds \]

Let’s deal with the second term first. We have:

\[
\frac{d}{dt} \left[ 3 \int_0^t s^2 Z(s) ds \right] = 3 \left[ \int_0^t s^2 Z(s) ds \right] = 3t^2 Z(t) dt
\]

We can now write the differential equation as:

\[ dW(t) = d \left[ t^3 Z(t) \right] - 3t^2 Z(t) dt \]

For the first term, we must use Itô’s Lemma. Let’s define:

\[ X(t) = t^3 Z(t) \]

The partial derivatives are:

\[ X_Z = t^3, \quad X_{ZZ} = 0, \quad X_t = 3t^2 Z(t) \]

This results in:

\[
dX(t) = X_Z dZ + \frac{1}{2} X_{ZZ} (dZ)^2 + X_t dt = t^3 dZ + \frac{1}{2} \times 0 \times (dZ)^2 + 3t^2 Z dt
\]

\[ = t^3 dZ(t) + 3t^2 Z(t) dt \]

Therefore:

\[
dW(t) = d \left[ t^3 Z(t) \right] - 3t^2 Z(t) dt = dX(t) - 3t^2 Z(t) dt = t^3 dZ(t) + 3t^2 Z(t) dt - 3t^2 Z(t) dt
\]

\[ = t^3 dZ(t) \]

Since there is no \( dt \) term, the drift is zero for \( dW \).

**Solution 92**

**B Chapter 20, Itô’s Lemma & Ornstein-Uhlenbeck Process**

Notice that the stochastic process for \( R(t) \) is an Ornstein-Uhlenbeck process with \( \lambda = 1, \alpha = 0.05, \) and \( \sigma = 0.10 \):

\[ X(t) = X(0)e^{-\lambda t} + \alpha \left( 1 - e^{-\lambda t} \right) + \sigma \int_0^t e^{-\lambda(t-s)} dZ(s) \quad \Leftrightarrow \quad dX(t) = \lambda [X - X(t)] dt + \sigma dZ(t) \]

\[ R(t) = R(0)e^{-t} + 0.05 \left( 1 - e^{-t} \right) + 0.10 \int_0^t e^{s-t} dZ(s) \quad \Leftrightarrow \quad dR(t) = [0.05 - R(t)] dt + 0.10 dZ(t) \]
The equation above for $dR(t)$ describes the Vasicek Model for the short rate, which appears in the Chapter 24 Review Note.

Since $X = R^2$, the partial derivatives of $X(t)$ are:

\[
X_R = 2R \\
X_RR = 2 \\
X_t = 0
\]

We can use Itô’s Lemma to find an expression for the differential of $X(t)$:

\[
dX(t) = X_RdR(t) + \frac{1}{2} X_{RR}[dR(t)]^2 + X_t dt
\]

Making use of the multiplication rules, we see that:

\[
[dR(t)]^2 = \left( [0.05 - R(t)]dt + 0.10dZ(t) \right)^2 = 0.01dt
\]

We can now find the differential of $X(t)$. To simplify the notation, we use $R$ for $R(t)$ and $X$ for $X(t)$ below:

\[
dX = X_RdR + \frac{1}{2} X_{RR}(dR)^2 + X_t dt
\]

\[
= 2RdR + \frac{1}{2} \times 2(dR)^2 + 0dt
\]

\[
= 2RdR + (dR)^2
\]

\[
= 2R([0.05 - R]dt + 0.10dZ) + 0.01dt
\]

\[
= (0.10R - 2R^2)dt + 0.2RdZ + 0.01dt
\]

\[
= (0.10R - 2R^2 + 0.01)dt + 0.2RdZ
\]

We are given that $X$ is equal to $R^2$:

\[
X = R^2 \quad \Rightarrow \quad \sqrt{X} = R
\]

Substituting for $R$, we have:

\[
dX = (0.10R - 2R^2 + 0.01)dt + 0.2RdZ
\]

\[
= (0.10\sqrt{X} - 2X + 0.01)dt + 0.2\sqrt{X}dZ
\]

**Solution 93**

E Chapter 20, Itô’s Lemma

We have:

\[
Y(t) = [X(t)]^{-1}
\]
The partial derivatives are:
\[ Y_X = -X^{-2} \quad Y_{XX} = 2X^{-3} \quad Y_t = 0 \]

From Itô’s Lemma and the multiplication rules:
\[
dY(t) = Y_X dX + \frac{1}{2} Y_{XX} (dX)^2 + Y_t dt
\]
\[
= -X^{-2} \left[ 4(5 - X)dt + 6dZ \right] + \frac{1}{2} \times 2X^{-3} \times 36dt + 0
\]
\[
= -X^{-2} \left[ 20dt - 4Xd + 6dZ \right] + 36X^{-3} dt
\]
\[
= \left[ 4X^{-1} - 20X^{-2} + 36X^{-3} \right] dt - 6X^{-2} dZ
\]
\[
= \left[ 4Y - 20Y^2 + 36Y^3 \right] dt - 6Y^2 dZ
\]

From the expression above, we can determine \( \alpha(Y) \) and \( \beta(Y) \):
\[
\alpha(Y) = 4Y - 20Y^2 + 36Y^3 \quad \Rightarrow \quad \alpha(0.5) = 4(0.5) - 20(0.5)^2 + 36(0.5)^3 = 1.5
\]
\[
\beta(Y) = -6Y^2 \quad \Rightarrow \quad \beta(1) = -6(1)^2 = -6
\]

The solution is:
\[
\alpha(0.5) - \beta(1) = 1.5 - (-6) = 7.5
\]

**Solution 94**

**E** Chapter 20, Synthetic Risk-free Asset from 2 Risky Assets

Let \( X \) be the percentage invested in Asset 1. Over the next instant, the percentage return on the investment is:
\[
X(0.08dt + 0.2dZ) + (1 - X)(0.0925dt - 0.3dZ)
\]

We can determine \( X \) so that the \( dZ \) terms cancel:
\[
0.2XdZ - 0.3(1 - X)dZ = 0
\]
\[
0.2X - 0.3(1 - X) = 0
\]
\[
0.2X - 0.3 + 0.3X = 0
\]
\[
0.5X = 0.3
\]
\[
X = 0.60
\]

The portion of the $1,000 to be invested in Asset 1 is:
\[
1,000 \times X = 1,000 \times 0.60 = 600
\]
Solution 95

C Chapter 20, Valuing a Claim on $S^a$

The expected value can be used to solve for $a$:

$$E \left[ S(T)^a \right] = S(t)^a e^{[a(\alpha - \delta) + 0.5a(a-1)\sigma^2](T-t)}$$

$$1.7 = 0.6^a e^{[a(0.08) + 0.5a(a-1)0.4^2](1-0)}$$

$$\ln(1.7) = a \ln(0.6) + a(0.08) + 0.08a(a-1)$$

$$\ln(0.6) = a \ln(0.6) + a(0.08) + 0.08a^2 - 0.08a$$

$$0 = 0.08a^2 + (\ln(0.6))a - \ln(1.7)$$

$$a = \frac{-\ln(0.6) \pm \sqrt{(\ln(0.6))^2 - 4(0.08)(-\ln(1.7))}}{2(0.08)}$$

$$a = -0.90928 \text{ or } a = 7.2946$$

Since $a$ is negative, its value must be $-0.90928$.

We can now calculate the expected return for the contingent claim:

$$\gamma = a(\alpha - r) + r = -0.90928(0.08 - 0.05) + 0.05 = 0.02272$$

Since we are given the expected value of the contingent claim at time 1, we can make use of it to find the time-0 price of the contingent claim:

$$F_0^T \left[ S(T)^a \right] = e^{-\gamma T} E \left[ S(T)^a \right] = e^{-0.02272} [1.7] = 1.662$$

Solution 96

B Chapter 20, Sharpe Ratio

The price follows geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \iff S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)}$$

We have:

$$\alpha_1 - 0 - 0.5\sigma_1^2 = 0.05 \quad \& \quad \sigma_1 = 0.2 \implies \alpha_1 = 0.05 + 0.5(0.2)^2 = 0.07$$

$$\alpha_2 - 0 - 0.5\sigma_2^2 = 0.04 \quad \& \quad \sigma_2 = 0.3 \implies \alpha_2 = 0.04 + 0.5(0.3)^2 = 0.085$$
Since the two assets are perfectly correlated, they must have the same Sharpe ratio:

\[
\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}
\]

\[
\frac{0.07 - r}{0.2} = \frac{0.085 - r}{0.3}
\]

\[
0.021 - 0.3r = 0.017 - 0.2r
\]

\[
0.004 = 0.1r
\]

\[
r = 0.04
\]

Solution 97
C Chapter 20, Sharpe Ratio

The price follows geometric Brownian motion:

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \iff S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)}
\]

We have:

\[
\alpha_1 - 0.02 - 0.5\sigma_1^2 = 0.05 \quad \& \quad \sigma_1 = 0.2 \implies \alpha_1 = 0.05 + 0.02 + 0.5(0.2)^2 = 0.09
\]

\[
\alpha_2 - 0 - 0.5\sigma_2^2 = 0.04 \quad \& \quad \sigma_2 = 0.4 \implies \alpha_2 = 0.04 + 0.5(0.4)^2 = 0.12
\]

Since the two assets are perfectly correlated, they must have the same Sharpe ratio:

\[
\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}
\]

\[
\frac{0.09 - r}{0.2} = \frac{0.12 - r}{0.4}
\]

\[
0.036 - 0.4r = 0.024 - 0.2r
\]

\[
0.012 = 0.2r
\]

\[
r = 0.06
\]

Solution 98
C Chapter 20, Geometric Brownian Motion

This question is based on the “Remarks” that appear after the official solution to Question #19 on the Spring 2009 MFE Exam.

Statement I is false. The stock price consists of 2 components: the prepaid forward price and the present value of the dividends, and the stock price’s volatility is an average of the volatilities of the two components. The volatility of the dividends is zero, so as the dividends are paid, the average approaches the volatility of the prepaid forward. Geometric Brownian motion calls for constant volatility, so the stock price process cannot be a geometric Brownian motion.
Statement II is true. Let’s use $F_{t,T}^P$ to denote the prepaid forward price. Since the prepaid forward price follows geometric Brownian motion, we have:

$$\frac{dF_{t,T}^P}{F_{t,T}^P} = \mu dt + \sigma dZ(t) \iff F_{t,T}^P = F_{0,T}^P \times e^{(\mu - 0.5\sigma^2)t + \sigma Z(t)}$$

The forward price process is:

$$F_{t,T} = F_{t,T}^P \times e^{r(T-t)} = F_{0,T}^P \times e^{(\mu - 0.5\sigma^2)t + \sigma Z(t)} e^{r(T-t)}$$

$$= F_{0,T}^P \times e^{rT} e^{(\mu - 0.5\sigma^2)t + \sigma Z(t)} = F_{0,T}^P e^{(\mu - 0.5\sigma^2)t + \sigma Z(t)}$$

This can be recognized as geometric Brownian motion:

$$\frac{dF_{t,T}}{F_{t,T}} = (\mu - r)dt + \sigma dZ(t) \iff F_{t,T} = F_{0,T}^P \times e^{(\mu - 0.5\sigma^2)t + \sigma Z(t)}$$

Statement III is true. Since the prepaid forward price follows geometric Brownian motion, we again have:

$$\frac{dF_{t,T}^P}{F_{t,T}^P} = \mu dt + \sigma dZ(t) \iff F_{t,T}^P = F_{0,T}^P \times e^{(\mu - 0.5\sigma^2)t + \sigma Z(t)}$$

The stock price can be written in terms of the prepaid forward price:

$$F_{t,T}^P = S_t e^{-\delta(T-t)} \Rightarrow S_t = F_{t,T}^P e^{\delta(T-t)}$$

The stock price process is:

$$S_t = F_{t,T}^P \times e^{\delta(T-t)} = F_{0,T}^P \times e^{(\mu - 0.5\sigma^2)t + \sigma Z(t)} e^{\delta(T-t)}$$

$$= F_{0,T}^P \times e^{\delta T} e^{(\mu - \delta - 0.5\sigma^2)t + \sigma Z(t)} = S_0 e^{(\mu - \delta - 0.5\sigma^2)t + \sigma Z(t)}$$

This can be recognized as geometric Brownian motion:

$$\frac{dS_t}{S_t} = (\mu - \delta)dt + \sigma dZ(t) \iff S_t = S_0 \times e^{(\mu - \delta - 0.5\sigma^2)t + \sigma Z(t)}$$

**Solution 99**

**D  Chapter 12, Black-Scholes Formula**

The prepaid forward volatility is 20%:

$$\frac{dF_{t,0.5}^P(S)}{F_{t,0.5}^P(S)} = \mu dt + 0.20dZ(t) \Rightarrow \sigma_{PF} = 0.20$$
The prepaid forward prices of the stock and the strike price are:

\[ F_{0,T}^P(S) = 70 - 4.00e^{-0.07(4/12)} = 66.0923 \]
\[ F_{0,T}^P(K) = 75e^{-0.07(0.5)} = 72.4204 \]

We use the prepaid forward volatility in the Black-Scholes Formula:

\[ d_1 = \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma_{PF}^2T}{\sigma_{PF}\sqrt{T}} = \frac{\ln \left( \frac{66.0923}{72.4204} \right) + 0.5 \times 0.20^2 \times 0.5}{0.20\sqrt{0.5}} \]
\[ d_1 = -0.57584 \]
\[ d_2 = d_1 - \sigma_{PF}\sqrt{T} = -0.57584 - 0.20\sqrt{0.5} = -0.71726 \]

We have:

\[ N(-d_1) = N(0.57584) = 0.71764 \]
\[ N(-d_2) = N(0.71726) = 0.76339 \]

The value of the European put option is:

\[ P_{Eur} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) = Ke^{-rT}N(-d_2) - \left[ S_0 - PV_{0,T}(Div) \right] N(-d_1) \]
\[ = 72.4204 \times 0.76339 - 66.0923 \times 0.71764 \]
\[ = 7.8546 \]

**Solution 100**

**B Chapter 20, Sharpe Ratio**

The price follows geometric Brownian motion:

\[ \frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \quad \Leftrightarrow \quad S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)} \]

We have:

\[ \alpha_1 - 0.04 - 0.5\sigma^2 = 0.16 \quad \& \quad \sigma_1 = 0.2 \quad \Rightarrow \quad \alpha_1 = 0.16 + 0.04 + 0.5(0.2)^2 = 0.22 \]
\[ \alpha_2 - 0 - 0.5\sigma^2 = 0.31 \quad \& \quad \sigma_2 = 0.4 \quad \Rightarrow \quad \alpha_2 = 0.31 + 0.5(0.4)^2 = 0.39 \]
Since the two assets are perfectly correlated, they must have the same Sharpe ratio:

\[
\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}
\]

\[
0.22 - r = 0.39 - r
\]

\[
0.2 = 0.4
\]

\[
0.088 - 0.4r = 0.078 - 0.2r
\]

\[
0.010 = 0.2r
\]

\[
r = 0.05
\]

**Solution 101**

E Chapter 20, Claim on \( S^a \)

The delta and gamma of the claim are:

\[
\Delta = \frac{\partial S^4}{\partial S} = 4S^3 \quad \Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial (4S^3)}{\partial S} = 12S^2
\]

At time 3, the value of \( S(3) = 2 \), so the value of gamma is:

\[
\Gamma = 12S^2 = 12 \times 2^2 = 48
\]

**Solution 102**

D Chapter 20, Claim on \( S^a \)

The delta and gamma of the claim are:

\[
\Delta = \frac{\partial S^k}{\partial S} = kS^{k-1} \quad \Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial (kS^{k-1})}{\partial S} = k(k-1)S^{k-2}
\]

At time 3, the value of \( S(3) = 2 \), so:

\[
\Gamma = k(k-1)2^{k-2}
\]

\[
48 = k(k-1)2^{k-2}
\]

Since we are given that \( k \) is a positive integer between 0 and 6, let's try a few integers to see which one works. For \( k = 0 \), the right side of the equation above is 0. For \( k = 1 \), the right side is again 0. For \( k = 2 \), the right side is 2. For \( k = 3 \), the right side is 12. For \( k = 4 \), the right side is 48, so the correct value of \( k \) is 4.

From the process for the stock, we observe that the expected rate of price increase is:

\[
\alpha - \delta = 0.14
\]

Since we are given that the dividend yield is 6%, we can find the stock's expected return:

\[
\alpha - 0.06 = 0.14 \quad \Rightarrow \quad \alpha = 0.20
\]
We can now obtain the expected return on the derivative:
\[ \gamma = k(\alpha - r) + r = 4(0.20 - 0.12) + 0.12 = 0.44 \]

**Solution 103**

C Chapter 20, Power Option

Since \( S(t) \) is a geometric Brownian motion, so is \([S(t)]^a\):

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \Rightarrow \frac{d(S^a)}{S^a} = \left[ a(\alpha - \delta) + 0.5a(a - 1)\sigma^2 \right]dt + a\sigma dZ(t)
\]

The coefficient of \( dZ(t) \) is the asset’s volatility. For \([S(t)]^4\), let’s call this volatility \( \sigma^* \):

\[ \sigma^* = a\sigma = 4 \times 0.32 = 1.28 \]

The dividend yield on \([S(t)]^4\) is:

\[ \delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2 = 0.12 - 4(0.12 - 0.06) - 0.5(4)(4 - 1)(0.32)^2 = -0.7344 \]

The initial price of \([S(t)]^4\) is:

\[ [S(0)]^4 = 2^4 = 16 \]

We can use the Black-Scholes formula to find the value of a 1-year European call option with \([S(t)]^4\) as the underlying asset, and the strike price is:

\[ K^* = K^4 = 2^4 = 16 \]

The first step is to calculate \( d_1 \) and \( d_2 \):

\[
d_1 = \frac{\ln(S^4 / K^*) + \left( r - \delta^* + 0.5(\sigma^*)^2 \right) T}{(\sigma^*)\sqrt{T}} = \frac{\ln(16/16) + (0.12 - (-0.7344) + 0.5 \times 1.28^2) \times 1}{1.28 \sqrt{1}} = 1.30750
\]

\[ d_2 = d_1 - (\sigma^*)\sqrt{T} = 1.30750 - 1.28 \sqrt{1} = 0.02750 \]

We have:

\[ N(d_1) = N(1.30750) = 0.90448 \]

\[ N(d_2) = N(0.02750) = 0.51097 \]

The value of the European power call option is:

\[
C_{Eur} = [S(0)]^4 e^{-\delta^* T} N(d_1) - (K^*) e^{-rT} N(d_2) = 16e^{-(-0.7344) \times 1} \times 0.90448 - 16e^{-0.12 \times 1} \times 0.51097 = 22.9113
\]


Solution 104

Chapter 20, Power Option

Since \( S(t) \) is a geometric Brownian motion, so is \([S(t)]^a\):

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \Rightarrow \frac{d(S^a)}{S^a} = \left[a(\alpha - \delta) + 0.5a(a-1)\sigma^2\right]dt + a\sigma dZ(t)
\]

The coefficient of \( dZ(t) \) is the asset’s volatility. For \([S(t)]^4\), let’s call this volatility \( \sigma^* \):

\[
\sigma^* = a\sigma = 4 \times 0.32 = 1.28
\]

The dividend yield on \([S(t)]^4\) is:

\[
\delta^* = r - a(r - \delta) - 0.5a(a-1)\sigma^2 = 0.12 - 4(0.12 \times 0.06) - 0.5(4)(4-1)(0.32)^2 = -0.7344
\]

The initial price of \([S(t)]^4\) is:

\[
[S(0)]^4 = 2^4 = 16
\]

We can use the Black-Scholes formula to find the delta of a 1-year European call option with \([S(t)]^4\) as the underlying asset, and the strike price is:

\[
K^* = K^4 = 2^4 = 16
\]

We can find \( d_1 \):

\[
d_1 = \frac{\ln(S^4 / K^*) + \left(r - (\delta^*) + 0.5(\sigma^*)^2\right)T}{(\sigma^*)\sqrt{T}}
\]

\[
= \frac{\ln(16/16) + (0.12 - (-0.7344) + 0.5 \times 1.28^2) \times 1}{1.28\sqrt{1}} = 1.30750
\]

We have:

\[
N(d_1) = N(1.30750) = 0.90448
\]

The partial derivative of the power option with respect to the underlying asset of \([S(t)]^4\) is:

\[
\frac{\partial V}{\partial [S(t)]^4} = e^{-(\delta^*)}N(d_1) = e^{-(-0.7344)} \times 0.90448 = 1.88515
\]

But we need the partial derivative with respect to the stock price, which is the delta of the power option:

\[
\Delta = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial [S(t)]^4} \times \frac{\partial [S(t)]^4}{\partial S} = 1.88515 \times 4[S(t)]^3 = 1.88515 \times 4 \times 2^3 = 60.3247
\]

Therefore, the market-maker must purchase 60.3247 shares of stock.
Solution 105

E Chapter 20, Power Option

Since \( S(t) \) is a geometric Brownian motion, so is \([S(t)]^a\):

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \quad \Rightarrow \quad \frac{d[S^a]}{S^a} = \left[ a(\alpha - \delta) + 0.5a(a-1)\sigma^2 \right]dt + a\sigma dZ(t)
\]

The coefficient of \( dZ(t) \) is the asset’s volatility. For \([S(t)]^4\), let’s call this volatility \( \sigma^* \):

\[ \sigma^* = a\sigma = 4 \times 0.32 = 1.28 \]

The dividend yield on \([S(t)]^4\) is:

\[ \delta^* = r - a(r - \delta) - 0.5a(a-1)\sigma^2 = 0.12 - 4(0.12 - 0.06) - 0.5(4)(4-1)(0.32)^2 = -0.7344 \]

The initial price of \([S(t)]^4\) is:

\[ [S(0)]^4 = 2^4 = 16 \]

We can use the Black-Scholes formula to find the value of a 1-year European call option with \([S(t)]^4\) as the underlying asset, and the strike price is:

\[ K^* = K^4 = 2^4 = 16 \]

The first step is to calculate \( d_1 \) and \( d_2 \):

\[
d_1 = \frac{\ln(S^4 / K^*) + (r - (\delta^*) + 0.5(\sigma^*)^2)T}{(\sigma^* \sqrt{T})} = \frac{\ln(16/16) + (0.12 - (-0.7344) + 0.5 \times 1.28^2) \times 1}{1.28 \sqrt{1}} = 1.30750 \]

\[ d_2 = d_1 - (\sigma^* \sqrt{T}) = 1.30750 - 1.28 \sqrt{1} = 0.02750 \]

We have:

\[ N(d_1) = N(1.30750) = 0.90448 \]

\[ N(d_2) = N(0.02750) = 0.51097 \]

The value of the European power call option is:

\[
C_{Eur} = S^4 e^{-(\delta^*) T} \left[ N(d_1) - (K^*) e^{-rT} N(d_2) \right] = 16e^{-(0.7344) \times 1} \times 0.90448 - 16e^{-0.12 \times 1} \times 0.51097 = 22.9113
\]

The partial derivative of the power option with respect to the underlying asset of \([S(t)]^4\) is:

\[
\frac{\partial V}{\partial [S(t)]^4} = e^{-(\delta^*)} N(d_1) = e^{-(0.7344)} \times 0.90448 = 1.88515
\]
The partial derivative with respect to the stock price is the delta of the power option:

\[
\Delta = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial [S(t)]^a} \times \frac{\partial [S(t)]^a}{\partial S} = 1.88515 \times 4[S(t)]^3 = 1.88515 \times 4 \times 2^3 = 60.3247
\]

The elasticity of the power option is:

\[
\Omega = \frac{S\Delta}{V} = \frac{2 \times 60.3247}{22.9113} = 5.2659
\]

**Solution 106**

B Chapter 20, Power Option

We recognize this option as a power option with \( a = -4 \)

Since \( S(t) \) is a geometric Brownian motion, so is \([S(t)]^a\):

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \implies \frac{d([S(t)]^a)}{[S(t)]^a} = \left[ a(\alpha - \delta) + 0.5a(a-1)\sigma^2 \right] dt + a\sigma dZ(t)
\]

The coefficient of \( dZ(t) \) is the asset’s volatility parameter:

\[a\sigma = -4 \times 0.32 = -1.28\]

Although the volatility parameter in the Itô process is negative, the volatility used in the Black-Scholes formula is the standard deviation of the return. Since a standard deviation must be positive, we use the absolute value of the volatility parameter in the Black-Scholes formula. Let’s call this standard deviation \( \sigma^* \):

\[\sigma^* = |a\sigma| = |-4 \times 0.32| = |-1.28| = 1.28\]

The dividend yield on \([S(t)]^{-4}\) is:

\[\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2 = 0.12 - (-4)(0.12 - 0.06) - 0.5(-4)(-4 - 1)(0.32)^2 = -0.664\]

The initial price of \([S(t)]^{-4}\) is:

\[\left[S(0)\right]^{-4} = 2^{-4} = 0.0625\]

We know that \([S(t)]^{-4}\) exhibits geometric Brownian motion, and we know its volatility, dividend yield, and initial value. Therefore, we can use the Black-Scholes formula to find the value of a 1-year European call option with \([S(t)]^{-4}\) as the underlying asset, and the strike price is:

\[K^* = K^{-4} = 2^{-4} = 0.0625\]
The first step is to calculate $d_1$ and $d_2$:

$$d_1 = \frac{\ln(S^{-4}/K^*) + \left(r - (\delta^*) + 0.5(\sigma^*)^2\right)T}{(\sigma^*)\sqrt{T}}$$

$$= \frac{\ln(0.0625/0.0625) + (0.12 - (-0.664) + 0.5 \times 1.28^2) \times 1}{1.28\sqrt{1}} = 1.25250$$

$$d_2 = d_1 - (\sigma^*)\sqrt{T} = 1.25250 - 1.28\sqrt{1} = -0.02750$$

We have:

$$N(d_1) = N(1.25250) = 0.89481$$

$$N(d_2) = N(-0.02750) = 0.48903$$

The value of the European power call option is:

$$C_{Eur} = S^{-4}e^{-\delta^*T}N(d_1) - (K^*)e^{-rT}N(d_2)$$

$$= 0.0625e^{-(-0.664)\times 1} \times 0.89481 - 0.0625e^{-0.12\times 1} \times 0.48903 = 0.08153$$

**Solution 107**

**D** Chapter 20, Power Options

We can value this option as a call option that has $[S(t)]^4$ as the underlying asset and $4[S(t)]^2$ as the strike asset. We make use of the formula for the dividend yield of an asset having a price of $[S(t)]^g$, and we use $U$ to indicate the underlying asset and $K$ to indicate the strike asset:

$$\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2$$

$$\delta_U = 0.12 - 4(0.12 - 0) - 0.5(4)(4 - 1)(0.3)^2 = -0.90$$

$$\delta_K = 0.12 - 2(0.12 - 0) - 0.5(2)(2 - 1)(0.3)^2 = -0.21$$

The price processes for the underlying asset and the strike asset follow geometric Brownian motion:

$$\frac{d[S(t)]^4}{[S(t)]^4} = \alpha_U dt + \sigma_U dZ(t) = \alpha_U dt + 4(0.3)dZ(t) = \alpha_U dt + 1.2dZ(t)$$

$$\frac{d[4S(t)]^2}{[4S(t)]^2} = \alpha_K dt + \sigma_K dZ(t) = \alpha_K dt + 2(0.3)dZ(t) = \alpha_K dt + 0.6dZ(t)$$

There is no need to solve for $\alpha_U$ and $\alpha_K$ to answer this question. We do need the volatility parameters though:

$$\sigma_U = 1.2 \quad \text{and} \quad \sigma_K = 0.6$$
Since the price processes for the underlying and strike assets have $Z(t)$ as their only source of randomness, their returns are perfectly correlated: 

$$\rho = 1$$

We can now use the Black-Scholes formula for exchange options. We begin by finding the volatility parameter:

$$\sigma = \sqrt{\sigma_U^2 + \sigma_K^2 - 2\rho \sigma_U \sigma_K} = \sqrt{1.2^2 + 0.6^2 - 2 \times 1 \times 1.2 \times 0.6} = 0.6$$

The next step is to find $d_1$ and $d_2$ and $N(d_1)$ and $N(d_2)$:

$$d_1 = \frac{\ln \left( \frac{Ue^{-\delta \tau T}}{Ke^{-\delta K T}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{2^4 e^{-(-0.90)\times1}}{(4 \times 2^2) e^{-(-0.21)\times1}} \right) + \frac{0.6^2 \times 1}{2}}{0.6 \times \sqrt{T}} = 1.45000$$

$$d_2 = d_1 - \sigma \sqrt{T} = 1.45000 - 0.6 \sqrt{T} = 0.85000$$

$$N(1.45000) = 0.92647$$

$$N(0.85000) = 0.80234$$

The price of the exchange call option is:

$$\text{ExchangeCallPrice} = Ue^{-\delta \tau T} N(d_1) - Ke^{-\delta K T} N(d_2)$$

$$= 2^4 e^{-(-0.90)\times1} \times 0.92647 - (4 \times 2^2) e^{-(-0.21)\times1} \times 0.80234 = 20.6227$$

**Solution 108**

**A** Chapter 20, Quadratic Variation

The quadratic variation is the sum of the squared increments of the process.

(i) $W$ is not stochastic, and its differential is found below:

$$W(t) = t(t^2 - 1) = t^3 - t$$

$$dW = (3t^2 - 1)dt$$

Making use of the multiplication rule, $dt \times dt = 0$, we obtain the quadratic variation:

$$V_{3,7}^2(W) = \int_0^{3.7} \left[ dW \right]^2 = \int_0^{3.7} \left[ (3t^2 - 1)dt \right]^2 = \int_0^{3.7} (3t^2 - 1)^2(dt \times dt) = 0$$
(ii) Let’s consider the values of $X(t)$ as $t$ increases from 0 to 3.7:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$X(t)$</th>
<th>Change in $X(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(0, 1]</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2]</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(2, 3]</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>(3, 3.7]</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

The increments to $X$ are 0, except just after time 0, time 1, time 2, and time 3. At those 4 times, the increments are 1. An infinite number of zeros is summed, and their sum is zero. The four non-zero increments (of 1 each) sum to 4:

$$ V_{3.7}^2(X) = \int_0^{3.7} [dX]^2 = \lim_{n \to \infty} \sum_{i=1}^{n} \left\{ X \left( i \times \frac{3.7}{n} \right) - X \left( \left\lfloor i \right\rfloor \times \frac{3.7}{n} \right) \right\}^2 $$

$$ = 1^2 + 0^2 + \cdots + 0^2 + 1^2 + 0^2 + \cdots + 0^2 + 1^2 + 0^2 + \cdots + 0^2 = 4 $$

(iii) We can find $dY$ by finding the partial differential of each of the terms:

$$ dY = 4dt + 0.8dZ $$

Making use of the multiplication rules, we have:

$$ (dY)^2 = (4dt + 0.8dZ)^2 = 0.64(dZ)^2 = 0.64dt $$

The quadratic variation is:

$$ V_{3.7}^2(Y) = \int_0^{3.7} [dY]^2 = \int_0^{3.7} 0.64dt = 0.64 \times (3.7 - 0) = 2.368 $$

Since $0 < 2.368 < 4$, we have:

$$ V_{3.7}^2(W) < V_{3.7}^2(Y) < V_{3.7}^2(X) $$

**Solution 109**

A Chapter 20, Volatility of Prepaid Forward

The prepaid forward price at time $t$ is equal to the stock price minus the present value of the dividend:

$$ F_{t,0.5}^P(S) = S(t) - 4e^{-0.07 \left( \frac{4}{12} - t \right)} $$
We can take the differential and divide by $F_{t,0.5}^P(S)$. For $0 \leq t < \frac{4}{12}$, we have:

$$dF_{t,0.5}^P(S) = dS(t) - 0.28e^{-0.07(\frac{4}{12}-t)}dt$$

$$\frac{dF_{t,0.5}^P(S)}{F_{t,0.5}^P(S)} = \frac{dS(t)}{F_{t,0.5}^P(S)} - 0.28e^{-0.07(\frac{4}{12}-t)}dt$$

$$\frac{dF_{t,0.5}^P(S)}{F_{t,0.5}^P(S)} = \frac{dS(t)}{F_{t,0.5}^P(S)} - \frac{0.28e^{-0.07(\frac{4}{12}-t)}}{F_{t,0.5}^P(S)}dt + \frac{dS(t)}{F_{t,0.5}^P(S)}$$

$$\frac{dF_{t,0.5}^P(S)}{F_{t,0.5}^P(S)} = \frac{dS(t)}{F_{t,0.5}^P(S)} - \frac{S(t) - 0.28e^{-0.07(\frac{4}{12}-t)}}{F_{t,0.5}^P(S)}dt + \frac{S(t) - 0.28e^{-0.07(\frac{4}{12}-t)}}{F_{t,0.5}^P(S)}$$

From the second term in the final expression above, we can see that the volatility parameter for the prepaid forward is:

$$\sigma_{PF} = \frac{S(t)}{F_{t,0.5}^P(S)}$$

Since we know the time-0 values on the right side of the equation, we can find $\sigma_{PF}$:

$$\sigma_{PF} = \frac{S(t)}{70 - 4e^{-0.07(\frac{4}{12}-0)}} = 1.059126 \times 0.189 = 0.20017$$

The prepaid forward prices of the stock and the strike price are:

$$F_{0,T}^P(S) = 70 - 4.00e^{-0.07(4/12)} = 66.0923$$

$$F_{0,T}^P(K) = 75e^{-0.07(0.5)} = 72.4204$$

We use the prepaid forward volatility in the Black-Scholes Formula:

$$d_1 = \frac{\ln\left(\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right) + 0.5\sigma_{PF}^2T}{\sigma_{PF}\sqrt{T}} = \frac{\ln(\frac{66.0923}{72.4204}) + 0.5 \times 0.20017^2 \times 0.5}{0.20017\sqrt{0.5}}$$

$$d_1 = -0.57522$$

$$d_2 = d_1 - \sigma_{PF}\sqrt{T} = -0.57522 - 0.20017\sqrt{0.5} = -0.71676$$
We have:

\[ N(d_1) = N(-0.57522) = 0.28257 \]
\[ N(d_2) = N(-0.71676) = 0.23676 \]

The value of the European call option is:

\[
C_{Eur} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) = \left[ S_0 - PV_{0,T}(Div) \right] N(d_1) - K e^{-rT} N(d_2)
\]
\[ = 66.0923 \times 0.28257 - 72.4204 \times 0.23676 \]
\[ = 1.5294 \]

**Solution 110**

B  Chapter 20, Volatility of Prepaid Forward

The prepaid forward price at time \( t \) follows geometric Brownian motion:

\[
F_{t,0.5}^P(S) = F_{0,0.5}^P(S) e^{(\alpha_{PF} - 0.5\sigma_{PF}^2)t - \sigma_{PF} Z(t)} \quad \Leftrightarrow \quad \frac{dF_{t,0.5}^P(S)}{F_{t,0.5}^P(S)} = \alpha_{PF} dt + \sigma_{PF} dZ(t)
\]

From (v), we know that the expected return is \( \alpha_{PF} = 0.12 \). From statement (iv) and the exponent in the first expression above, we have:

\[
\alpha_{PF} - 0.5\sigma_{PF}^2 = 0.04
\]
\[
0.12 - 0.5\sigma_{PF}^2 = 0.04
\]
\[
\sigma_{PF} = 0.40
\]

The prepaid forward prices of the stock and the strike price are:

\[
F_{0,T}^P(S) = 70 - 4.00 e^{-0.07(4/12)} = 66.0923
\]
\[
F_{0,T}^P(K) = 75 e^{-0.07(0.5)} = 72.4204
\]

We use the prepaid forward volatility in the Black-Scholes Formula:

\[
d_1 = \frac{\ln \left( \frac{F_{0,T}^P(S)}{F_{0,T}^P(K)} \right) + 0.5\sigma_{PF}^2 T}{\sigma_{PF} \sqrt{T}} = \frac{\ln \left( \frac{66.0923}{72.4204} \right) + 0.5 \times 0.4^2 \times 0.5}{0.4 \sqrt{0.5}} = -0.18186
\]

\[
d_2 = d_1 - \sigma_{PF} \sqrt{T} = -0.18186 - 0.4 \sqrt{0.5} = -0.46470
\]

We have:

\[ N(d_1) = N(-0.18186) = 0.42785 \]
\[ N(d_2) = N(-0.46470) = 0.32107 \]
The value of the European call option is:

\[
C_{Eur} \left( F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T \right) = \left[ S_0 - PV_{0,T}(Div) \right] N(d_1) - Ke^{-rT} N(d_2)
\]

\[
= 66.0923 \times 0.42785 - 72.4204 \times 0.32107 = 5.0256
\]

**Solution 111**

**C** Chapter 20, Forward Exchange Contract

When the contract is initiated, the prepaid forward prices of the payoffs to each party must be equal. Therefore, at time \( t \):

\[
e^{-0.02(5-t)} A(t) = e^{-0.06(5-t)} B(t) \beta(t)
\]

We can rearrange the equation above to solve for \( \beta(t) \):

\[
\beta(t) = \frac{e^{-0.02(5-t)} A(t)}{e^{-0.06(5-t)} B(t)}
\]

We make use of the following geometric Brownian motion equivalency:

\[
dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t) \iff S(t) = S(0)e^{(\alpha-\delta-0.5\sigma^2)t+\sigma Z(t)}
\]

Substituting \( A(t) \) and \( B(t) \) for \( S(t) \) as needed, we have:

\[
\beta(t) = \frac{e^{-0.02(5-t)} A(t)}{e^{-0.06(5-t)} B(t)} = e^{0.04(5-t)} \frac{A(0)e^{[0.08-0.5(0.20)^2]t+0.2Z(t)}}{B(0)e^{[0.09-0.5(0.40)^2]t+0.4Z(t)}}
\]

\[
= e^{0.20} e^{-0.04t} \frac{5}{4} e^{0.05t-0.2Z(t)} = 1.25e^{0.20+0.01t-0.2Z(t)}
\]

**Solution 112**

**D** Chapter 20, Claim on \( S^\alpha \)

The question asks for the ratio of the expected value to the prepaid forward price:

\[
X \quad \frac{E \left[ S(T)^\alpha | S(t) \right]}{F_{0,T}^P \left[ S(T)^\alpha \right]} = \frac{S(t)^\alpha e^{(\gamma-\delta\gamma)(T-t)}}{e^{-(\delta\gamma)(T-t)} \cdot S(t)^\alpha} = e^{\gamma(T-t)} = e^{\gamma(2-0)} = e^{2\gamma}
\]

The ratio of the expected value of the stock to the prepaid forward price of the stock is:

\[
\frac{E \left[ S(2) | S(0) \right]}{F_{0,2}^P \left[ S(2) \right]} = \frac{S(0)e^{(\alpha-\delta)2}}{S(0)e^{-\delta \times 2}} = e^{2\alpha}
\]
We can now use (i) and (ii) to solve for $\alpha$:

$$\frac{0.78663}{0.64404} = e^{2\alpha} \quad \Rightarrow \quad \alpha = 0.10$$

We can use (ii) and (iii) to solve for $r$:

$$\frac{F_{0.2} [S(2)]}{F_{0.2}^P [S(2)]} = \frac{S(0)e^{2(r-\delta)}}{S(0)e^{-2\delta}} = e^{2r}$$

$$\frac{0.71177}{0.64404} = e^{2r} \quad \Rightarrow \quad r = 0.05$$

We can now find the expected return on the contingent claim:

$$\gamma = a(\alpha - r) + r = 3(0.10 - 0.05) + 0.05 = 0.20$$

The ratio is:

$$\frac{X}{Y} = e^{2\gamma} = e^{2 \times 0.20} = 1.4918$$

**Solution 113**

**A** Chapter 20, Sharpe Ratio and 2 Stocks

From (i), we have:

$$\delta_X = 0.05 \quad \delta_Y = 0.02$$

From (ii), we have:

$$\alpha_X - \delta_X = 0.03 \quad \Rightarrow \quad \alpha_X = 0.03 + \delta_X = 0.03 + 0.05 = 0.08 \quad \sigma_X = 0.10$$

From (iii), we have:

$$\mu = \alpha_Y - \delta_Y - 0.5\sigma_Y^2$$

$$\sigma_Y = -0.20$$

Stocks X and Y must have the same Sharpe ratio:

$$\frac{\alpha_X - r}{\sigma_X} = \frac{\alpha_Y - r}{\sigma_Y}$$

$$\frac{0.08 - 0.07}{0.10} = \frac{\alpha_Y - 0.07}{-0.20}$$

$$\alpha_Y = 0.05$$

We can now find $\mu$:

$$\mu = \alpha_Y - \delta_Y - 0.5\sigma_Y^2 = 0.05 - 0.02 - 0.5 \times (-0.20)^2 = 0.01$$
Solution 114

B Chapter 20, Sharpe Ratio and 2 Stocks

Stock 1 does not pay dividends, so its parameters are:
\[ \alpha_1 = \mu \quad \text{and} \quad \sigma_1 = 10\mu \]

Stock 2 has a dividend yield of 0.03, so its parameters are:
\[ \begin{align*}
\alpha_2 &= \delta_2 - 0.5\sigma_2^2 = 0.02 \quad \sigma_2 = 0.10 \\
\alpha_2 &= 0.03 - 0.5 \times (0.10)^2 = 0.02 \\
\alpha_2 &= 0.055
\end{align*} \]

Since the two stocks have the same source of uncertainty, they must have the same Sharpe ratio:
\[ \frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2} \]
\[ \frac{\mu - 0.04}{10\mu} = \frac{0.055 - 0.04}{0.10} \]
\[ \mu - 0.04 = 1.5\mu \]
\[ \mu = -0.08 \]

Solution 115

D Chapter 20, Ornstein-Uhlenbeck Process

The key to this question is recognizing that the process is an Ornstein-Uhlenbeck process:
\[ \begin{align*}
dX(t) &= \lambda \times [\alpha - X(t)]dt + \sigma dZ(t) \\
X(t) &= X(0)e^{-\lambda t} + \alpha \left(1 - e^{-\lambda t}\right) + \sigma \int_0^t e^{-\lambda(t-s)}dZ(s)
\end{align*} \]

The parameters of the process are:
\[ \lambda = 4 \quad \alpha = 0 \quad \sigma = 3 \]

Therefore the solution is:
\[ X(t) = X(0)e^{-4t} + 0 \times \left(1 - e^{-4t}\right) + 3 \int_0^t e^{-4(t-s)}dZ(s) = e^{-4t} \left[X(0) + 3 \int_0^t e^{4s}dZ(s)\right] \]

In this form, we can observe that:
\[ A = 4 \quad B = X(0) = 5 \quad C = 3 \quad D = -4 \]

The sum of \(A, B, C,\) and \(D\) is:
\[ A + B + C + D = 4 + 5 + 3 - 4 = 8 \]
Solution 116

B Chapter 20, Portfolio Returns

The instantaneous percentage increase of the fund is the weighted average of the return on the stock (including its dividend yield) and the return on the risk-free asset, minus the asset fee:

\[
\frac{dW(t)}{W(t)} = 0.80 \left[ \frac{dS(t)}{S(t)} + \delta dt \right] + 0.20 r dt - 0.01 dt
\]

\[
= 0.80 \left[ 0.09 dt + 0.25 dZ(t) + 0.03 dt \right] + 0.20(0.07) dt - 0.01 dt
\]

\[
= 0.10 dt + 0.20 dZ(t)
\]

Since the fund’s value follows geometric Brownian motion, we can convert the partial differential equation into an expression for the value of the fund:

\[
\frac{dW(t)}{W(t)} = (\alpha_W - \delta_W) dt + \sigma_W dZ(t) \quad \Rightarrow \quad W(t) = W(0)e^{(\alpha_W - \delta_W - 0.5\sigma_W^2) t + \sigma_W Z(t)}
\]

\[
\frac{dW(t)}{W(t)} = 0.10 dt + 0.20 dZ(t) \quad \Rightarrow \quad W(t) = W(0)e^{(0.10 - 0.5 \times 0.20^2) t + 0.20 Z(t)}
\]

Simplifying the lower right-hand expression above, we have:

\[
W(t) = W(0)e^{0.08 t + 0.20 Z(t)}
\]

Solution 117

A Chapter 20, Risk-Neutral Pricing

The risk-neutral price process is:

\[
\frac{dS(t)}{S(t)} = (r - \delta) dt + \sigma d\tilde{Z}(t)
\]

From statement (iii), we can obtain the risk-free rate and the volatility parameter:

\[
r - \delta = 0.045 \quad \Rightarrow \quad r = 0.045 + \delta = 0.045 + 0.05 = 0.095
\]

\[
\sigma = 0.30
\]

We can use the following equivalency:

\[
\frac{dS(t)}{S(t)} = (r - \delta) dt + \sigma d\tilde{Z}(t) \quad \Rightarrow \quad S(t) = S(0)e^{(r - \delta - 0.5\sigma^2) t + \sigma \tilde{Z}(t)}
\]

\[
\frac{dS(t)}{S(t)} = 0.045 dt + 0.30 d\tilde{Z}(t) \quad \Rightarrow \quad S(t) = S(0)e^{(0.045 - 0.5 \times 0.3^2) t + 0.3 \tilde{Z}(t)} = e^{0.3 \tilde{Z}(t)}
\]
The expected value of the derivative under the risk-neutral probability measure is:
\[
E\left[ S(2)\{\ln S(2)\}^2 \right] = E\left[ e^{0.3Z(2)} \times \{\ln e^{0.3Z(2)}\}^2 \right]
\]
\[
= E\left[ e^{0.3Z(2)} \times \{0.3Z(2)\}^2 \right] = E\left[ 0.09e^{0.3Z(2)} \times \{Z(2)\}^2 \right]
\]
\[
= 0.09E\left[ \{Z(2)\}^2 \times e^{0.3Z(2)} \right]
\]

The value of \( \tilde{Z}(2) \) has a risk-neutral distribution that is the same as the distribution of \( Z \times \sqrt{2} \), for some standard normal random variable \( Z \). Therefore, we have:
\[
E\left[ S(2)\{\ln S(2)\}^2 \right] = 0.09E\left[ \{\tilde{Z}(2)\}^2 \times e^{0.3\tilde{Z}(2)} \right] = 0.09E\left[ \{Z(\sqrt{2})\}^2 \times e^{0.3Z(\sqrt{2})} \right]
\]
\[
= 0.09E\left[ 2Z^2 \times e^{0.3\sqrt{2}Z} \right] = 0.18E\left[ Z^2 \times e^{0.3\sqrt{2}Z} \right]
\]
\[
= 0.18E\left[ Z^2 \times e^{0.18Z} \right] = 0.18[1 + 0.18]e^{0.5\times0.18}
\]
\[
= 0.23240
\]

Although the amount of the payment becomes known at time 2, the payment is not actually made until time 3, so the time-0 price is obtained by discounting the expected value at the risk-free rate for 3 years:
\[
0.23240 \times e^{-3r} = 0.23240 \times e^{-3\times0.095} = 0.17477
\]

Solution 118

B Chapter 20, Gap Power Option

The usual put-call parity expression for gap calls and gap puts is:
\[
\text{GapCall} + K_1e^{-rT} = Se^{-\delta T} + \text{GapPut}
\]

In this case, the underlying asset is \( S^4 \), so we replace \( S \) with \( S^4 \) and we replace \( \delta \) with the dividend yield of \( S^4 \), which we denote \( \delta^* \):
\[
\text{GapCall} + K_1e^{-rT} = S^4e^{-\delta^*T} + \text{GapPut}
\]
\[
-25.90 + 850e^{-0.08(0.5)} = 5^4 e^{-\delta^*\times0.5} + 106.91
\]
\[
\delta^* = -0.1800
\]

We have another formula for \( \delta^* \), and we can use it to solve for \( \delta \):
\[
\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2
\]
\[
-0.1800 = 0.08 - 4(0.08 - \delta) - 0.5(4)(4 - 1)(0.10)^2
\]
\[
\delta = 0.0300
\]
Solution 119

B  Chapter 20, Claim on $S^a$

The usual relationship is:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta) dt + \sigma dZ(t) \quad \Rightarrow \quad \frac{d(S^a)}{S^a} = \left[ a(\alpha - \delta) + 0.5a(a-1)\sigma^2 \right] dt + a\sigma dZ(t)$$

In this case, the underlying stock moves inversely with the standard Brownian motion (as indicated by the negative sign in statement (i), so we have:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta) dt - \sigma dZ(t) \quad \Rightarrow \quad \frac{d(S^a)}{S^a} = \left[ a(\alpha - \delta) + 0.5a(a-1)\sigma^2 \right] dt - a\sigma dZ(t)$$

From statement (ii), we have the following 2 equations and 2 unknowns:

$$\begin{align*}
a(\alpha - \delta) + 0.5a(a-1)\sigma^2 &= -0.02 \\
-a\sigma &= 0.8
\end{align*}$$

We can simplify this system of equations as follows:

$$\begin{align*}
0.25a + 0.5a(a-1)\sigma^2 &= -0.02 \\
a &= \frac{-0.8}{\sigma}
\end{align*}$$

Substituting the second expression into the first, we have:

$$\begin{align*}
0.25a + 0.5a(a-1)\sigma^2 &= -0.02 \\
0.25\left(\frac{-0.8}{\sigma}\right) + 0.5\left(\frac{-0.8}{\sigma}\right)\left[\frac{-0.8}{\sigma} - 1\right]\sigma^2 &= -0.02 \\
\frac{-0.2}{\sigma} - 0.4\left[-0.8 - \sigma\right] &= -0.02 \\
0.2 + 0.4\sigma[-0.8 - \sigma] &= 0.02\sigma \\
0.2 - 0.32\sigma - 0.4\sigma^2 &= 0.02\sigma \\
-0.4\sigma^2 - 0.34\sigma + 0.2 &= 0 \\
\sigma &= \frac{0.34 \pm \sqrt{(-0.34)^2 - 4(-0.4)(0.2)}}{2(-0.04)} \\
\sigma &= -12.5 \quad \text{or} \quad \sigma = 0.4
\end{align*}$$

Since we are told that $\sigma$ is a positive constant, we conclude that $\sigma = 0.4$.
Solution 120

B  Chapter 20, Sharpe Ratio and Arbitrage

The Sharpe ratio of Stock X is greater than that of Stock Q:

\[
\frac{0.03 - 0.05}{-0.20} \quad \frac{0.07 - 0.05}{0.40} \quad 0.10 > 0.05
\]

An arbitrage strategy can be constructed with the following 3 steps:

1. Buy the following quantity of shares of Stock X:

\[
\frac{1}{\sigma_X S_X} = \frac{1}{-0.20 \times 20} = -0.25
\]

Since the quantity to buy is negative, we sell 0.25 shares of Stock X.

2. Sell the following quantity of Stock Q:

\[
\frac{1}{\sigma_Q S_Q} = \frac{1}{0.4 \times 5} = 0.5
\]

We sell 0.5 shares of Stock Q.

3. Lend the following quantity of dollars at the risk-free rate:

\[
\frac{1}{\sigma_Q} - \frac{1}{\sigma_X} = \frac{1}{0.40} - \frac{1}{-0.20} = 7.50
\]

Since the quantity to lend is positive, we lend $7.50 at the risk-free rate.

Since arbitrage produces riskless profits, the arbitrageur would like to implement the strategy above on as large a scale as possible, but lending is limited to $60. The strategy described above requires lending $7.50, so the strategy can be scaled up by a factor of:

\[
\frac{60}{7.50} = 8
\]

Scaling the strategy by a scale of 8 means that instead of purchasing \(-0.25\) shares of Stock X, the number of shares purchased is:

\[8 \times (-0.25) = -2\]

Solution 121

A  Chapter 20, Standard Brownian Motion

Statement B is true [Second sentence on page 651 of the Derivatives Markets textbook].

Statement C is true [Final bullet point on page 650 of the Derivatives Markets textbook].

Statement D is true, because \(Z(0) = 0\) and Brownian motion exhibits the infinite crossings property [Final sentence of the section preceding “Arithmetic Brownian Motion” on page 653 of the Derivatives Markets textbook].
Statement E is true, because \( \{Z(t+s) - Z(t)\} \) has a normal distribution described by \( N(0,s) \) [Second bullet point on page 650 of the Derivatives Markets textbook]. Since the other statements are true, Statement A must be the false statement.

**Solution 122**

A Chapter 20, Geometric Brownian Motion Equivalencies

We make use of the following equivalency:

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \iff \ln \left( \frac{S(t + h)}{S(t)} \right) \sim N(\alpha - \delta - 0.5\sigma^2)h, \sigma^2 h
\]

To avoid a cluttered appearance, we use \( S_t \) to represent \( S(t) \) in the solution below. We can rewrite the expression for \( G \), so that it is the product of independent random variables:

\[
G = \prod_{j=1}^{N} S\left( \frac{jT}{N} \right) = \left[ S_{\frac{T}{N}} \times S_{\frac{2T}{N}} \times \cdots \times S_{T} \right]^{1/N}
\]

\[
= \left[ S_0 \frac{S_{\frac{T}{N}}}{S_0} \times S_0 \frac{S_{\frac{2T}{N}}}{S_0} \times \cdots \times S_0 \frac{S_{T}}{S_0} \right]^{1/N}
\]

\[
= \left( S_0 \right) \left[ \left( \frac{S_{\frac{T}{N}}}{S_0} \right)^N \times \left( \frac{S_{\frac{2T}{N}}}{S_{\frac{T}{N}}} \right)^{(N-1)} \times \cdots \times \left( \frac{S_{T}}{S_{\frac{(N-1)T}{N}}} \right) \right]^{1/N}
\]

Taking the natural log, we have:

\[
\ln G = \ln(S_0) + \ln \left( \frac{S_{\frac{T}{N}}}{S_0} \right) + \frac{(N-1)}{N} \ln \left( \frac{S_{\frac{2T}{N}}}{S_{\frac{T}{N}}} \right) + \cdots + \frac{1}{N} \ln \left( \frac{S_{T}}{S_{\frac{(N-1)T}{N}}} \right)
\]
The variance, \( \text{Var}[\ln G] \), is:

\[
\text{Var}[\ln S_0] + \text{Var}[\ln \left( \frac{S_T}{S_0} \right)] \leq \frac{1}{N} \left( \frac{N-1}{N} \right)^2 \text{Var}[\ln \left( \frac{S_T}{S_N} \right)] + \cdots + \frac{1}{N} \left( \frac{1}{N} \right)^2 \text{Var}[\ln \left( \frac{S_T}{S_{(N-1)T}} \right)]
\]

\[
= 0 + \frac{T \sigma^2}{N} \left( \frac{N-1}{N} \right)^2 \text{Var}[\ln \left( \frac{S_T}{S_N} \right)] + \cdots + \frac{T \sigma^2}{N} \left( \frac{1}{N} \right)^2 \text{Var}[\ln \left( \frac{S_T}{S_{(N-1)T}} \right)]
\]

\[
= \frac{T \sigma^2}{N^3} \left[ N^2 + (N-1)^2 + \cdots + 1^2 \right] = \frac{T \sigma^2}{N^3} \left[ \frac{N(N+1)(2N+1)}{6} \right]
\]

\[
= \frac{(N+1)(2N+1)T \sigma^2}{6N^2}
\]

**Solution 123**

**E**  Chapter 20, Sharpe Ratio and Arbitrage

The Sharpe ratio of Stock 1 is greater than that of Stock 2:

\[
\frac{0.03 - 0.05}{-0.20} > \frac{0.07 - 0.05}{0.40} \Rightarrow \frac{0.10}{0.05} > \frac{0.20}{0.40}
\]

Since both assets follow geometric Brownian motion, a strategy that involves purchasing \( \frac{1}{\sigma_1 S_1} \) shares of Stock 1 and selling \( \frac{1}{\sigma_2 S_2} \) shares of Stock 2 results in an arbitrage profit of:

\[
\left[ \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} \right] dt = \left[ \frac{0.03 - 0.05}{-0.20} - \frac{0.07 - 0.05}{0.40} \right] dt = 0.05 dt
\]

Since \( \sigma_1 \) is negative, the amount of Stock 1 to purchase is negative:

\[
\frac{1}{\sigma_1 S_1} = \frac{1}{-0.20 \times 20} = -0.25
\]

Therefore the strategy calls for selling Stock 1.

Since \( \sigma_2 \) is positive, the amount of Stock 2 to sell is positive:

\[
\frac{1}{\sigma_2 S_2} = \frac{1}{0.40 \times 5} = 0.5
\]

Therefore, the strategy calls for selling Stock 2.

The proceeds from selling Stock 1 and Stock 2 are positive, and these funds are lent at the risk-free rate. Selling Stock 1, selling Stock 2, and lending at the risk-free rate is described by the strategy in Choice E.
Solution 124

C Chapter 20, Arbitrage

To keep the notation from becoming cluttered, we’ve dropped (t) from the processes below. For example, we write Q(t) as Q.

The incremental change in Asset Q is:

\[ dQ = \alpha_Q dt + QdZ_1 + 2QdZ_2 \]

We can create a zero-cost, risk-free portfolio by taking the following steps:

1. Purchase 1 unit of Asset Q.
2. Sell \( \frac{Q}{0.20S_1} \) units of Stock 1. [To remove the \( QdZ_1 \) term]
3. Sell \( \frac{2Q}{0.40S_2} \) units of Stock 2. [To remove the \( 2QdZ_2 \) term]
4. Lend \( \left( \frac{Q}{0.20} + \frac{2Q}{0.40} - Q \right) \) at the risk-free rate. [To invest the net funds from the three steps above]

The resulting portfolio is:

\[
\begin{align*}
dQ &= \left[ \alpha_Q dt + QdZ_1 + 2QdZ_2 \right] - \left[ \frac{0.12Q}{0.20} dt + QdZ_1 \right] - \left[ \frac{0.10 \times 2Q}{0.40} dt + 2QdZ_2 \right] \\
&\quad + \left( \frac{Q}{0.2} + \frac{2Q}{0.4} - Q \right) 0.08dt \\
&= \alpha_Q dt + QdZ_1 + 2QdZ_2 - \frac{0.12Q}{0.20} dt - QdZ_1 - \frac{0.10 \times 2Q}{0.40} dt - 2QdZ_2 \\
&\quad + \left( \frac{Q}{0.2} + \frac{2Q}{0.4} - Q \right) 0.08dt \\
&= \alpha_Q dt + \frac{0.12Q}{0.20} dt - \frac{0.10 \times 2Q}{0.40} dt + \left( \frac{Q}{0.2} + \frac{2Q}{0.4} - Q \right) 0.08dt \\
&= (\alpha_Q - 0.38) Q dt
\end{align*}
\]

Since the portfolio is a risk-free, zero-cost portfolio, its return must be zero. Therefore, its drift must be zero:

\[
(\alpha_Q - 0.38) Q = 0 \\
\alpha_Q - 0.38 = 0 \\
\alpha_Q = 0.38
\]
Solution 125
C Chapter 20, Pure Brownian Motion

The expected value of the product is:

\[
E\left[Z(5)Z(10)\right] = E\left[Z(5)\{Z(10) - Z(5) + Z(5)\}\right]
\]

\[
= E\left[Z(5)\{Z(10) - Z(5)\} + Z(5)Z(5)\right]
\]

\[
= E\left[Z(5)\{Z(10) - Z(5)\}\right] + E\left[Z(5)Z(5)\right]
\]

\[
= E\left[Z(5)\right]E\left[Z(10) - Z(5)\right] + E\left[Z(5)^2\right]
\]

\[
= 8 \times 0 + E\left[Z(5)^2\right] = E\left[Z(5)^2\right]
\]

The variance of \(Z(5)\) is equal to the amount of time elapsed from time 1 to time 5:

\[
Var\left[Z(5)\right] = 5 - 1 = 4
\]

We can use the following expression for variance to solve for the expected value of the square:

\[
Var[X] = E\left[X^2\right] - (E[X])^2
\]

\[
Var\left[Z(5)\right] = E\left[Z(5)^2\right] - (E\left[Z(5)\right])^2
\]

\[
4 = E\left[Z(5)^2\right] - (8)^2
\]

\[
E\left[Z(5)^2\right] = 68
\]

Since the expected value of the product is equal to the expected value of the square of \(Z(5)\), we have:

\[
E\left[Z(5)Z(10)\right] = E\left[Z(5)^2\right] = 68
\]
Chapter 21 – Solutions

Solution 1

Chapter 21, Differential Equation for a Riskless Asset

The terminal boundary condition is a red herring, as it is not needed to answer this question.

Let’s take the derivative of the solution to the differential equation:

\[ S(t) = Ae^{-0.05(T-t)} + b \]

\[ \frac{dS(t)}{dt} = 0.05Ae^{-0.05(T-t)} \]

\[ \frac{dS(t)}{dt} = 0.05[S(t) - b] \]

\[ \frac{dS(t)}{dt} = 0.05S(t) - 0.05b \]

In the question, we are given that:

\[ \frac{dS(t)}{dt} = 0.05S(t) \]

Therefore, the value of \( b \) must be zero.

Solution 2

Chapter 21, Differential Equation for a Riskless Asset

In this question, \( 0.5D \) is the continuous fixed dividend, and \( D \) is twice the continuous fixed dividend.

As \( t \) approaches \( T \):

\[ \lim_{t \to T} S(t) = \lim_{t \to T} S(t) \left[ \int_t^T 0.5De^{-0.04(s-t)}ds + 12e^{-0.04(T-t)} + b \right] = 12 + b \]

The question tells us that the terminal boundary condition is:

\[ S(T) = 12 \]

Therefore:

\[ b = 0 \]
We can take the derivative of the solution to the differential equation:

\[ S(t) = \int_t^T 0.5De^{-0.04(s-t)}ds + 12e^{-0.04(T-t)} + 0 \]

\[ S(t) = 0.5De^{0.04t} \int_t^T e^{-0.04s}ds + 12e^{-0.04(T-t)} \]

\[ \frac{dS(t)}{dt} = (0.04)0.5De^{0.04t} \int_t^T e^{-0.04s}ds + 0.5De^{0.04t} \left( -e^{-0.04t} \right) + (0.04)12e^{-0.04(T-t)} \]

\[ \frac{dS(t)}{dt} = (0.04)0.5De^{0.04t} \int_t^T e^{-0.04(s-t)}ds + (0.04)12e^{-0.04(T-t)} + 0.5De^{0.04t} \left( -e^{-0.04t} \right) \]

\[ \frac{dS(t)}{dt} = (0.04)S(t) - 0.5D \]

\[ \frac{dS(t)}{dt} + 0.5D = (0.04)S(t) \]

The question tells us that:

\[ \frac{dS(t)}{dt} + 8 = 0.04S(t) \]

We can now solve for \( D \):

\[ 0.5D = 8 \]

\[ D = 16 \]

**Solution 3**

C Chapter 21, Differential Equation for a Riskless Asset

Substituting 40 for \( S \), 0.10 for \( r \), and 2 for \( t \):

\[ S(2) = \int_2^3 8e^{-0.10(s-2)}ds + 40e^{-0.10(3-2)} \]

\[ = 8e^{0.10(2)} \int_2^3 e^{-0.10s}ds + 40e^{-0.10} \]

\[ = 8e^{0.10(2)} \left[ e^{-0.10(3)} - e^{-0.10(2)} \right] + 40e^{-0.10} \]

\[ = 8 \left[ e^{-0.10} - \frac{1}{-0.10} \right] + 40e^{-0.10} \]

\[ = 8 \left[ \frac{1 - e^{-0.10}}{0.10} \right] + 40e^{-0.10} \]

\[ = 43.81 \]
Solution 4

A Chapter 21, Differential Equation for a Riskless Asset

The pricing formula is:

\[
S(t) = \int_t^T \delta S(s) e^{-0.10(s-t)} ds + \bar{S}e^{-0.10(T-t)}
\]

\[
= \delta e^{0.10t} \int_t^T S(s) e^{-0.10s} ds + \bar{S}e^{-0.10(T-t)}
\]

Let's take the derivative with respect to \(t\):

\[
\frac{dS(t)}{dt} = 0.10\delta e^{0.10t} \int_t^T S(s) e^{-0.10s} ds + \delta e^{0.10t} \left(-S(t)e^{-0.10t}\right) + 0.10\bar{S}e^{-0.10(T-t)}
\]

\[
\frac{dS(t)}{dt} = 0.10 \left[ \delta e^{0.10t} \int_t^T S(s) e^{-0.10s} ds + \bar{S}e^{-0.10(T-t)} \right] - \delta S(t)
\]

\[
\frac{dS(t)}{dt} = 0.10S(t) - \delta S(t)
\]

We can substitute for the derivative of \(S(t)\):

\[
0.06S(t) = 0.10S(t) - \delta S(t)
\]

\[
\delta = 0.04
\]

Solution 5

A Chapter 21, Differential Equation for a Riskless Asset

The pricing formula for \(S(t)\) satisfies the differential equation on the right:

\[
S(t) = \int_t^T 0.04S(s) e^{-0.10(s-t)} ds + \bar{S}e^{-0.10(T-t)} \iff \frac{dS(t)}{dt} + 0.04S(t) = 0.10S(t)
\]

We can make the following observations based on the pricing formula for the underlying asset:

- The risk-free rate \(r\) is 10%.
- The dividend yield on the underlying asset is 4%.
- The volatility parameter \(\sigma\) is zero since the asset is riskless.

The quickest way to answer this question is to use the following formula from the Chapter 20 Review Note:

\[
\delta^* = r - a(r - \delta) - 0.5a(a - 1)\sigma^2 = 0.10 - 3(0.10 - 0.04) - 0.5(3)(2)(0) = -0.08
\]

Therefore, the solution is \(-8\%\).

Alternatively, let’s begin by noting that:

\[
\frac{dS(t)}{dt} + 0.04S(t) = 0.10S(t) \implies \frac{dS(t)}{dt} = 0.06S(t)
\]
Let’s denote the value of the claim as \( V(t) \):

\[
V(t) = [S(t)]^3
\]

Since the claim is risk-free, it must satisfy the differential equation:

\[
\frac{dV(t)}{dt} + D(t) = rV(t)
\]

We can now solve for \( D(t) \), the rate at which dividends are paid by the claim:

\[
\frac{dV(t)}{dt} + D(t) = rV(t)
\]

\[
\frac{dS^3}{dS} \times \frac{dS}{dt} + D(t) = 0.10S^3
\]

\[
3S^2 \times 0.06S + D(t) = 0.10S^3
\]

\[
D(t) = -0.08S^3
\]

\[
D(t) = -0.08V
\]

Since \( D(t) = \delta^* \times V(t) \), the lease rate of the claim is \( \delta^* = -0.08 \).

**Solution 6**

**Chapter 21, Black-Scholes Equation**

The expressions for the value of the derivative and the partial derivatives are:

\[
V = e^{0.07t} \left[ \ln S - 0.02t + 0.08 \right]
\]

\[
V_S = e^{0.07t} S^{-1}
\]

\[
V_{SS} = -e^{0.07t} S^{-2}
\]

\[
V_t = 0.07e^{0.07t} \{ \ln S - 0.02t + 0.08 \} - 0.02e^{0.07t}
\]
From the Black-Scholes equation, we have:

\[
0.5\sigma^2S^2V_{SS} + (r - \delta)SV_S + V_t = rV \\
-\frac{0.5\beta^2S^2e^{0.07t}}{S^2} + \frac{(0.07 - 0.03)Se^{0.07t}}{S} + V_t = 0.07V \\
-0.5\beta^2e^{0.07t} + 0.04e^{0.07t} + 0.07e^{0.07t}(\ln S - 0.02t + 0.08) - 0.02e^{0.07t} = 0.07V \\
e^{0.07t}\left[-0.5\beta^2 + 0.02 + 0.07(\ln S - 0.02t + 0.08)\right] = 0.07e^{0.07t}\left[\ln S - 0.02t + 0.08\right] \\
-0.5\beta^2 + 0.02 + 0.07[\ln S - 0.02t + 0.08] = 0.07[\ln S - 0.02t + 0.08] \\
-0.5\beta^2 + 0.02 = 0 \\
0.5\beta^2 = 0.02 \\
\beta^2 = 0.04 \\
\beta = 0.2 \quad \text{or} \quad \beta = -0.2
\]

Since 0.2 is not one of the available choices, it must be the case that \( \beta = -0.2 \).

**Solution 7**

**D** Chapter 21, The Black-Scholes Equation

We can use the Black-Scholes equation to find the value of the European option:

\[
0.5\sigma^2S^2V_{SS} + (r - \delta)SV_S + V_t = rV \\
0.5(0.32)^2(40)^2(0.008) + (0.095 - 0.00)(40)(0.869) - 2.238 = 0.095V \\
V = 18.1006
\]

The value now of continuous payments of $5 per year for 4 years is:

\[
\frac{5}{e^{-0.095(4)}} - \frac{1}{e^{0.095}} = 16.6389
\]

Adding the two components of the derivative together, we have:

\[
18.1006 + 16.6389 = 34.7395
\]

**Solution 8**

**D** Chapter 21, B-S Equation and Expected Return on Option

We can use the Black-Scholes equation to find the value of the option:

\[
0.5\sigma^2S^2V_{SS} + (r - \delta)SV_S + V_t = rV \\
0.5(0.30)^2(50)^2(0.020) + (0.08 - 0.02)(50)(0.743) - 3.730 = 0.08V \\
V = 9.3625
\]
Since \( \alpha - \delta = 0.13 \), the instantaneous expected return is:

\[
\alpha_{Option} = \frac{(\alpha - \delta)SV_S + 0.5 \sigma^2 S^2 V_{SS} + V_t}{V}
\]

\[
= \frac{0.13(50)(0.743) + 0.5(0.30)^2(50)^2(0.020) - 3.730}{9.3625}
\]

\[
= 0.3578
\]

**Solution 9**

B Chapter 21, Black-Scholes Equation

Since European options do not normally pay dividends, we assume that the option does not pay dividends:

\( D(t) = 0 \)

We can use the Black-Scholes equation to find \( rV \):

\[
0.5 \sigma^2 S^2 V_{SS} + (r - \delta)SV_S + V_t + D(t) = rV
\]

\[
0.5(0.30)^2(50)^2(0.020) + (0.08 - 0.02)(50)(0.743) - 3.730 + 0 = rV
\]

\[0.749 = rV\]

The expected change per unit of time under the risk-neutral distribution is:

\[
E^*[dV + D(t)dt] = rV
\]

\[
E^*[dV + 0] = 0.749
\]

\[
E^*[dV] = 0.749
\]

**Solution 10**

E Chapter 21, Black-Scholes Equation

The expressions for the value of the derivative and the partial derivatives are:

\[
V = e^{rt} \ln S
\]

\[
V_S = e^{rt} S^{-1}
\]

\[
V_{SS} = -e^{rt} S^{-2}
\]

\[
V_t = re^{rt} \ln S
\]
From the Black-Scholes equation, we have:

\[
0.5\sigma^2 S^2 V_{SS} + (r - \delta)SV_S + V_t = rV
\]

\[
0.5(0.2)^2 S^2 (e^{rt} S^{-2}) + (0.09 - \delta)S(e^{rt} S^{-1}) + re^{rt} \ln S = r\left(e^{rt} \ln S\right)
\]

\[
0.5(0.2)^2 S^2 (e^{rt} S^{-2}) + (0.09 - \delta)S(e^{rt} S^{-1}) = 0
\]

\[-0.5(0.2)^2 e^{rt} + (0.09 - \delta)e^{rt} = 0
\]

\[-0.5(0.2)^2 + (0.09 - \delta) = 0
\]

\[0.09 - \delta = 0.5(0.04)
\]

\[\delta = 0.07
\]

**Solution 11**

**E Chapter 21, Black-Scholes Equation**

For the derivative, we have:

\[D(t) = (\delta^*)V(t) = 0.05V(t)
\]

The expressions for the value of the derivative and the partial derivatives are:

\[V = 2e^{0.5rt} \ln S
\]

\[V_S = 2e^{0.5rt} S^{-1}
\]

\[V_{SS} = -2e^{0.5rt} S^{-2}
\]

\[V_t = re^{0.5rt} \ln S
\]

From the Black-Scholes equation, we have:

\[
0.5\sigma^2 S^2 V_{SS} + (r - \delta)SV_S + V_t + D(t) = rV
\]

\[
0.5\sigma^2 S^2 \left(\frac{-2e^{0.05t}}{S^2}\right) + (0.1 - 0.055)S \left(\frac{2e^{0.05t}}{S}\right) + 0.1e^{0.05t} \ln S + 0.05V = 0.1\left(2e^{0.05t} \ln S\right)
\]

\[-0.5\sigma^2 \times 2e^{0.05t} + 0.045 \times 2e^{0.05t} + 0.1e^{0.05t} \ln S + 0.05\left(2e^{0.05t} \ln S\right) = 2 \times 0.1e^{0.05t} \ln S
\]

\[-\sigma^2 e^{0.05rt} + 0.09e^{0.05t} = 0
\]

\[-\sigma^2 + 0.09 = 0
\]

\[\sigma^2 = 0.09
\]

\[\sigma = 0.30
\]
Solution 12  

D Chapters 12 and 21, Sharpe Ratio

The stock and the call option must have the same Sharpe ratio at all times, including time 2:

\[
\frac{0.12 - 0.095}{\sigma} = \frac{\gamma - 0.095}{\sigma_C}
\]

The cost of the shares required to delta-hedge the call option is the number of shares required, \( \Delta \), times the cost of each share, \( S \). Therefore, based on statement (iv) in the question, we have:

\[\Delta \times S = 78\]

We can find the volatility of the call option in terms of the volatility of the underlying stock:

\[
\sigma_C = \sigma \times \Omega_C = \sigma \times \frac{S \times \Delta}{V} = \sigma \times \frac{78}{20} = 3.9\sigma
\]

We can now solve for \( \gamma \):

\[
\frac{0.12 - 0.095}{\sigma} = \frac{\gamma - 0.095}{3.9\sigma} \quad \Rightarrow \quad \gamma = 0.1925
\]

Solution 13  

B Chapters 12 and 21, Sharpe Ratio

The stock and the call option must have the same Sharpe ratio at all times, including time 3:

\[
\frac{\alpha - 0.095}{\sigma} = \frac{0.26 - 0.095}{\sigma_C}
\]

The cost of the shares required to delta-hedge the call option is the number of shares required, \( \Delta \), times the cost of each share, \( S \). Therefore, based on statement (iv) in the question, we have:

\[\Delta \times S = 78\]

We can find the volatility of the call option in terms of the volatility of the underlying stock:

\[
\sigma_C = \sigma \times \Omega_C = \sigma \times \frac{S \times \Delta}{V} = \sigma \times \frac{78}{20} = 3.9\sigma
\]

We can now solve for \( \alpha \):

\[
\frac{\alpha - 0.095}{\sigma} = \frac{0.26 - 0.095}{3.9\sigma} \quad \Rightarrow \quad \alpha = 0.1373
\]
Solution 14

E Chapters 12 and 21, Sharpe Ratio

The stock and the call option must have the same Sharpe ratio at all times, including time 3:

\[
\frac{\alpha - 0.095}{\sigma} = \frac{0.36 - 0.095}{\sigma_C}
\]

The cost of the shares required to delta-hedge the call option is the number of shares required, \(\Delta\), times the cost of each share, \(S\). Therefore, based on statement (iv) in the question, we have:

\[\Delta \times S = 82\]

We can find the volatility of the call option in terms of the volatility of the underlying stock:

\[\sigma_C = \sigma \times \Omega_C = \sigma \times \frac{S \times \Delta}{V} = \sigma \times \frac{82}{20} = 4.1\sigma\]

We can now solve for \(\alpha\):

\[
\frac{\alpha - 0.095}{\sigma} = \frac{0.36 - 0.095}{4.1\sigma} \quad \Rightarrow \quad \alpha = 0.1596
\]

The general form for a dividend-paying stock that follows geometric Brownian motion is:

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)
\]

We can use the differential equation provided in the question to find the dividend yield:

\[
\frac{dS(t)}{S(t)} = 0.12dt + \sigma dZ(t) \quad \Rightarrow \quad 0.12 = \alpha - \delta
\]

\[0.12 = 0.1596 - \delta \]

\[\delta = 0.0396\]
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Chapter 22 – Solutions

Solution 1
A Chapter 22, Cash Call Options

The payoff is the same as the payoff resulting from owning 50 cash calls. To find the value of the cash calls, we must find the values of $d_2$ and $N(d_2)$:

$$d_2 = \frac{\ln(S_t/K) + [r - \delta - 0.5\sigma^2](T-t)}{\sigma \sqrt{T-t}}$$

$$= \frac{\ln(50/52) + [0.08 - 0.03 - 0.5(0.3)^2](0.5)}{0.3\sqrt{0.5}} = -0.17310$$

$$N(d_2) = N(-0.17310) = 0.43129$$

The value of 50 cash calls is:

$$50 \times \text{CashCall}(K) = 50e^{-r(T-t)}N(d_2) = 50e^{-0.08(0.5)}(0.43129) = 20.7189$$

Solution 2
D Chapter 22, Asset Call Options

The payoff is the same as the payoff resulting from owning 5 asset calls. To find the value of the asset calls, we must find the values of $d_1$ and $N(d_1)$:

$$d_1 = \frac{\ln(S_t/K) + [r - \delta + 0.5\sigma^2](T-t)}{\sigma \sqrt{T-t}}$$

$$= \frac{\ln(50/52) + [0.08 - 0.03 + 0.5(0.3)^2](0.5)}{0.3\sqrt{0.5}} = 0.03903$$

$$N(d_1) = N(0.03903) = 0.51557$$

The value of 5 asset calls is:

$$5 \times \text{AssetCall}(K) = 5 \times S_t e^{-\delta(T-t)}N(d_1) = 5 \times 50e^{-0.03(0.5)}(0.51557) = 126.9735$$

Solution 3
B Chapter 22, All-or-Nothing Options

We can use the value of the gap call to find the value of an asset call with a strike price of $52:

$$\text{GapCall} = \text{AssetCall}(K_2) - K_1 \times \text{CashCall}(K_2)$$

$$3.43 = \text{AssetCall}(52) - 53 \times 0.41$$

$$\text{AssetCall}(52) = 25.16$$
Since the combination of an asset call with a strike price of $52 and an asset put with a strike price of $52 is certain to have a payoff of 1 share of stock at time 0.5, the combination of their prices must be equal to the prepaid forward price of 1 share of stock:

\[
AssetCall(52) + AssetPut(52) = F_{0.0.5}^P(S)
\]

\[
25.16 + AssetPut(52) = 50e^{-0.03 \times 0.5}
\]

\[
AssetPut(52) = 24.0956
\]

### Solution 4

**B** Chapter 22, All-or-Nothing Options

We can use the value of the gap put to find the value of a cash put with a strike price of $52:

\[
GapPut = K_1 \times CashPut(K_2) - AssetPut(K_2)
\]

\[
6.00 = 53 \times CashPut(52) - 20.68
\]

\[
CashPut(K_2) = 0.5034
\]

Since the combination of a cash call and a cash put is certain to pay off for $1 at the end of the year, the current value of the combination must be the present value of $1:

\[
CashCall(52) + CashPut(52) = e^{-r(T-t)}
\]

\[
CashCall(52) + 0.5034 = e^{-0.08 \times 1}
\]

\[
CashCall(52) = 0.4197
\]

### Solution 5

**B** Chapter 22, All-or-Nothing Options

We can use the asset-or-nothing call price find the value of \( d_1 \):

\[
AssetCall(K) = S_t e^{-\delta(T-t)} N(d_1)
\]

\[
27.84 = 50e^{-0.03(1)} N(d_1)
\]

\[
N(d_1) = 0.57376
\]

\[
d_1 = 0.18596
\]

The value of \( d_2 \) is:

\[
d_2 = d_1 - \sigma \sqrt{T-t} = 0.18596 - 0.3\sqrt{1} = -0.11404
\]
The Sharpe ratio of the asset call is equal to the Sharpe ratio of the stock:

\[ 0.20 = \frac{\alpha - r}{\sigma} \]

We can now find \( \hat{d}_2 \):

\[
\hat{d}_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}} = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - r + r - \delta - 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}}
\]

\[ = \frac{\ln\left(\frac{S_t}{K}\right) + (r - \delta - 0.5\sigma^2)(T - t)}{\sigma \sqrt{T - t}} + \frac{\alpha - r}{\sigma} \sqrt{T - t} = d_2 + \frac{\alpha - r}{\sigma} \sqrt{T - t}
\]

\[ = -0.11404 + 0.2\sqrt{1} = 0.08596 \]

The probability that the stock price is greater than $52 is equal to \( N(\hat{d}_2) \):

\[ N(d_2) = N(0.08596) = 0.53425 \]

**Solution 6**

A Chapter 22, All-or-Nothing Options

For Stock B, we can use the cash call price to determine \( K \):

\[ \text{CashCall}(K) = e^{-r(T - t)} N(d_2) \]

\[ 0.49 = e^{-0.12(0.75)} N(d_2) \]

\[ N(d_2) = 0.53615 \]

\[ d_2 = 0.09074 \]

\[ \ln\left(\frac{50}{K}\right) + [0.12 - 0.03 - 0.5(0.2)^2](0.75) = 0.09074 \]

\[ K = 51.87341 \]

We can use \( K \) to obtain \( d_1 \) for Stock A:

\[ d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + [r - \delta + 0.5\sigma^2](T - t)}{\sigma \sqrt{T - t}} \]

\[ = \frac{\ln\left(\frac{50}{51.87341}\right) + [0.12 - 0.07 + 0.5(0.2)^2](0.75)}{0.2\sqrt{0.75}} = 0.09074 \]

We can now determine the price of the asset put on Stock A:

\[ \text{AssetPut}(K) = S_t e^{-\delta(T - t)} N(-d_1) \]

\[ = 50e^{-0.07(0.75)} N(-0.09074) \]

\[ = 50e^{-0.07(0.75)} \times 0.46385 \]

\[ = 22.00630 \]
Solution 7

B Chapter 22, All-or-Nothing Options

For the square of the final stock price to be less than 100, the final stock price must be
less than 10:

\[ [S(1)]^2 < 100 \quad \Leftrightarrow \quad S(1) < 10 \]

Therefore, the option described in the question is 100 cash-or-nothing put options that
have a strike price of 10. The current value of the option is:

\[ 100 \times \text{CashPut}(S, K, T) = 100 \times e^{-rT} N(-d_2) = 100 \times e^{-0.03 \times 1} N(-d_2) \]

To find the delta of the option, we must find the derivative of the price with respect to the
stock price:

\[
\frac{\partial \left(100 \times e^{-0.03} N(-d_2)\right)}{\partial S} = 100e^{-0.03} \frac{\partial (N(-d_2))}{\partial S} = -100e^{-0.03} N'(d_2) \frac{\partial d_2}{\partial S}
\]

The derivative of \(d_2\) with respect to the stock price is:

\[
\frac{\partial d_2}{\partial S} = \frac{\partial (d_1 - \sigma \sqrt{T})}{\partial S} = \left(\frac{\ln \left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) + \frac{\sigma^2}{2} \frac{T}{2}}{\sigma \sqrt{T}} - \sigma \sqrt{T}\right) = \frac{Ke^{-rT} \times e^{-\delta T}}{Se^{-\delta T} \times Ke^{-rT}} \sigma \sqrt{T} = \frac{1}{S \sigma \sqrt{T}}
\]

The current values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln \left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) + \frac{\sigma^2}{2} \frac{T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left(\frac{10e^{-0.02}}{10e^{-0.03}}\right) + 0.3^2}{0.30 \sqrt{1}} = 0.18333
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = 0.18333 - 0.30 \sqrt{1} = -0.11667
\]

The density function for the standard normal random variable is:

\[ N'(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} \]
We can now calculate the delta of the option:

\[
-100e^{-0.03} N'(-d_2) \frac{\partial d_2}{\partial S} = -100e^{-0.03} \frac{1}{\sqrt{2\pi}} e^{-0.5d_2^2} \times \frac{1}{3}
\]

\[
= -100e^{-0.03} \frac{1}{\sqrt{2\pi}} \times e^{-0.5 \times (-0.11667)^2} \times \frac{1}{3} = -12.81753
\]

Since the option was sold, the resulting delta of the position is:

\[
-1 \times (-12.81753) = 12.81753
\]

Therefore, 12.81753 shares of stock must be sold to hedge the position. Or, said differently, the quantity to be purchased is \(-12.81753\).

**Solution 8**

**A Chapter 22, All-or-Nothing Options**

The option can be replicated by purchasing 100 cash calls with a strike price of $100 and selling 50 cash calls with a strike price of $150. Therefore, the price of the option is:

\[
100 \times \text{CashCall}(100,100,1) - 50 \times \text{CashCall}(100,150,1)
\]

First we find the value of the cash call with a strike price of $100:

\[
d_1 = \frac{\ln \left( \frac{S e^{-rT}}{K e^{-rT}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{100 e^{-0.03 \times 1}}{100 e^{-0.09 \times 1}} \right) + 0.3^2 \times 1}{0.3 \sqrt{1}} = 0.35000
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.35000 - 0.3 = 0.05000
\]

\[
N(d_2) = N(0.05000) = 0.51994
\]

\[
\text{CashCall}(100,100,1) = e^{-rT} N(d_2) = e^{-0.09 \times 1}(0.51994) = 0.47519
\]

Next, we calculate the value of the cash call with a strike price of $150:

\[
d_1 = \frac{\ln \left( \frac{S e^{-rT}}{K e^{-rT}} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{100 e^{-0.03 \times 1}}{150 e^{-0.09 \times 1}} \right) + 0.3^2 \times 1}{0.3 \sqrt{1}} = -1.00155
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = -1.00155 - 0.3 = -1.30155
\]

\[
N(d_2) = N(-1.30155) = 0.09654
\]

\[
\text{CashCall}(100,150,1) = e^{-rT} N(d_2) = e^{-0.09 \times 1}(0.09654) = 0.088231
\]

The value of the option described in the question is:

\[
100 \times \text{CashCall}(100,100,1) - 50 \times \text{CashCall}(100,150,1) = 100 \times 0.47519 - 50 \times 0.088231 = 43.1074
\]
**Solution 9**

C Chapter 22, Collect-on-Delivery Call

The COD call consists of a regular call option and the sale of $P$ cash calls. Let’s begin by finding the value of the regular call.

The first step is to calculate $d_1$ and $d_2$:

$$d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = \frac{\ln(100 / 105) + (0.07 - 0.02 + 0.5 \times 0.2^2) \times 2}{0.2\sqrt{2}}$$

$$d_1 = 0.32248$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.32248 - 0.20\sqrt{2} = 0.03964$$

We have:

$$N(d_1) = N(0.32248) = 0.62646$$

$$N(d_2) = N(0.03964) = 0.51581$$

The value of the European call option is:

$$C_{Eur} = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$= 100e^{-0.02(2)} \times 0.62646 - 105e^{-0.07(2)} \times 0.51581 = 13.10513$$

The value of a cash call is:

$$CashCall(105) = e^{-r(T-t)}N(d_2) = e^{-0.07(2)} \times 0.51581 = 0.44842$$

The initial cost of a COD call is zero:

$$C_{Eur} - P \times CashCall(K) = 0$$

$$13.10513 - P \times 0.44842 = 0$$

$$P = 29.2249$$

If the final stock price is $130, then the payout is:

$$S(T) - K - P = 130 - 105 - 29.2249 = -4.2249$$

**Solution 10**

D Chapter 22, Collect-on-Delivery Call

The COD call consists of a regular call option and the sale of $P$ cash calls. Let’s begin by finding the value of the regular call.

The first step is to calculate $d_1$ and $d_2$:

$$d_1 = \frac{\ln(S / K) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = \frac{\ln(100 / 105) + (0.07 - 0.02 + 0.5 \times 0.2^2) \times 2}{0.2\sqrt{2}}$$

$$d_1 = 0.32248$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.32248 - 0.20\sqrt{2} = 0.03964$$
We have:

\[ N(d_1) = N(0.32248) = 0.62646 \]
\[ N(d_2) = N(0.03964) = 0.51581 \]

The value of the European call option is:

\[
C_{Eur} = S e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)
\]
\[
= 100e^{-0.02(2)} \times 0.62646 - 105e^{-0.07(2)} \times 0.51581 = 13.10513
\]

The value of a cash call is:

\[
CashCall(105) = e^{-r(T-t)} N(d_2) = e^{-0.07(2)} \times 0.51581 = 0.44842
\]

The initial cost of a COD call is zero:

\[
C_{Eur} - P \times CashCall(K) = 0
\]
\[
13.10513 - P \times 0.448424 = 0
\]
\[
P = 29.2249
\]

The COD call consists of a regular call option and the sale of \(P\) cash calls.

The delta of the European call option is:

\[
\Delta_{Call} = e^{-\delta(T-t)} N(d_1) = e^{-0.02(2)} \times 0.62646 = 0.601896
\]

The delta of the cash call is:

\[
\frac{\partial}{\partial S} CashCall(K) = \frac{\partial}{\partial S} \left[ e^{-r(T-t)} N(d_2) \right] = e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S}
\]
\[
= e^{-r(T-t)} \times \frac{1}{\sqrt{2\pi}} e^{-0.5(d_2)^2} \times \frac{1}{S\sigma\sqrt{T}}
\]
\[
= e^{-0.07(2)} \times \frac{1}{\sqrt{2\pi}} e^{-0.5(0.03964)^2} \times \frac{1}{100(0.2)\sqrt{2}}
\]
\[
= 0.012252
\]

The delta of the COD call is the delta of the regular call minus \(P\) times the delta of the cash call:

\[
0.601896 - 29.2249 \times 0.012252 = 0.2438
\]

Since the market-maker sold the COD call, the market-maker must purchase 0.2438 shares in order to delta-hedge the position.
Solution 11

E  Chapter 22, Supershares

The payoff from having 1 share of the index at time 1 can be replicated with 50 of Supershare 1, 120 of Supershare 2, and 170 of Supershare 3. Therefore, the prepaid forward price of 1 share of the index is:

\[ F_{0,1}^P(\text{Index}) = 50 \times 0.4298 + 120 \times 0.2269 + 170 \times 0.2481 = 90.8950 \]

The index pays dividends at a rate of 4%, so purchasing the index now results in the ownership of \( e^{0.04} \) shares at the end of 1 year. Therefore, the current price of the index is:

\[ e^{0.04} \times 90.8950 = 94.6045 \]

Solution 12

E  Chapter 22, Supershares

The payoff from having $1 at time 1 can be replicated with 1 of Supershare 1, 1 of Supershare 2, and 1 of Supershare 3. Therefore, the price now of $1 at the end of the year is:

\[ e^{-r(1)} = 1 \times 0.4298 + 1 \times 0.2269 + 1 \times 0.2481 \]

\[ r = 10.00\% \]

Solution 13

D  Chapter 22, Supershares

The asset put option pays $50 if the stock price is $50, and it pays $120 if the stock price is $120. Therefore, the current value of the asset put option is:

\[ 50 \times 0.4298 + 120 \times 0.2269 = 48.718 \]

Solution 14

C  Chapter 22, Supershares

We do not need to know the dividend yield on the index to solve this problem.

We can establish a system of 3 equations and 3 unknowns that allows us to solve for the supershare prices:

\[ e^{-0.08 \times 0.5} = SS_1 + SS_2 + SS_3 \]
\[ 12.49 = 65 \times SS_1 \]
\[ 0.67 = SS_3 \]
The solution to this system is:

\[
SS_1 = \frac{12.49}{65} = 0.1922
\]
\[
SS_2 = e^{-0.04} - 0.1922 - 0.67 = 0.0986
\]
\[
SS_3 = 0.67
\]

The product is:

\[
SS_1 \times SS_2 \times SS_3 = 0.1922 \times 0.0986 \times 0.67 = 0.01270
\]

Solution 15

C  Chapter 22, Asset-or-Nothing Options

The investor can replicate the payoff by purchasing an asset call and selling 100 cash puts. Therefore, the current value of the payoff is:

\[
AssetCall(100) - 100 \times CashPut(100) = S_t e^{-\delta(T-t)} N(d_1) - 100 \times e^{-r(T-t)} N(-d_2)
\]
\[
= e^{-r} \left[ 90 e^{-\delta} N(d_1) - 100 N(-d_2) \right]
\]
\[
= 100 e^{-r} \left[ N(d_1) - N(-d_2) \right] \quad \text{(since } 90 e^{-\delta} = 100)\]

Once again, we can make use of the substitution \(90 e^{-\delta} = 100\) in determining the values of \(d_1\) and \(d_2\):

\[
d_1 = \ln(S/K) + (r - \delta + 0.5 \sigma^2)(T-t) / \sigma \sqrt{T-t} = \ln \left( \frac{90}{90 e^{(r-\delta)}} \right) + (r - \delta + 0.5 \times \sigma^2) \times 1 / \sigma \sqrt{1}
\]
\[
d_1 = 0.5 \sigma
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = 0.5 \sigma - \sigma \sqrt{1} = -0.5 \sigma
\]

Therefore:

\[
d_1 = -d_2
\]

The current value of the payoff is:

\[
100 e^{-r} \left[ N(d_1) - N(-d_2) \right] = 100 e^{-r} \left[ N(d_1) - N(d_1) \right] = 0
\]

Solution 16

B  Chapter 22, Cash-Or-Nothing Call Option

A European call option has the same gamma as an otherwise equivalent European put option. Therefore, the gamma of a call option with the payoff described below is 0.035:

\[
S_T - 60 \quad \text{if } S_T > 60
\]
Exam MFE/3F Solutions  Chapter 22 – Exotic Options II

The gap call option described in the question has a gamma of 0.03, and its payoff is:

\[ S_T - 55 \quad \text{if } S_T > 60 \]

The cash-or-nothing call option has the following payoff:

\[ 1,000 \quad \text{if } S_T > 60 \]

The cash-or-nothing call option can be replicated by purchasing 200 of the gap call options and selling 200 of the call options:

\[ 200[GapCall - Call] = 200[(S_T - 55) - (S_T - 60)] = 200 \times 5 = 1,000 \quad \text{if } S_T > 60 \]

The gamma of the position is the gamma of a position consisting of 200 long gap calls and 200 short calls is:

\[ 200[0.03 - 0.035] = 200 \times (-0.005) = -1.00 \]

Solution 17

A  Chapter 22, Cash-Or-Nothing Call Option

A European call option has the same vega as an otherwise equivalent European put option. Therefore, the vega of a call option with the payoff described below is 0.185:

\[ S_T - 60 \quad \text{if } S_T > 60 \]

The gap call option described in the question has a vega of 0.160, and its payoff is:

\[ S_T - 55 \quad \text{if } S_T > 60 \]

The cash-or-nothing call option has the following payoff:

\[ 1,000 \quad \text{if } S_T > 60 \]

The cash-or-nothing call option can be replicated by purchasing 200 of the gap call options and selling 200 of the call options:

\[ 200[GapCall - Call] = 200[(S_T - 55) - (S_T - 60)] = 200 \times 5 = 1,000 \quad \text{if } S_T > 60 \]

The vega of the position is the vega of a position consisting of 200 long gap calls and 200 short calls is:

\[ 200[0.160 - 0.185] = 200 \times (-0.025) = -5.00 \]

Solution 18

D  Chapter 22, Cash-Or-Nothing Call Option

The cash-or-nothing put option pays:

\[ 1,000 \quad \text{if } S_T < 60 \]

A gap put option with a strike price of 55 and a trigger price of 60 pays:

\[ 55 - S_T \quad \text{if } S_T < 60 \]
A regular put option with a strike price of 60 pays:

\[ 60 - S_T \quad \text{if } S_T < 60 \]

The cash-or-nothing put option can be replicated by purchasing 200 of the put options and selling 200 of the gap put options:

\[ 200[\text{Put} - \text{GapPut}] = 200[(60 - S_T) - (55 - S_T)] = 200 \times 5 = 1,000 \quad \text{if } S_T < 60 \]

We use put-call parity for gap calls and gap puts to find the theta of the gap put:

\[
\begin{align*}
\text{GapCall} + K_1 e^{-r(T-t)} &= S e^{-\delta(T-t)} + \text{GapPut} \\
\text{GapCall} + 55e^{-0.12(0.75-T)} &= 60e^{-0.07(0.75-T)} + \text{GapPut} \\
\theta_{\text{GapCall}} + (0.12)55e^{-0.12(0.75-t)} &= (0.07)60e^{-0.07(0.75-t)} + \theta_{\text{GapPut}} \\
-3.5 + (0.12)55e^{-0.12(0.75-t)} &= (0.07)60e^{-0.07(0.75-t)} + \theta_{\text{GapPut}}
\end{align*}
\]

Evaluating at time \( t = 0 \), we have:

\[ -3.5 + (0.12)55e^{-0.12(0.75-0)} = (0.07)60e^{-0.07(0.75-0)} + \theta_{\text{GapPut}} \]

\[ \theta_{\text{GapPut}} = -1.4532 \]

The theta of the cash-or-nothing put is the theta of a position consisting of 200 long puts and 200 short gap puts:

\[ 200[\theta_{\text{Put}} - \theta_{\text{GapPut}}] = 200[-1.0 - (-1.4532)] = 200 \times 0.4532 = 90.65 \]

**Solution 19**

**C** Chapter 22, Cash-Or-Nothing Call Option

The call option has the payoff described below:

\[ S_T - 60 \quad \text{if } S_T > 60 \]

The gap call option has the payoff described below:

\[ S_T - 55 \quad \text{if } S_T > 60 \]

The cash-or-nothing call option has the following payoff:

\[ 1,000 \quad \text{if } S_T > 60 \]

The cash-or-nothing call option can be replicated by purchasing 200 of the gap call options and selling 200 of the call options:

\[ 200[\text{GapCall} - \text{Call}] = 200[(S_T - 55) - (S_T - 60)] = 200 \times 5 = 1,000 \quad \text{if } S_T > 60 \]

Let \( C \) denote the cost of the regular call option. The value of the replicating portfolio is:

\[ 200(1.5C - C) = 100C \]
We can now determine the weights of the gap calls and calls in the portfolio. Note that the portfolio consists of a short position in the calls, so the calls have a negative weight:

\[
\omega_{\text{GapCall}} = \frac{200(1.5C)}{100C} = 3 \\
\omega_{\text{Call}} = \frac{-200(C)}{100C} = -2
\]

The elasticity of the replicating portfolio is:

\[
\omega_{\text{GapCall}} \times \Omega_{\text{GapCall}} + \omega_{\text{Call}} \times \Omega_{\text{Call}} = 3 \times 6 - 2 \times 7 = 18 - 14 = 4
\]

Solution 20

C Chapter 22, Early Asset-or-Nothing Put

Since \( K = 1,000(1 - 0.35) = 650 \), the value of \( d_1 \) is:

\[
d_1 = \frac{\ln \left( \frac{S_1}{K} \right) + [r - \delta + 0.5\sigma^2](T - t)}{\sigma \sqrt{T - t}} = \frac{\ln (1,000/650) + [0.12 - 0.05 + 0.5(0.22)^2](1 - 0)}{0.22\sqrt{1 - 0}} = 2.38629
\]

The value of \( N(-d_1) \) is:

\[
N(-d_1) = N(-2.38629) = 0.00851
\]

The price of one of the asset-or-nothing put options is:

\[
AssetPut(K) = S_1 e^{-\delta(T-t)}N(-d_1) = 1,000e^{-0.05(1-0)} \times 0.00851 = 8.094962
\]

Therefore, the price of one million of the options is:

\[
1,000,000 \times 8.094962 = 8,094,962
\]

Solution 21

B Chapter 22 Delta-Hedging Gap Call Options

The payoff of each gap call can be written as the payoff of a regular call minus 20 cash calls:

\[
\text{GapCall Payoff} = S_1 - 70 \quad \text{if } S_1 > 50 \\
\quad = (S_1 - 50) - 20 \quad \text{if } S_1 > 50
\]

Therefore, the price of a gap call option is equal to a price of a regular call minus the price of 20 cash calls:

\[
\text{GapCall} = \text{Call}(50) - 20 \times \text{CashCall}(50)
\]
Therefore, the delta of one gap call option is:

\[ \Delta_{\text{GapCall}} = \Delta_{\text{Call}(50)} - 20\Delta_{\text{CashCall}(50)} \]

To find the deltas of the regular call and the cash call, we need to calculate \( d_1 \) and \( d_2 \):

\[
d_1 = \frac{\ln \left( \frac{Se^{-\delta T}}{K} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{50e^{-0.05\times1}}{50e^{-0.05\times1}} \right) + \frac{0.5^2 \times 1}{2}}{0.50\sqrt{1}} = 0.35000 \Rightarrow N(d_1) = 0.63683
\]

\[ d_2 = d_1 - 0.50\sqrt{T} = 0.35000 - 0.50\sqrt{1} = -0.15000 \]

The deltas of the regular call and the cash call are:

\[
\Delta_{\text{Call}} = e^{-\delta(T-t)} N(d_1) = e^{0\times1} (0.63683) = 0.63683
\]

\[
\Delta_{\text{CashCall}(50)} = \frac{e^{-r(T-t)N'(d_2)}}{S\sigma\sqrt{T-t}} = \frac{e^{-0.05\times1 \times -0.5(-0.15)^2}}{50 \times 0.5\sqrt{2\pi \times 1}} = 0.015001
\]

Now we can calculate the delta of the gap call option:

\[ \Delta_{\text{GapCall}} = \Delta_{\text{Call}(50)} - 20 \times \Delta_{\text{CashCall}(50)} = 0.63683 - 20 \times 0.015001 = 0.33664 \]

Since the market-maker sells 1,000 of the gap call options, the market-maker multiplies the delta of one gap call option by 1,000 to determine the number of shares that must be purchased in order to delta-hedge the position:

\[ 1,000 \times 0.33664 = 336.64 \]

**Solution 22**

**D** Chapter 22, Asset-or-Nothing Power Option

Since \( S(t) \) is a geometric Brownian motion, so is \([S(t)]^a\):

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \Rightarrow \frac{d([S(t)]^a)}{[S(t)]^a} = \left[ \alpha(\alpha - \delta) + 0.5\alpha(\alpha - 1)\sigma^2 \right] dt + \alpha \sigma dZ(t)
\]

The coefficient of \( dZ(t) \) is the asset’s volatility. For \([S(t)]^4\), let’s call this volatility \( \sigma^* \):

\[ \sigma^* = \alpha \sigma = 4 \times 0.32 = 1.28 \]

The dividend yield on \([S(t)]^4\) is:

\[ \delta^* = r - \alpha(r - \delta) - 0.5\alpha(\alpha - 1)\sigma^2 = 0.12 - 4(0.12 - 0.06) - 0.5(4)(4 - 1)(0.32)^2 = -0.7344 \]

The initial price of \([S(t)]^4\) is:

\[ [S(0)]^4 = 2^4 = 16 \]
We use \([S(1)]^4\) as the underlying asset, and the trigger price that corresponds to this underlying asset is 16:

\[
S(1) > 2 \Rightarrow [S(1)]^4 > 16 \Rightarrow K^* = 16
\]

The first step is to calculate \(d_1\):

\[
d_1 = \frac{\ln(S^4 / K^*) + \left( r - (\delta^*) + 0.5(\sigma^*)^2 \right) T}{(\sigma^*)\sqrt{T}}
\]

\[
= \frac{\ln(16/16) + (0.12 - (-0.7344) + 0.5 \times 1.28^2) \times 1}{1.28\sqrt{1}} = 1.30750
\]

We have:

\[
N(d_1) = N(1.30750) = 0.90448
\]

The value of the asset-or-nothing power call option is:

\[
AssetCall(16) = [S(0)]^4 e^{-(\delta^*)T} N(d_1) = 16e^{(-0.7344) \times 1} \times 0.90448 = 30.1623
\]

**Solution 23**

**B Chapter 22, Asset-Or-Nothing Call Option**

A European call option has the same gamma as an otherwise equivalent European put option. Therefore, the gamma of a call option with the payoff described below is 0.035:

\[
S_T - 60 \quad \text{if} \quad S_T > 60
\]

The gap call option described in the question has a gamma of 0.03, and its payoff is:

\[
S_T - 55 \quad \text{if} \quad S_T > 60
\]

The asset-or-nothing call option has the following payoff:

\[
S_T \quad \text{if} \quad S_T > 60
\]

We need to use the call and the gap call to build an option that pays off for some amount of stock when the final stock price is higher than 60. Purchasing 60 of the gap calls and selling 55 of the call options leaves us with a payout equal to that of 5 asset calls:

\[
60 \times \text{GapCall} - 55 \times \text{Call} = 60(S_T - 55) - 55(S_T - 60) = 5S_T \quad \text{if} \quad S_T > 60
\]

Therefore, the gamma of 5 asset calls is:

\[
60 \times \Gamma_{GapCall} - 55 \times \Gamma_{Call} = 60 \times 0.03 - 55 \times 0.035 = -0.125
\]

The gamma of 1 asset call is found by dividing by 5:

\[
\frac{-0.125}{5} = -0.025
\]
Chapter 23 – Solutions

Solution 1

Chapter 11, Estimating Volatility

Since we have 6 weeks of data, we can calculate 5 weekly returns. Each weekly return is calculated as a continuously compounded rate:

\[ \eta_i = \ln \left( \frac{S_i}{S_{i-1}} \right) \]

The next step is to calculate the average of the returns:

\[ \bar{\eta} = \frac{1}{k} \sum_{i=1}^{k} \eta_i \]

The returns and their average are shown in the third column below:

<table>
<thead>
<tr>
<th>Date</th>
<th>Price</th>
<th>( \eta_i = \ln \left( \frac{S_i}{S_{i-1}} \right) )</th>
<th>( (\eta_i - \bar{\eta})^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10/11/2006</td>
<td>85</td>
<td>-0.048202</td>
<td>0.004381</td>
</tr>
<tr>
<td>10/18/2006</td>
<td>81</td>
<td>-0.048202</td>
<td>0.004381</td>
</tr>
<tr>
<td>10/25/2006</td>
<td>87</td>
<td>0.071459</td>
<td>0.002859</td>
</tr>
<tr>
<td>11/01/2006</td>
<td>80</td>
<td>-0.083881</td>
<td>0.010378</td>
</tr>
<tr>
<td>11/08/2006</td>
<td>86</td>
<td>0.072321</td>
<td>0.002952</td>
</tr>
<tr>
<td>11/15/2006</td>
<td>93</td>
<td>0.078252</td>
<td>0.003632</td>
</tr>
</tbody>
</table>

\[ \bar{\eta} = 0.017990 \quad \sum_{i=1}^{5} (\eta_i - \bar{\eta})^2 = 0.024201 \]

The fourth column shows the squared deviations and the sum of squares.

The estimate for the standard deviation of the weekly returns is:

\[ \hat{\sigma}_w = \sqrt{\frac{\sum_{i=1}^{k} (\eta_i - \bar{\eta})^2}{k - 1}} \]

\[ \hat{\sigma} = \hat{\sigma}_w \sqrt{\frac{1}{h}} = \sqrt{0.024201} \]

We adjust the weekly volatility to obtain the annual volatility:

\[ \hat{\sigma} = \hat{\sigma}_w \sqrt{\frac{1}{h}} = 0.077784 \sqrt{52} = 0.561 \]
This problem isn’t very difficult if you are familiar with the statistical function on your calculator. We recommend using the TI-30XS MultiView.

Using the TI-30XS MultiView, the procedure is:

- [data] [data] 4 (to clear the data table)
- (enter the data below)
- L1  | L2  | L3
- 85  | 81  | ————
- 81  | 87  |
- 87  | 80  |
- 80  | 86  |
- 86  | 93  |

(place cursor in the L3 column)

- [data] → (to highlight FORMULA)
- 1  [ln] [data] 2 / [data] 1 [enter]
- [2nd] [quit] [2nd] [stat] 1

DATA: (highlight L3)   FRQ: (highlight one)   (select CALC)   [enter]
- 3 (to obtain Sx)

$Sx \times \sqrt{52}$ [enter] The result is 0.560909261

On the TI-30X IIS, the steps are:

- [2nd] [STAT] (Select 1-VAR) [ENTER]
- [DATA]

- X1= $\ln\left(\frac{81}{85}\right)$ [ENTER] ↓↓ (Hit the down arrow twice)
- X2= $\ln\left(\frac{87}{81}\right)$ [ENTER] ↓↓
- X3= $\ln\left(\frac{80}{87}\right)$ [ENTER] ↓↓
- X4= $\ln\left(\frac{86}{80}\right)$ [ENTER] ↓↓
- X5= $\ln\left(\frac{93}{86}\right)$ [ENTER]

- [STATVAR] → → (Arrow over to Sx)

- $\sqrt{52}$ [ENTER]

The result is: 0.560909261

To exit the statistics mode:
- [2nd] [EXITSTAT] [ENTER]
Solution 2

D Chapter 11, Estimating Volatility

Since we have 7 months of data, we can calculate 6 monthly returns. Each monthly return is calculated as a continuously compounded rate:

\[ r_i = \ln \left( \frac{S_i}{S_{i-1}} \right) \]

The next step is to calculate the average of the returns:

\[ \bar{r} = \frac{1}{k} \sum_{i=1}^{k} r_i \]

The returns and their average are shown in the third column below:

<table>
<thead>
<tr>
<th>Date</th>
<th>Price</th>
<th>[ r_i = \ln \left( \frac{S_i}{S_{i-1}} \right) ]</th>
<th>( (r_i - \bar{r})^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>110</td>
<td>0.095310</td>
<td>0.004215</td>
</tr>
<tr>
<td>3</td>
<td>112</td>
<td>0.018019</td>
<td>0.000153</td>
</tr>
<tr>
<td>4</td>
<td>105</td>
<td>-0.064539</td>
<td>0.009011</td>
</tr>
<tr>
<td>5</td>
<td>113</td>
<td>0.073427</td>
<td>0.001852</td>
</tr>
<tr>
<td>6</td>
<td>115</td>
<td>0.017544</td>
<td>0.000165</td>
</tr>
<tr>
<td>7</td>
<td>120</td>
<td>0.042560</td>
<td>0.000148</td>
</tr>
</tbody>
</table>

The fourth column shows the squared deviations and the sum of squares.

The estimate for the standard deviation of the monthly returns is:

\[ \hat{\sigma}_h = \sqrt{\frac{\sum_{i=1}^{k} (r_i - \bar{r})^2}{k-1}} \]

\[ \hat{\sigma}_{12} = \sqrt{\frac{0.015544}{5}} = 0.055757 \]

We adjust the monthly volatility to obtain the annual volatility:

\[ \hat{\sigma} = \hat{\sigma}_h \sqrt{\frac{1}{\frac{1}{12}}} = 0.055757 \sqrt{12} = 0.193 \]

This problem isn’t very difficult if you are familiar with the statistical function of your calculator.

On the TI-30X IIS, the steps are:

1. \[2^{nd}\] [STAT] (Select 1-VAR) [ENTER]
[DATA]

\[
\begin{align*}
X_1 &= \ln \left( \frac{110}{100} \right) \quad \text{[ENTER]} \quad \downarrow \downarrow \quad \text{(Hit the down arrow twice)} \\
X_2 &= \ln \left( \frac{112}{110} \right) \quad \text{[ENTER]} \quad \downarrow \\
X_3 &= \ln \left( \frac{105}{112} \right) \quad \text{[ENTER]} \quad \downarrow \\
X_4 &= \ln \left( \frac{113}{105} \right) \quad \text{[ENTER]} \quad \downarrow \\
X_5 &= \ln \left( \frac{115}{113} \right) \quad \text{[ENTER]} \quad \downarrow \\
X_6 &= \ln \left( \frac{120}{115} \right) \quad \text{[ENTER]}
\end{align*}
\]

[STATVAR] \rightarrow \rightarrow \quad \text{(Arrow over to } S_x \text{)}
\times \sqrt{12} \quad \text{[ENTER]}

The result is: 0.193149327
To exit the statistics mode:
[2\text{nd}] \text{[EXITSTAT]} \quad \text{[ENTER]}

On the BA II Plus calculator, the steps are:

[2\text{nd}][\text{DATA}] \quad [2\text{nd}][\text{CLR WORK}]

\[
\begin{align*}
110/100 &= \text{LN} \quad \text{[ENTER]} \quad \downarrow \downarrow \quad \text{(Hit the down arrow twice)} \\
112/110 &= \text{LN} \quad \text{[ENTER]} \quad \downarrow \\
105/112 &= \text{LN} \quad \text{[ENTER]} \quad \downarrow \\
113/105 &= \text{LN} \quad \text{[ENTER]} \quad \downarrow \\
115/113 &= \text{LN} \quad \text{[ENTER]} \quad \downarrow \\
120/115 &= \text{LN} \quad \text{[ENTER]}
\end{align*}
\]

[2\text{nd}][\text{STAT}] \quad \downarrow \downarrow \downarrow \times \sqrt{12} =

The result is: 0.19314933
To exit the statistics mode: [2\text{nd}][\text{QUIT}]

Solution 3

D Chapter 11, Estimating Volatility

Although the textbook’s example does not use dividends, it is not difficult to calculate the monthly returns when a discrete dividend is present.
Since we have 7 months of data, we can calculate 6 monthly returns. Each monthly return is calculated as a continuously compounded rate. The discrete dividend must be included in the return calculations:

\[ \eta_i = \ln \left( \frac{S_i + Div_i}{S_{i-1}} \right) \quad \text{where: } Div_i = \text{Dividend paid at time } i \]

In the table below, the fourth return is bolded, because it includes the $5 dividend:

\[ \eta_5 = \ln \left( \frac{S_5 + Div_5}{S_4} \right) = \ln \left( \frac{113 + 5}{105} \right) = 0.116724 \]

The next step is to calculate the average of the returns:

\[ \bar{\eta} = \frac{\sum_{i=1}^{k} \eta_i}{k} \]

The returns and their average are shown in the third column below:

<table>
<thead>
<tr>
<th>Date</th>
<th>Price</th>
<th>( \eta_i = \ln \left( \frac{S_i + Div_i}{S_{i-1}} \right) )</th>
<th>( (\eta_i - \bar{\eta})^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>110</td>
<td>0.095310</td>
<td>0.003330</td>
</tr>
<tr>
<td>3</td>
<td>112</td>
<td>0.018019</td>
<td>0.000384</td>
</tr>
<tr>
<td>4</td>
<td>105</td>
<td>-0.064539</td>
<td>0.010433</td>
</tr>
<tr>
<td>5</td>
<td>113</td>
<td><strong>0.116724</strong></td>
<td>0.006260</td>
</tr>
<tr>
<td>6</td>
<td>115</td>
<td>0.017544</td>
<td>0.000402</td>
</tr>
<tr>
<td>7</td>
<td>120</td>
<td>0.042560</td>
<td>0.000025</td>
</tr>
</tbody>
</table>

\[ \bar{\eta} = 0.037603 \quad \sum_{i=1}^{6} (\eta_i - \bar{\eta})^2 = 0.020834 \]

The fourth column shows the squared deviations and the sum of squares.

The estimate for the standard deviation of the monthly returns is:

\[ \sigma_h = \sqrt{\frac{\sum_{i=1}^{k} (\eta_i - \bar{\eta})^2}{k-1}} \]

\[ \sigma_{\frac{1}{12}} = \sqrt{\frac{0.020834}{5}} = 0.064550 \]

We adjust the monthly volatility to obtain the annual volatility:

\[ \sigma = \sigma_h \sqrt{\frac{1}{h}} = 0.064550 \sqrt{12} = 0.2236 \]

*This problem isn’t very difficult if you are familiar with the statistical function of your calculator.*
On the TI-30X IIS, the steps are:

1. Press `[2nd] [STAT]` (Select 1-VAR) `[ENTER]`
2. Press `[DATA]`
3. For each X1, X2, X3, X4, X5, X6, enter the values as follows:
   - X1 = \( \ln \left( \frac{110}{100} \right) \) [ENTER] (Hit the down arrow twice)
   - X2 = \( \ln \left( \frac{112}{110} \right) \) [ENTER] (Hit the down arrow twice)
   - X3 = \( \ln \left( \frac{105}{112} \right) \) [ENTER] (Hit the down arrow twice)
   - X4 = \( \ln \left( \frac{118}{105} \right) \) [ENTER] (Hit the down arrow twice)
   - X5 = \( \ln \left( \frac{115}{113} \right) \) [ENTER] (Hit the down arrow twice)
   - X6 = \( \ln \left( \frac{120}{115} \right) \) [ENTER]
4. Press `[STATVAR]` → → (Arrow over to Sx)
5. Press \( \times \sqrt{12} \) [ENTER]

The result is: 0.223608532

To exit the statistics mode: [2nd] [EXITSTAT] [ENTER]

On the BA II Plus calculator, the steps are:

1. Press `[2nd] [DATA]` [2nd] [CLR WORK]
2. For each X1, X2, X3, X4, X5, X6, enter the values as follows:
   - X1 = \( \ln \left( \frac{110}{100} \right) \) [ENTER] (Hit the down arrow twice)
   - X2 = \( \ln \left( \frac{112}{110} \right) \) [ENTER] (Hit the down arrow twice)
   - X3 = \( \ln \left( \frac{105}{112} \right) \) [ENTER] (Hit the down arrow twice)
   - X4 = \( \ln \left( \frac{118}{105} \right) \) [ENTER] (Hit the down arrow twice)
   - X5 = \( \ln \left( \frac{115}{113} \right) \) [ENTER] (Hit the down arrow twice)
   - X6 = \( \ln \left( \frac{120}{115} \right) \) [ENTER]
3. Press `[2nd] [STAT]` [2nd] [CLR WORK] \( \times \sqrt{12} \) [ENTER]

The result is: 0.22360853

To exit the statistics mode: [2nd] [QUIT]
Solution 4

D Chapter 18, Estimated Standard Deviation

The quickest way to do this problem is to input the data into a calculator.

Using the TI-30X IIS, the procedure is:

- [2nd][STAT] (Select 1-VAR) [ENTER]
- [DATA]
- X1= ln(104/100.00) [ENTER] ↓↓ (Hit the down arrow twice)
- X2= ln(97/104) [ENTER] ↓↓
- X3= ln(95/97) [ENTER] ↓↓
- X4= ln(103/95) [ENTER]
- [STATVAR] → → (Arrow over to Sx)
- ×\sqrt{12} [ENTER]

The result is: 0.229314157

To exit the statistics mode: [2nd][EXITSTAT] [ENTER]

Alternatively, using the BA II Plus calculator, the procedure is:

- [2nd][DATA] [2nd][CLR WORK]
- X01= 104/100 = LN [ENTER] ↓↓ (Hit the down arrow twice)
- X02= 97/104 = LN [ENTER] ↓↓
- X03= 95/97 = LN [ENTER] ↓↓
- X04= 103/95 = LN [ENTER]
- [2nd][STAT] ↓↓↓ ×\sqrt{12} =

The result is: 0.22931416

To exit the statistics mode: [2nd][QUIT]

Solution 5

D Chapter 18, Estimated Parameters of the Lognormal Distribution

The quickest way to do this problem is to input the data into a calculator.

Using the TI-30X IIS, the procedure is:

- [2nd][STAT] (Select 1-VAR) [ENTER]
- [DATA]
\[
X_1 = \ln\left(\frac{104}{100.00}\right) \quad \text{(Hit the down arrow twice)}
\]
\[
X_2 = \ln\left(\frac{97}{104}\right) \quad \text{↓↓}
\]
\[
X_3 = \ln\left(\frac{95}{97}\right) \quad \text{↓↓}
\]
\[
X_4 = \ln\left(\frac{103}{95}\right) \quad \text{[ENTER]}
\]

\[
[\text{STATVAR}] \to \to \quad \text{(Arrow over to Sx)}
\]
\[
\times \sqrt{12} \quad \text{[ENTER]} \quad \text{(The result is 0.229314157)}
\]
\[
[\text{STATVAR}] \to \quad \text{(Arrow over to } \bar{X} \text{)}
\]
\[
\times 12 \quad \text{[ENTER]} \quad \text{(The result is 0.088676407)}
\]

To exit the statistics mode: [2nd] [EXITSTAT] [ENTER]

Alternatively, using the BA II Plus calculator, the procedure is:

\[
[\text{2nd}][\text{DATA}] \quad [\text{2nd}][\text{CLR WORK}]
\]
\[
X01 = \frac{104}{100} = \ln \quad \text{[ENTER]} \quad \text{(Hit the down arrow twice)}
\]
\[
X02 = \frac{97}{104} = \ln \quad \text{[ENTER]} \quad \text{↓↓}
\]
\[
X03 = \frac{95}{97} = \ln \quad \text{[ENTER]} \quad \text{↓↓}
\]
\[
X04 = \frac{103}{95} = \ln \quad \text{[ENTER]}
\]

\[
[\text{2nd}][\text{STAT}] \quad \text{↓↓↓} \times \sqrt{12} = \quad \text{(The result is 0.22931416)}
\]
\[
[\text{2nd}][\text{STAT}] \quad \text{↓↓} \times 12 = \quad \text{(The result is 0.08867641)}
\]

To exit the statistics mode: [2nd][QUIT]

Therefore, the annualized estimates for the mean and standard deviation of the normal distribution are:

\[
\hat{\alpha} - \delta - 0.5\hat{\sigma}^2 = 0.088676
\]
\[
\hat{\sigma} = 0.22931
\]

The estimate for the annualized expected return is:

\[
\hat{\alpha} = \frac{\bar{X}}{h} + \delta + 0.5\hat{\sigma}^2 = 0.088676 + 0 + 0.5 \times (0.22931)^2 = 0.114969
\]

We know that the stock price at the end of month 4 is $103. The expected value of the stock at the end of month 12 is:

\[
E[S_T] = S_t e^{(\alpha - \delta)(T - t)}
\]
\[
E[S_1] = S_{\frac{1}{12}} e^{(0.114969 - 0)(1 - 4/12)} = 103e^{0.114969 \times 2/3} = 111.20
\]
Solution 6

B Chapter 18, Estimated Parameters of the Lognormal Distribution

The quickest way to do this problem is to input the data into a calculator.

Using the TI-30XS MultiView, the procedure is:

1. [data] [data] 4 (to clear the data table)
2. (enter the data below)
    
    | L1  | L2  | L3 |
    |-----|-----|----|
    | 50.00 | 54.00 |  |
    | 54.00 | 48.00 |  |
    | 48.00 | 55.00 |  |
    | 55.00 | 61.00 |  |
    | 61.00 | 56.00 |  |
    | 56.00 | 54.00 |  |

3. (place cursor in the L3 column) [data] → (to highlight FORMULA)
4. 1 [ln] [data] 2 / [data] 1 ) [enter]
5. [2nd] [quit] [2nd] [stat] 1
6. Data: (highlight L3) FRQ: (highlight one) [enter]
7. 2 (to obtain $\bar{x}$)
8. $\bar{x} \times 12$ [enter] The result is 0.153922082
9. [2nd] [stat] 3
10. 3 (to obtain Sx)
11. $Sx \times \sqrt{12}$ [enter] The result is 0.368883615

Using the TI-30X IIS, the procedure is:

1. [2nd][STAT] (Select 1-VAR) [ENTER]
2. [DATA]
3. X1= ln(54/50) [ENTER] ↓↓ (Hit the down arrow twice)
4. X2= ln(48/54) [ENTER] ↓↓
5. X3= ln(55/48) [ENTER] ↓↓
6. X4= ln(61/55) [ENTER] ↓↓
7. X5= ln(56/61) [ENTER] ↓↓
8. X6= ln(54/56) [ENTER]
9. [STATVAR] → → (Arrow over to Sx)
10. $\times \sqrt{12}$ [ENTER] (The result is 0.368883615)
[STATVAR] → (Arrow over to $\bar{x}$)
$\times 12$ [ENTER] (The result is 0.153922082)

To exit the statistics mode: [2nd] [EXITSTAT] [ENTER]

Alternatively, using the BA II Plus calculator, the procedure is:

[2nd][DATA] [2nd][CLR WORK]
X01= 54/50 = LN [ENTER] ↓↓ (Hit the down arrow twice)
X02= 48/54 = LN [ENTER] ↓↓
X03= 55/48 = LN [ENTER] ↓↓
X04= 61/55 = LN [ENTER] ↓↓
X05= 56/61 = LN [ENTER] ↓↓
X06= 54/56 = LN [ENTER]

[2nd][STAT] ↓↓ $\times \sqrt{12}$ = (The result is 0.36888361)
[2nd][STAT] ↓↓ $\times 12$ = (The result is 0.15392208)

To exit the statistics mode: [2nd][QUIT]

Therefore, the annualized estimates for the mean and standard deviation of the normal distribution are:

$\hat{\alpha} - \delta - 0.5\hat{\sigma}^2 = 0.153922$
$\hat{\sigma} = 0.368884$

The estimate for the annualized expected return is:

$\hat{\alpha} = \frac{\bar{r}}{h} + \delta + 0.5\hat{\sigma}^2 = 0.153922 + 0 + 0.5 \times (0.368884)^2 = 0.221960$

Solution 7

Chapter 18, Annualized Expected Return

First, we determine the weekly mean and the weekly standard deviation using a calculator, and then we determine the annualized standard deviation and the annualized expected return.
Using the TI-30X IIS calculator, press [2nd] [STAT] and choose 1-VAR and press [ENTER] [DATA]. Perform the following sequence:

\[
X1 = \ln(51.0/50.5) \quad \text{[ENTER]} \\
X2 = \ln(52.0/51.0) \quad \text{[ENTER]} \\
X3 = \ln(51.5/52.0) \quad \text{[ENTER]} \\
X4 = \ln(51.0/51.5) \quad \text{[ENTER]} \\
X5 = \ln(52.0/51.0) \quad \text{[ENTER]} \\
X6 = \ln(52.5/52.0) \quad \text{[ENTER]}
\]

Press [STATVAR] and → and the weekly mean is shown as 0.0064733. Press → again and the weekly standard deviation is shown as 0.0132659. Multiply this number by the square root of 52 to get the annualized standard deviation of 0.09566. To exit the statistics mode, press [2nd] [EXITSTAT] [ENTER].

To determine the annualized expected return, we multiply the weekly mean return by 52 and we add to that amount one half of the annualized variance:

\[
0.006473 \times 52 + 0.5 \times 0.09566^2 = 0.34119
\]

Alternatively, we could use the BA II Plus Professional calculator. Press [2nd] [DATA] [2nd] [CLR WORK], and then perform the following sequence:

\[
X01 = 51.0/50.5 = \ln \quad \text{[ENTER]} \\
X02 = 52.0/51.0 = \ln \quad \text{[ENTER]} \\
X03 = 51.5/52.0 = \ln \quad \text{[ENTER]} \\
X04 = 51.0/51.5 = \ln \quad \text{[ENTER]} \\
X05 = 52.0/51.0 = \ln \quad \text{[ENTER]} \\
X06 = 52.5/52.0 = \ln \quad \text{[ENTER]}
\]

Press [2nd] [STAT] and [2nd] [SET] until you see 1-V. Then press ↓ to see that 6 data points have been entered. Press ↓ again and the weekly mean is shown as 0.0064733. Press ↓ again and the weekly standard deviation is shown as 0.0132656. Multiply this number by the square root of 52 to get the annualized standard deviation of 0.09566. To exit the statistics mode, press [2nd] [QUIT].

To determine the annualized expected return, we multiply the weekly mean return by 52 and we add to that amount one half of the annualized variance:

\[
0.006473 \times 52 + 0.5 \times 0.09566^2 = 0.34119
\]

**Solution 8**

C Chapter 23, Volatility Skew

*There is no such thing as a volatility wink.*
The speculator wants the volatility to be low for the option she is purchasing and high for the option she is selling. Therefore, she wants the volatility to be low when the strike price is high, and she wants the volatility to be high when the strike price is low. This describes a volatility smirk.

**Solution 9**

**E Chapter 23, Historical Volatility**

There is not enough information available for us to determine \( \bar{r} \), so we cannot estimate the volatility as:

\[
\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{k} (r_i - \bar{r})^2}{h(k-1)}}
\]

Therefore, we'll use the alternative method explained in Chapter 23 of the textbook.

The original volatility estimate is based on 10 prices, so there are 9 returns and \( k = 9 \):

\[
\hat{\sigma}_H = \frac{1}{\sqrt{h}} \times \sqrt{\frac{9}{k-1} \sum_{i=1}^{9} (r_i)^2}
\]

\[
0.22 = \frac{1}{\sqrt{1/252}} \times \sqrt{\frac{9}{9-1} \sum_{i=1}^{9} (r_i)^2}
\]

\[
\sum_{i=1}^{9} (r_i)^2 = 0.0015365
\]

After a week passes, the analyst can add 5 more returns into the sum of squared returns:

\[
\sum_{i=1}^{14} (r_i)^2 = \sum_{i=1}^{9} (r_i)^2 + \sum_{i=10}^{14} (r_i)^2 = 0.0015365 + \sum_{i=10}^{14} (r_i)^2
\]

\[
= 0.0015365 + \left[ \ln \left( \frac{102}{100} \right) \right]^2 + \left[ \ln \left( \frac{104}{102} \right) \right]^2 + \left[ \ln \left( \frac{106}{104} \right) \right]^2 + \left[ \ln \left( \frac{105}{106} \right) \right]^2 + \left[ \ln \left( \frac{101}{105} \right) \right]^2
\]

\[
= 0.0015365 + 0.0027304 + 0.0042669
\]

\[
= 0.0042669
\]

Now we can calculate the volatility using all 15 prices. There are 14 returns, so \( k = 14 \):

\[
\hat{\sigma}_H = \frac{1}{\sqrt{h}} \times \sqrt{\frac{14}{k-1} \sum_{i=1}^{14} (r_i)^2}
\]

\[
= \frac{1}{\sqrt{1/252}} \times \sqrt{\frac{14}{14-1} \sum_{i=1}^{14} (r_i)^2}
\]

\[
= \frac{1}{\sqrt{1/252}} \times \sqrt{0.0042669} = 0.2876
\]
Chapter 24 – Solutions

Solution 1
A Chapter 24, Duration-Hedging

The number of 10-year bonds that must be purchased to duration-hedge the position is:

\[ N = \left( \frac{T_2 - t}{T_1 - t} \right) \frac{P(t, T_2)}{P(t, T_1)} = \frac{(3 - 0) \times 85.40}{(10 - 0) \times 73.50} = -0.349 \]

Purchasing –0.349 of the 10-year bonds is equivalent to selling 0.349 of the 10-year bonds.

Solution 2
C Chapter 24, Delta-Hedging

The number of 10-year bonds that must be purchased to delta-hedge the position is:

\[ N = \frac{P_r(r, t, T_2)}{P_r(r, t, T_1)} = \frac{-1.93}{-3.18} = -0.607 \]

Purchasing –0.607 of the 10-year bonds is equivalent to selling 0.607 of the 10-year bonds.

Solution 3
E Chapter 24, Rendleman-Bartter Model

In the Rendleman-Bartter model, the short rate follows geometric Brownian motion:

\[ dr = ar\,dt + \sigma dZ \quad \Leftrightarrow \quad d[\ln r] = (a - 0.5\sigma^2)dt + \sigma dZ \]

Choice E is of this form with:

\[ a - 0.5\sigma^2 = 0.01 \quad \text{and} \quad \sigma = 0.02 \]

Solution 4
C Chapter 24, Vasicek Model

The process describes the Vasicek model with:

\[ a = 0.2 \]
\[ b = 0.1 \]
\[ \sigma = 0.02 \]

The Sharpe ratio is zero, so:

\[ \phi = 0 \]
We can use these values to find the value that the yield approaches as the time to maturity goes to infinity:

\[ \bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.1 + 0 \left( \frac{0.02}{0.2} \right) - 0.5 \frac{0.02^2}{0.2^2} = 0.095 \]

**Solution 5**

**B Chapter 24, Vasicek Model**

The process describes the Vasicek model with:

\[ a = 0.2 \]
\[ b = 0.1 \]
\[ \sigma = 0.02 \]

The Sharpe ratio is zero, so:

\[ \phi = 0 \]

We can use these values to find the price of the 10-year zero-coupon bond:

\[ \bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.1 + 0 \left( \frac{0.02}{0.2} \right) - 0.5 \frac{0.02^2}{0.2^2} = 0.095 \]

\[ B(t,T) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.2(10)}}{0.2} = 4.32332 \]

\[ A(t,T) = e^{\bar{r} \times [B(t,T) - (T-t)] - \frac{\{B(t,T)^2 \times \sigma^2\}}{4a}} = e^{0.095(4.32332 - 10) - \frac{4.32332^2 \times 0.02^2}{4 \times 0.2}} = 0.57774 \]

\[ P(r,t,T) = A(t,T)e^{-B(t,T)r} = 0.57774e^{-4.32332 \times 0.08} = 0.40881 \]

Since the bond matures for $100, its price is:

\[ 100 \times 0.40881 = 40.881 \]

**Solution 6**

**D Chapter 24, Vasicek Model & Forward Interest Rates**

The process describes the Vasicek model with:

\[ a = 0.2 \]
\[ b = 0.1 \]
\[ \sigma = 0.02 \]

We can use these values to find the price of a 10-year zero-coupon bond:

\[ B(0,10) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.2(10)}}{0.2} = 4.32332 \]

\[ A(0,10) = 0.57774 \]

\[ P(r,0,10) = A(0,10)e^{-B(0,10)r} = 0.57774e^{-4.32332 \times 0.08} = 0.40881 \]
We can also find the price of a 9-year zero-coupon bond:

\[
B(0,9) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.2(9)}}{0.2} = 4.17351
\]

\[
A(0,9) = 0.62674
\]

\[
P(r,0,9) = A(0,9)e^{-B(0,9)r} = 0.62674e^{-4.17351 \times 0.08} = 0.44883
\]

The forward interest rate, agreed upon now, that applies from time 9 to time 10 can be found with the following formula:

\[
R_0(T,T+s) = \frac{1}{P_0(T,T+s)} - 1 = \frac{P(r,0,T)}{P(r,0,T+s)} - 1
\]

\[
R_0(9,10) = \frac{P(r,0,9)}{P(r,0,10)} - 1 = \frac{0.44883}{0.40881} - 1 = 0.0979
\]

Since the amount of the loan is $100, the payment at the end of 10 years is:

\[
100 \times (1 + 0.0979) = 109.79
\]

**Solution 7**

**B Chapter 24, Rendleman-Bartter Model**

In the Rendleman-Bartter Model, the short-term rate follows geometric Brownian motion. From the Chapter 18 Review Note, we know that the probability that the short rate is greater than 10% at the end of 3 months can be found as:

\[
\text{Prob}[r(t) > 0.10] = N(\hat{d}_2)
\]

Although we have -0.25 for the volatility parameter in the geometric Brownian motion, we use the absolute value of -0.25 in the formula for \( \hat{d}_2 \), because the \( \sigma \) in the formula for \( \hat{d}_2 \) is the standard deviation of the stock's return, and it must therefore be positive:

\[
\hat{d}_2 = \ln\left(\frac{r(t)}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T-t) = \ln\left(\frac{0.09}{0.10}\right) + (0.10 - 0.5 \times 0.25^2) \times 0.25
\]

\[
= -0.70538
\]

The value of \( N(\hat{d}_2) \) is:

\[
N(\hat{d}_2) = N(-0.70538) = 0.24029
\]
Solution 8

A Chapter 24, Cox-Ingersoll-Ross Model

The process describes the Cox-Ingersoll-Ross model with:

\[ a = 0.18 \]
\[ b = 0.1 \]
\[ \sigma = 0.17 \]

The Sharpe ratio is zero, so \( \bar{\phi} \) is zero:

\[ \phi(r,t) = 0 \]
\[ \frac{\bar{\phi} \sqrt{r}}{\sigma} = 0 \]
\[ \bar{\phi} = 0 \]

We can use these values to find \( \gamma \), the yield as the maturity goes to infinity:

\[ \gamma = \sqrt{(a - \bar{\phi})^2 + 2\sigma^2} = \sqrt{(0.18 - 0)^2 + 2 \times 0.17^2} = 0.30033 \]
\[ \bar{r} = \frac{2ab}{(a - \bar{\phi} + \gamma)} = \frac{2(0.18)(0.10)}{(0.18 - 0 + 0.30033)} = 0.0749 \]

Solution 9

C Chapter 24, Forward Prices

We wish to find the price, agreed upon now to be paid in 2 years, for a 1-year bond:

\[ P_0(2,3) = F_{0,2}[P(2,3)] \]

This price is the ratio of the 3-year bond price to the 2-year bond price:

\[ P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} \]
\[ P_0(2,3) = \frac{P(0,3)}{P(0,2)} = \frac{0.7722}{0.8573} = 0.9007 \]

Solution 10

C Chapter 24, Black Model

This method of presenting forward price volatilities did not appear in the textbook’s description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.

The forward price volatilities are:

\[ 0.100 = \frac{\text{Var}\{\ln[P_t(1.2)]\}}{t} \]
\[ 0.105 = \frac{\text{Var}\{\ln[P_t(2.3)]\}}{t} \]
\[ 0.120 = \frac{\text{Var}\{\ln[P_t(3.4)]\}}{t} \]
The appropriate volatility for an option that expires in 2 years on a bond that matures in 3 years is:
\[ \sigma = 0.105 \]

The bond forward price is:
\[ F = P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)} = \frac{P(0, 3)}{P(0, 2)} = 0.9007 \]

The values of \( d_1 \) and \( d_2 \) are:
\[
\begin{align*}
    d_1 &= \frac{\ln \left( \frac{F}{K} \right) + 0.5 \sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{0.9007}{0.92} \right) + 0.5(0.105)^2 (2)}{0.105 \sqrt{2}} = -0.06827 \\
    d_2 &= d_1 - \sigma \sqrt{T} = -0.06827 - 0.105 \sqrt{2} = -0.21676
\end{align*}
\]

We have:
\[
\begin{align*}
    N(d_1) &= N(-0.06827) = 0.47279 \\
    N(d_2) &= N(-0.21676) = 0.41420
\end{align*}
\]

The Black formula for the call price is:
\[
C = P(0, T) \left[ F \times N(d_1) - K \times N(d_2) \right]
\]
\[
= 0.8573 \left[ 0.9007 \times 0.47279 - 0.92 \times 0.41420 \right]
\]
\[
= 0.03840
\]

**Solution 11**

B Chapter 24, Black Model

*This method of presenting forward price volatilities did not appear in the textbook’s description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.*

The forward price volatilities are:
\[
\begin{align*}
    0.100 &= \sqrt{\frac{\text{Var} \left\{ \ln [P_t(1,2)] \right\}}{t}} \\
    0.105 &= \sqrt{\frac{\text{Var} \left\{ \ln [P_t(2,3)] \right\}}{t}} \\
    0.120 &= \sqrt{\frac{\text{Var} \left\{ \ln [P_t(3,4)] \right\}}{t}}
\end{align*}
\]

The appropriate volatility for an option that expires in 2 years on a bond that matures in 3 years is:
\[ \sigma = 0.105 \]

The bond forward price is:
\[ F = P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)} = \frac{P(0, 3)}{P(0, 2)} = 0.9007 \]
The values of $d_1$ and $d_2$ are:

$$
\begin{align*}
    d_1 &= \frac{\ln \left( \frac{F}{K} \right) + 0.5\sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{0.9007}{0.92} \right) + 0.5(0.105)^2 (2)}{0.105\sqrt{2}} = -0.06827 \\
    d_2 &= d_1 - \sigma \sqrt{T} = -0.06827 - 0.105\sqrt{2} = -0.21676
\end{align*}
$$

We have:

$$
\begin{align*}
    N(-d_1) &= N(0.06827) = 0.52721 \\
    N(-d_2) &= N(0.21676) = 0.58580
\end{align*}
$$

The Black formula for the put price is:

$$
P = P(0,T)\left[K \times N(-d_2) - F \times N(-d_1)\right]
$$

$$
= 0.8573[0.92 \times 0.58580 - 0.9007 \times 0.52721]
$$

$$
= 0.05492
$$

**Solution 12**

**D** Chapter 24, Black Model

This method of presenting forward price volatilities did not appear in the textbook’s description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.

The price of the caplet is:

$$
\text{Caplet price} = (1.1075) \times \text{(Put Option Price)}
$$

The put option:

- has a zero-coupon bond that expires at time 3 as its underlying asset
- expires at time 2
- has a strike price of:

$$
K = \frac{1}{1 + K_R} = \frac{1}{1.1075} = 0.9029
$$

The forward price volatilities are:

$$
0.100 = \sqrt{\frac{\text{Var} \left[ \ln \left[ P_t(1,2) \right] \right]}{t}} \quad 0.105 = \sqrt{\frac{\text{Var} \left[ \ln \left[ P_t(2,3) \right] \right]}{t}} \quad 0.120 = \sqrt{\frac{\text{Var} \left[ \ln \left[ P_t(3,4) \right] \right]}{t}}
$$

The appropriate volatility for an option that expires in 2 years on a bond that matures in 3 years is:

$$
\sigma = 0.105
$$
The bond forward price is:

\[ F = P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)} = \frac{P(0, 3)}{P(0, 2)} = 0.9007 \]

The values of \( d_1 \) and \( d_2 \) are:

\[
\begin{align*}
d_1 &= \frac{\ln \left( \frac{F}{K} \right) + 0.5\sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{0.9007}{0.9029} \right) + 0.5(0.105)^2 (2)}{0.105 \sqrt{2}} = 0.05782 \\
d_2 &= d_1 - \sigma \sqrt{T} = 0.05782 - 0.105 \sqrt{2} = -0.09067
\end{align*}
\]

We have:

\[
\begin{align*}
N(-d_1) &= N(-0.05782) = 0.47695 \\
N(-d_2) &= N(0.09067) = 0.53612
\end{align*}
\]

The Black formula for the put price is:

\[
P = P(0, T) \left[ K \times N(-d_2) - F \times N(-d_1) \right] \\
= 0.8573[0.9029 \times 0.53612 - 0.9007 \times 0.47695] \\
= 0.04670
\]

The price of the caplet is:

\[
\text{Caplet price} = (1.1075 \times \text{(Put Option Price)}) = 1.1075 \times 0.04670 = 0.05172
\]

**Solution 13**

**C**  Chapter 24, Black Model

*This method of presenting forward price volatilities did not appear in the textbook’s description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.*

The price of the caplet is:

\[
\text{Caplet price} = (1.12 \times \text{(Put Option Price)})
\]

The put option:

- has a zero-coupon bond that expires at time 4 as its underlying asset
- expires at time 3
- has a strike price of:

\[
K = \frac{1}{1 + K_R} = \frac{1}{1.12} = 0.8929
\]
The forward price volatilities are:

\[
0.100 = \sqrt{\frac{\text{Var}\left[\ln\left(P_t(1.2)\right)\right]}{t}} \quad 0.105 = \sqrt{\frac{\text{Var}\left[\ln\left(P_t(2.3)\right)\right]}{t}} \quad 0.120 = \sqrt{\frac{\text{Var}\left[\ln\left(P_t(3.4)\right)\right]}{t}}
\]

The appropriate volatility for an option that expires in 3 years on a bond that matures in 4 years is:

\[\sigma = 0.12\]

The bond forward price is:

\[
F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 4)}{P(0, 3)} = \frac{0.6956}{0.7722} = 0.9008
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.9008}{0.8929}\right) + 0.5(0.12)^2(3)}{0.12\sqrt{3}} = 0.14655
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = 0.14655 - 0.12\sqrt{3} = -0.06130
\]

We have:

\[
N(-d_1) = N(-0.14655) = 0.44174
\]

\[
N(-d_2) = N(0.06130) = 0.52444
\]

The Black formula for the put price is:

\[
P = P(0, T)\left[K \times N(-d_2) - F \times N(-d_1)\right]
\]

\[
= 0.7722[0.8929 \times 0.52444 - 0.9008 \times 0.44174]
\]

\[
= 0.054308
\]

The price of the caplet is:

\[
\text{Caplet price} = (1.12) \times (\text{Put Option Price}) = 1.12 \times 0.054308 = 0.060825
\]

**Solution 14**

**Chapter 24, Floorlet in the Black Model**

This method of presenting forward price volatilities did not appear in the textbook’s description of the Black model, but the volatilities are presented this way in Problem 24.2 at the end of the textbook chapter.

The price of the floorlet is:

\[
\text{Floorlet price} = (1.12) \times (\text{Call Option Price})
\]
The call option:
- has a zero-coupon bond that expires at time 4 as its underlying asset
- expires at time 3
- has a strike price of:

\[ K = \frac{1}{1 + K_R} = \frac{1}{1.12} = 0.8929 \]

The forward price volatilities are:

\[ 0.100 = \sqrt{\frac{\text{Var} \left( \ln[P_t(1,2)] \right)}{t}} \quad 0.105 = \sqrt{\frac{\text{Var} \left( \ln[P_t(2,3)] \right)}{t}} \quad 0.120 = \sqrt{\frac{\text{Var} \left( \ln[P_t(3,4)] \right)}{t}} \]

The appropriate volatility for an option that expires in 3 years on a bond that matures in 4 years is:

\[ \sigma = 0.12 \]

The bond forward price is:

\[ F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{P(0, 4)}{P(0, 3)} = \frac{0.6956}{0.7722} = 0.9008 \]

The values of \( d_1 \) and \( d_2 \) are:

\[ d_1 = \frac{\ln \left( \frac{F}{K} \right) + 0.5 \sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{0.9008}{0.8929} \right) + 0.5(0.12)^2(3)}{0.12 \sqrt{3}} = 0.14655 \]
\[ d_2 = d_1 - \sigma \sqrt{T} = 0.14655 - 0.12 \sqrt{3} = -0.06130 \]

We have:

\[ N(d_1) = N(0.14655) = 0.55826 \]
\[ N(d_2) = N(-0.06130) = 0.47556 \]

The Black formula for the call price is:

\[ C = P(0, T) \left[ F \times N(d_1) - K \times N(d_2) \right] \]
\[ = 0.7722 \left[ 0.9008 \times 0.55826 - 0.8929 \times 0.47556 \right] \]
\[ = 0.060444 \]

The price of a floorlet with a notional amount of $1 is:

\[ \text{Floorlet price} = (1.12) \times \text{(Call Option Price)} = 1.12 \times 0.060444 = 0.067697 \]

Since the notional amount is $1,000, we multiply by $1,000 to obtain the answer:

\$1,000 \times 0.067697 = 67.697 \]
Solution 15

E  Chapter 24, Forward Rate Agreements

By purchasing the caplet and selling the floorlet, the investor has created a forward rate agreement that locks in the rate $K_R$ from time 2 to time 3. The investor’s time 3 cash flow is the payment from the caplet minus the payment from the floorlet:

$$Max[0, R_2 - K_R] - Max[0, K_R - R_2] = R_2 - K_R$$

A forward rate agreement, which can be entered into at zero cost, provides a cash flow at time 3 of:

$$R_2 - R_0(2,3)$$

Both $R_0(2,3)$ and $K_R$ are known values at time 0. Since the cost of both strategies is the same (zero), to preclude arbitrage, we must have:

$$K_R = R_0(2,3)$$

We can find the forward rate that applies from time 2 to time 3:

$$K_R = R_0(2,3) = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.8573}{0.7722} - 1 = 0.1102$$

Solution 16

C  Chapter 24, Binomial Interest Rate Model

The price of the 2-year zero-coupon bond is:

$$P(0,2) = E^* \left[ e^{-\sum_{i=0}^{1} \eta_i} \right] = 0.5 \times e^{-(0.12+0.16)} + 0.5 \times e^{-(0.12+0.08)} = 0.7873$$

Solution 17

A  Chapter 24, Binomial Interest Rate Model

The price of the 3-year zero-coupon bond is:

$$P_0(0,3) = E^* \left[ e^{-\sum_{i=0}^{2} \eta_i} \right]$$

$$= 0.25 \times e^{-(0.12+0.16+0.20)}$$
$$+ 0.25 \times e^{-(0.12+0.16+0.12)}$$
$$+ 0.25 \times e^{-(0.12+0.08+0.12)}$$
$$+ 0.25 \times e^{-(0.12+0.08+0.04)}$$
$$= 0.7005$$
Solution 18

Chapter 24, Binomial Interest Rate Model

The terminology can be a bit confusing. This same option could also be described as a 2-year put on a 3-year zero-coupon bond, with the understanding that the 3-year bond will be a 1-year bond at the end of 2 years. In this case, since a 1-year bond will have matured by the end of 2 years, it is apparent that the 1-year bond in the question refers to a 1-year bond at the end of 2 years.

The tree of prices for the 3-year bond is:

\[
\begin{array}{cc}
0.8187 & 1.0000 \\
0.7267 & 0.8869 & 1.0000 \\
0.7005 & 0.8869 & 1.0000 \\
0.8528 & 0.9608 & 1.0000 \\
\end{array}
\]

There is no need to calculate the first two columns of the tree to answer this question, but they are shown above for illustrative purposes.

The third column is easily obtained by discounting a 1-year bond at the possible interest rates at time 2:

\[
e^{-0.20} = 0.8187 \\
e^{-0.12} = 0.8869 \\
e^{-0.04} = 0.9608
\]

The put option pays off only if the price falls to $0.8187. If this occurs, the option payoff is:

\[
V_2 = 0.85 - 0.8187 = 0.0313
\]

There is a 0.25 risk-neutral probability of this occurring, so the price of the put option is:

\[
V_0 = E^*[V_2 \times e^{-\sum_{i=0}^{2-1} \eta_i}] = 0.25(0.0313)e^{-(0.12+0.16)} = 0.0059
\]
Solution 19

Chapter 24, Binomial Interest Rate Model

The tree of prices for the 3-year bond is:

<table>
<thead>
<tr>
<th></th>
<th>0.9048</th>
<th>1.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8523</td>
<td>0.9418</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.8361</td>
<td>0.9418</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.9233</td>
<td>0.9802</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

There is no need to calculate the first two columns of the tree to answer this question, but they are shown above for illustrative purposes.

The third column is easily obtained by discounting a 1-year bond at the possible interest rates at time 2:

\[ e^{-0.10} = 0.9048 \]
\[ e^{-0.06} = 0.9418 \]
\[ e^{-0.02} = 0.9802 \]

The call option pays off if the final interest rate is 0.06 or lower. The possible payoffs are:

- \( r_2 = 10\% \) \( \Rightarrow \ V_2 = 0.0000 \)
- \( r_2 = 6\% \) \( \Rightarrow \ V_2 = 0.9418 - 0.92 = 0.0218 \)
- \( r_2 = 2\% \) \( \Rightarrow \ V_2 = 0.9802 - 0.92 = 0.0602 \)

Each of the 4 paths has a 0.25 risk-neutral probability of occurring. This leads to the following call price:

\[
V_0 = E^x \left[ V_2 \times e^{-\sum_{i=0}^{2-1} \eta_i} \right]
\]

\[
= 0.25(0.0000)e^{-(0.06+0.08)}
+ 0.25(0.0218)e^{-(0.06+0.08)}
+ 0.25(0.0218)e^{-(0.06+0.04)}
+ 0.25(0.0602)e^{-(0.06+0.04)}
= 0.02327
\]
Solution 20

B Chapter 24, Black-Derman-Toy Model

*The amount of the loan does not affect the forward rate.*

The price of a 3-year zero-coupon bond is:

\[
P_0(0,3) = E^* \left[ \prod_{i=0}^{2} \frac{1}{1 + r_i} \right]
\]

\[
= 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times \frac{1}{1.1889} \\
+ 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times \frac{1}{1.1489} \\
+ 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1293} \times \frac{1}{1.1489} \\
+ 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1293} \times \frac{1}{1.1173}
\]

\[
= 0.6930
\]

The price of a 2-year zero-coupon bond is:

\[
P_0(0,2) = E^* \left[ \prod_{i=0}^{1} \frac{1}{1 + r_i} \right]
\]

\[
= 0.5 \times \frac{1}{1.10} \times \frac{1}{1.1517} + 0.5 \times \frac{1}{1.10} \times \frac{1}{1.1293}
\]

\[
= 0.7972
\]

The contract rate on the forward rate agreement is the forward rate:

\[
R_{0}(2,3) = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.7972}{0.6930} - 1 = 0.1503
\]

Solution 21

A Chapter 24, Risk-Neutral Vasicek Model

Greg uses the Vasicek model for the short rate. The risk-neutral version of Greg’s model is:

\[
dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}
\]

\[
= [a(b - r) + \sigma\phi]dt + \sigma d\tilde{Z}
\]

\[
= [0.2(0.1 - r) + 0.05 \times 0.08] dt + 0.05 d\tilde{Z}
\]

\[
= 0.2(0.12 - r)dt + 0.05d\tilde{Z}
\]

Colleen’s model will produce the same price as Greg’s if the risk-neutral version of her model is the same as the risk-neutral version of Greg’s model.
The risk-neutral version of the model shown in Choice A is:
\[
\begin{align*}
\frac{dr}{dt} &= (a(r-b) + \sigma \phi)dt + \sigma d\tilde{Z} = [0.2(0.08 - r) + 0.05 \times 0.16]dt + 0.05d\tilde{Z} \\
&= 0.2(0.12 - r)dt + 0.05d\tilde{Z}
\end{align*}
\]
Therefore, the risk-neutral version of Choice A is the same as the risk-neutral version of Greg’s model, and Choice A is correct.

For the sake of thoroughness, let’s consider the other choices as well. As shown below, the risk-neutral version of Choice B does not match the risk-neutral version of Greg’s model:
\[
\begin{align*}
\frac{dr}{dt} &= (a(r-b) + \sigma \phi)dt + \sigma d\tilde{Z} = [0.2(0.08 - r) + 0.05 \times 0.32]dt + 0.05d\tilde{Z} \\
&= 0.2(0.16 - r)dt + 0.05d\tilde{Z}
\end{align*}
\]
As shown below, the risk-neutral version of Choice C does not match the risk-neutral version of Greg’s model:
\[
\begin{align*}
\frac{dr}{dt} &= (a(r-b) + \sigma \phi)dt + \sigma d\tilde{Z} = [0.2(0.12 - r) + 0.05 \times 0.08]dt + 0.05d\tilde{Z} \\
&= 0.2(0.14 - r)dt + 0.05d\tilde{Z}
\end{align*}
\]
Choices D and E are unlikely to be correct, because they are based on the Cox-Ingersoll-Ross model. Since we have already shown that Choice A is correct, and the question tells us that only one model produces the same price as Greg’s model, Choices D and E must be incorrect.

Solution 22

B  Chapter 24, Black-Derman-Toy Model

Although the cap payments are made at the end of each year, we will value them at the beginning of each year using the following formula:

\[
T\text{-year caplet payoff at time } (T - 1) = \frac{\text{Max}[0, R_{T-1} - K_R]}{1 + R_{T-1}} \times \text{Notional}
\]

A 2-year cap consists of a 1-year caplet and a 2-year caplet.

The payoff for the 1-year caplet is zero since 10% is less than 14%.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 15.17%:

\[
\begin{align*}
\text{2-year caplet payoff at time } 1 &= \frac{\text{Max}[0, R_1 - K_R]}{1 + R_1} \times 100 = \frac{0.1517 - 0.14}{1.1517} \times 100 \\
&= 1.0159
\end{align*}
\]
This payment is made at time 1 and it has a 50% probability of occurring, so the value of the 2-year cap is:

\[
\frac{0.5 \times 1.0159}{1.10} = 0.462
\]

**Solution 23**

**Chapter 24, Black-Derman-Toy Model**

Although the cap payments are made at the end of each year, we will value them at the beginning of each year using the following formula:

\[
T\text{-year caplet payoff at time } (T - 1) = \frac{\text{Max}[0, R_{T-1} - K]}{1 + R_{T-1}} \times \text{Notional}
\]

A 3-year cap consists of a 1-year caplet, a 2-year caplet, and a 3-year caplet.

The payoff for the 1-year caplet is zero since 10% is less than 14%.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 15.17%:

\[
R_1 = 0.1517 \Rightarrow \frac{0.1517 - 0.14}{1.1517} \times 100 = 1.0159
\]

The payoff for the 3-year caplet is positive when the short-term rate exceeds 14%:

\[
R_2 = 0.1889 \Rightarrow \frac{0.1889 - 0.14}{1.1889} \times 100 = 4.1130
\]

\[
R_2 = 0.1489 \Rightarrow \frac{0.1489 - 0.14}{1.1489} \times 100 = 0.7747
\]

The possible payments from the cap are illustrated in the tree below:

```
4.1130
1.0159
0.0000
0.7747
0.0000
0.0000
```

The value of the 2-year caplet is:

\[
\text{Value of 2-year caplet} = \frac{0.5 \times 1.0159}{1.10} = 0.4618
\]
The value of the 3-year caplet is:

\[
\text{Value of the 3-year caplet} = 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times 4.1130 \\
+ 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1517} \times 0.7747 \\
+ 0.25 \times \frac{1}{1.10} \times \frac{1}{1.1293} \times 0.7747 \\
= 1.1204
\]

The value of the 3-year cap is:

\[
\text{Value of 3-year cap} = (1\text{-year caplet}) + (2\text{-year caplet}) + (3\text{-year caplet}) \\
= 0.000 + 0.4618 + 1.1204 = 1.5822
\]

Solution 24

A Chapter 24, Black-Derman-Toy Model

There are two ways to answer this question.

Method 1

First, we consider the quick way. The short-term rates in one period are \( R_1 \) and \( e^{2\sigma_1 \sqrt{h}} R_1 \):

\[
R_1 = 0.1293 \\
e^{2\sigma_1 \sqrt{h}} R_1 = 0.1517
\]

We can solve for \( \sigma_1 \), which is the yield volatility for the 2-year bond:

\[
\frac{e^{2\sigma_1 \sqrt{h}} R_1}{R_1} = \frac{0.1517}{0.1293} \\
\sigma_1 = \ln\left(\frac{0.1517}{0.1293}\right) \times 0.5 = 0.0799
\]

This method only works for the 2-year bond! For longer term bonds, we must use the method shown below.

Method 2

The second method takes a bit longer. First, we create the tree of prices for the 2-year bond:

\[
0.8683 \\
0.7972 \\
0.8855
\]
The calculations to obtain these prices are:

\[ P(1,2,r_u) = \frac{1}{1.1517} = 0.8683 \]

\[ P(1,2,r_d) = \frac{1}{1.1293} = 0.8855 \]

\[ P(0,2) = \frac{1}{1.10} \times 0.5(0.8683 + 0.8855) = 0.7972 \]

*We don’t need to calculate \( P(0,2) \) to answer this question, but we included it for completeness.*

The formula for the one-year yield volatility for a \( T \)-year zero-coupon bond is:

\[
\text{Yield volatility} = 0.5 \times \ln \left( \frac{P(1,T,r_u)^{-1/(T-1)} - 1}{P(1,T,r_d)^{-1/(T-1)} - 1} \right)
\]

The yield volatility of the 2-year bond is:

\[
\text{Yield volatility} = 0.5 \times \ln \left( \frac{0.8683^{-1/(2-1)} - 1}{0.8855^{-1/(2-1)} - 1} \right) = 0.5 \times \ln \left( \frac{0.1517}{0.1293} \right) = 0.0799
\]

**Solution 25**

**B** Chapter 24, Black-Derman-Toy Model

The tree of prices for the 3-year bond is:

\[
\begin{align*}
0.8411 & \quad 0.7430 \\
0.6930 & \quad 0.8704 \\
0.7816 & \quad 0.8950
\end{align*}
\]

The calculations to obtain these prices are:

\[ P(2,3,r_{uu}) = \frac{1}{1.1889} = 0.8411 \]

\[ P(2,3,r_{ud}) = P(2,3,r_{du}) = \frac{1}{1.1489} = 0.8704 \]

\[ P(2,3,r_{dd}) = \frac{1}{1.1173} = 0.8950 \]

\[ P(1,3,r_u) = \frac{0.5(0.8411 + 0.8704)}{1.1517} = 0.7430 \]

\[ P(1,3,r_d) = \frac{0.5(0.8704 + 0.8950)}{1.1293} = 0.7816 \]

\[ P(0,3) = \frac{0.5(0.7430 + 0.7816)}{1.10} = 0.6930 \]
We don’t need to calculate $P(0,3)$ to answer this question, but we included it for completeness.

The formula for the one-year yield volatility for a $T$-year zero-coupon bond is:

\[
\text{Yield volatility} = 0.5 \times \ln \left[ \frac{P(1, T, r_u)^{-1/(T-1)} - 1}{P(1, T, r_d)^{-1/(T-1)} - 1} \right]
\]

The yield volatility of the 3-year bond is:

\[
\text{Yield volatility} = 0.5 \times \ln \left[ \frac{0.7430^{-1/(3-1)} - 1}{0.7816^{-1/(3-1)} - 1} \right] = 0.100
\]

Solution 26

D Chapter 24, Vasicek Model

This question is based on Table 24.1 in the textbook.

The process can be written as:

\[
dr = 0.25(0.10 - r)dt + 0.01dZ
\]

This is the Vasicek model with:

- $a = 0.25$
- $b = 0.10$
- $\sigma = 0.01$

The gap between $r$ and $b$ is expected to close at a rate equal to $a$ times the gap:

\[
X = 0.25(0.10 - 0.05) = 0.0125
\]

Solution 27

E Chapter 24, Continuous-Time Interest Rate Models


Choice A is true, because the Rendleman-Bartter model assumes that the short-rate follows geometric Brownian motion. See the sentence preceding Equation 24.24 on page 785.

Choice B is true, because a shortcoming of the Vasicek model is that interest rates can be negative. See the last sentence of the third paragraph on page 786.

Choice C is true, because a shortcoming of the Vasicek model is that the variance does not change with the short-rate. See the last sentence of the third paragraph on page 786.

Choice D is true, because in the CIR model the variance increases with the short-rate. See the second bullet point on the lower half of page 787.
Choice E is false, because the Rendleman-Bartter model is not mean-reverting. See the last paragraph on page 785.

Solution 28

E Chapter 24, Vasicek Model

As noted in the Chapter 24 Review Note, the textbook contains a typo in the formula for $A(t,T)$ when $a = 0$. Since this typo was not noted in the textbook’s errata at press time, we use the incorrect version of the formula (as it appears in the textbook) below. The correct price is actually $68.44.

The process describes the Vasicek model with:

$a = 0$

$\sigma = 0.1$

The value of $b$ is unknown, but it has no impact on the model since $a$ is zero.

The Sharpe ratio is:

$\phi = 0.15$

We can use these values to find the price of the 5-year zero-coupon bond. When $a = 0$, we have:

$B(0,T) = T = 5$

$A(0,T) = e^{0.5\sigma^2T^2 + \frac{\sigma^2}{6}T^3} = e^{0.5(0.1)(0.15)^2(5)^2 + \frac{(0.1)^2(5)^3}{6}} = 1.485622$

$P(r,0,T) = A(0,T)e^{-B(0,T)r} = 1.485622e^{-5 \times 0.08} = 0.995842$

Since the bond matures for $100, its price is:

$100 \times 0.995842 = 99.5842$

Solution 29

D Chapter 24, Black-Derman-Toy Model

The value of $r_0$ is the same as the yield on the 1-year zero-coupon bond, and the yield volatility of the 2-year bond is $\sigma_1$:

$r_0 = 8\%$

$\sigma_1 = 12\%$

The first part of the interest rate tree is:

$r_u = R_1e^{2\sigma_1} \quad \Rightarrow \quad 8\%$

$r_d = R_1 \quad \Rightarrow \quad R_1$
To obtain the value of $r_u$, we need to determine the value of $R_1$. The tree must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.08} \left(0.5 \left(\frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}\right)\right)$$

$$\left(\frac{1}{1.09}\right)^2 = \frac{1}{1.08} \left(0.5 \left(\frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}\right)\right)$$

$$1.818028 = \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}$$

There are 2 ways to answer this question.

**Method 1**

Since we are provided with the possible answers, a perfectly valid approach during the exam is to use trial & error to see which one works. We use the following relationships:

$$R_1 e^{0.24} = r_u \quad \Rightarrow \quad R_1 = r_u e^{-0.24}$$

Let’s try Choice C first:

$$\frac{1}{1 + 0.1017 e^{-0.24}} + \frac{1}{1.1017} = 1.833614$$

The value 1.833614 is greater than the correct answer of 1.818028, so let’s try a higher interest rate. Let’s try Choice D:

$$\frac{1}{1 + 0.1122 e^{-0.24}} + \frac{1}{1.1122} = 1.818017$$

The value of 1.818017 is within rounding tolerance to the correct answer of 1.818028, so Choice D is the correct answer.

**Method 2**

The second method is more time consuming:

$$1.818028 = \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}$$

$$1.818028 \left(1 + R_1 e^{0.24}\right)\left(1 + R_1\right) = 1 + R_1 + 1 + R_1 e^{0.24}$$

$$1.818028 \left(1 + R_1 e^{0.24} + R_1 + R_1^2 e^{0.24}\right) = 2 + 2.271249R_1$$

$$1.818028 + 4.129196R_1 + 2.311168R_1^2 = 2 + 2.271249R_1$$

$$2.311168R_1^2 + 1.857947R_1 - 0.181971 = 0$$
We can use the quadratic formula to find $R_1$:

$$R_1 = \frac{-1.857947 \pm \sqrt{1.857947^2 - 4(2.311168)(-0.181971)}}{2(2.311168)}$$

$$R_1 = 0.088253 \quad \text{or} \quad R_1 = -0.892153$$

We use the positive value of $R_1$ since the negative value doesn’t make sense for interest rates. The value of $r_u$ is:

$$r_u = R_1 e^{2\sigma_1} = 0.088253 e^{2 \times 0.12} = 0.112192$$

**Solution 30**

A Chapter 24, Black-Derman-Toy Model

The value of $r_0$ is the same as the yield on the 1-year zero-coupon bond, and the yield volatility of the 2-year bond is $\sigma_1$:

$$r_0 = 8\% \quad \sigma_1 = 12\%$$

The first part of the interest rate tree is:

$$r_u = R_1 e^{2\sigma_1} \quad \Rightarrow \quad 8\% \quad R_1$$

$$r_d = R_1$$

To obtain the value of $r_u$, we need to determine the value of $R_1$. The tree must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.08} \left( \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1} \right)$$

$$\left( \frac{1}{1.09} \right)^2 = \frac{1}{1.08} \left( \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1} \right)$$

$$1.818028 = \frac{1}{1 + R_1 e^{0.24}} + \frac{1}{1 + R_1}$$

$$1.818028 \left(1 + R_1 e^{0.24}\right) \left(1 + R_1\right) = 1 + R_1 + 1 + R_1 e^{0.24}$$

$$1.818028 \left(1 + R_1 e^{0.24} + R_1 + R_1^2 e^{0.24}\right) = 2 + 2.271249 R_1$$

$$1.818028 + 4.129196 R_1 + 2.311168 R_1^2 = 2 + 2.271249 R_1$$

$$2.311168 R_1^2 + 1.857947 R_1 - 0.181971 = 0$$
We can use the quadratic formula to find $R_1$:

$$R_1 = \frac{-1.857947 \pm \sqrt{1.857947^2 - 4(2.311168)(-0.181971)}}{2(2.311168)}$$

$$R_1 = 0.088253 \quad \text{or} \quad R_1 = -0.892153$$

We use the positive value of $R_1$. The values of $r_u$ and $r_d$ are:

$$r_u = R_1 e^{2\sigma_1} = 0.088253 e^{2 \times 0.12} = 0.112192$$
$$r_d = R_1 = 0.088253$$

The first part of the interest rate tree is:

<table>
<thead>
<tr>
<th>Rate</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.2192%</td>
<td>50%</td>
</tr>
<tr>
<td>8.0000%</td>
<td></td>
</tr>
<tr>
<td>8.8253%</td>
<td></td>
</tr>
</tbody>
</table>

The cap pays off only if the interest rate reaches 11.2192%. There is a 50% probability of this occurring, so the value of the interest rate cap is:

$$0.5 \times \frac{100 \times (0.112192 - 0.1)}{1.08 \times 1.112192} = 0.5075$$

**Solution 31**

**B Chapter 24, Black-Derman-Toy Model**

The value of $r_0$ is the same as the yield on the 1-year zero-coupon bond, so $r_0 = 7\%$. The first part of the interest rate tree is:

$$r_u = 0.076821 e^{2\sigma_1}$$
$$7\%$$
$$r_d = 0.076821$$
The value of $\sigma_1$ must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.07} \left(0.5\right) \left( \frac{1}{1 + 0.076821 e^{2\sigma_1}} + \frac{1}{1 + 0.076821 e^{2\sigma_1}} \right)$$

$$\left( \frac{1}{1.08} \right)^2 = \frac{1}{1.07} \left(0.5\right) \left( \frac{1}{1 + 0.076821 e^{2\sigma_1}} + \frac{1}{1 + 0.076821 e^{2\sigma_1}} \right)$$

$$1.834705(1 + 0.076821 e^{2\sigma_1})(1 + 0.076821) = 1 + 0.076821(1 + 0.076821 e^{2\sigma_1})$$

$$1.834705(1.076821 + 0.082722 e^{2\sigma_1}) = 2.076821 + 0.076821 e^{2\sigma_1}$$

$$0.074950 e^{2\sigma_1} = 0.101172$$

$$2\sigma_1 = \ln(1.349855)$$

$$\sigma_1 = 0.149998$$

The yield volatility of the 2-year zero-coupon bond is:

$$\sigma_1 = 0.149998$$

**Solution 32**

**D** Chapter 24, Black-Derman-Toy Model

Each node is $e^{2\sigma_1 \sqrt{t}}$ times the one below it. This means that 13.51% is $e^{4\sigma_2}$ times as large as 7.17%:

$$0.1351 = 0.0717 e^{4\sigma_2}$$

$$\sigma_2 = \ln \left( \frac{0.1351}{0.0717} \right) \times \frac{1}{4} = 0.15838$$

The interest rate for the node above 7.17% is:

$$X = 0.0717 e^{2\sigma_2} = 0.0717 e^{2 \times 0.15838} = 0.09842$$

**Solution 33**

**E** Chapter 24, Duration-Hedging

*This question is similar to Problem 24.4 on at the end of the textbook chapter.*

Let’s assume that the bonds both mature for $1. We’ll scale up the result at the end.

The prices of the bonds are:

$$P(0,2) = \frac{1}{e^{2(0.08)}} = 0.85214379$$

$$P(0,7) = \frac{1}{e^{7(0.08)}} = 0.57120906$$
The quantity of 7-year bonds that must be purchased is:

\[ N = \frac{(T_2 - t) \times P(t, T_2)}{(T_1 - t) \times P(t, T_1)} = \frac{2 \times P(0, 2)}{7 \times P(0, 7)} = \frac{2 \times e^{-2(0.08)}}{7 \times e^{-7(0.08)}} = -0.426236 \]

Since \( N \) is negative, 0.426236 7-year bonds are sold.

The amount that is lent is:

\[ W = -P(t, T_2) - N \times P(t, T_1) = -0.85214379 + 0.426236 \times 0.57120906 = -0.60867413 \]

Since \( W \) is negative, 0.60867413 is borrowed at the short-term rate of 8%.

This hedged position has an initial value of zero. After 1 day, the short-term rate increases to 8.5%, and the new value of the position is:

\[ e^{-\left(\frac{2}{365}\right)0.085} - 0.426236e^{-\left(\frac{7}{365}\right)0.085} - 0.60867413e^{\left(\frac{1}{365}\right)0.08} = -0.00009662 \]

Let’s scale this up by $1,000,000:

\[ 1,000,000 \times (-0.00009662) = -96.62 \]

The investor loses $96.62 on the hedge.

**Solution 34**

**C** Chapter 24, Risk-Neutral Vasicek Model

The short-rate follows the Vasicek model:

\[ dr = a(b - r)dt + \sigma dZ \]

From the realistic process provided in the question, we observe that \( b = 0.12 \):

\[ dr = a(0.12 - r)dt + \sigma dZ \quad \Rightarrow \quad b = 0.12 \]

The risk-neutral process is given by:

\[ dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z} = [a(b - r) + \sigma\phi]dt + \sigma d\tilde{Z} \]

From risk-neutral process provided in the question, we observe that \( \sigma = 0.03 \) and \( a = 0.3 \):

\[ dr = 0.3(0.125 - r)dt + 0.03d\tilde{Z} \quad \Rightarrow \quad \sigma = 0.03 \text{ and } a = 0.3 \]
We use the coefficient of $dt$ in the risk-neutral process to determine the Sharpe ratio $\phi$:

$$a(b - r) + \phi = 0.3(0.125 - r)$$

$$0.3(0.12 - r) + 0.03\phi = 0.3(0.125 - r)$$

$$0.12 - r + 0.1\phi = 0.125 - r$$

$$0.12 + 0.1\phi = 0.125$$

$$\phi = \frac{0.125 - 0.12}{0.1} = 0.05$$

The values of $B(0,10)$ and $q(0.08,0,10)$ are:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.3 \times 10}}{0.3} = 3.16738$$

$$q(0.08,0,10) = B(0,10)\sigma(r) = B(0,10)\sigma = 3.16738 \times 0.03 = 0.09502$$

We can now determine the expected return on the bond:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$0.05 = \frac{\alpha(0.08,0,10) - 0.08}{0.09502}$$

$$\alpha(0.08,0,10) = 0.08475$$

Solution 35

D Chapter 24, Risk-Neutral Cox-Ingersoll-Ross Model

The short-rate follows the Cox-Ingersoll-Ross model:

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ$$

From the realistic process provided in the question, we observe that $b = 0.08$:

$$dr = a(0.08 - r)dt + \sigma\sqrt{r}dZ \quad \Rightarrow \quad b = 0.08$$

The risk-neutral process is given by:

$$dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}$$

$$= [a(b - r) + \sigma\sqrt{r} \frac{\sqrt{r}}{\sigma}]dt + \sigma\sqrt{r}d\tilde{Z}$$

$$= [a(b - r) + r\tilde{\phi}]dt + \sigma\sqrt{r}d\tilde{Z}$$

From risk-neutral process provided in the question, we observe that $\sigma = 0.04$:

$$dr = 0.2(0.096 - r)dt + 0.04\sqrt{r}d\tilde{Z} \quad \Rightarrow \quad \sigma = 0.04$$
We use the coefficient of $dt$ in the risk-neutral process to determine $a$ and $\bar{\phi}$:

\[
\begin{align*}
  a(b - r) + r\bar{\phi} &= 0.2(0.096 - r) \\
  ab - (a - \bar{\phi})r &= 0.0192 - 0.2r \\
  0.08a - (a - \bar{\phi})r &= 0.0192 - 0.2r
\end{align*}
\]

From the equation above, we observe that:

\[
\begin{align*}
  0.08a &= 0.0192 \quad \Rightarrow \quad a = 0.24 \\
  a - \bar{\phi} &= 0.2 \quad \Rightarrow \quad \bar{\phi} = a - 0.2 = 0.24 - 0.2 = 0.04
\end{align*}
\]

In the CIR model, the Sharpe ratio for a bond of any maturity is:

\[
\phi(r,t) = \bar{\phi} \frac{\sqrt{r}}{\sigma} = 0.04 \frac{\sqrt{0.07}}{0.04} = 0.2646
\]

All bonds have the same Sharpe ratio, so the Sharpe ratio of the 9-year bond is 0.2646.

**Solution 36**

A Chapter 24, Delta-Gamma Approximation

The process describes the Vasicek model with:

\[
a = 0.3, \quad b = 0.1, \quad \sigma = 0.12
\]

The short-term interest rate increases by one standard deviation, so under the Vasicek model it increases by:

\[
\sigma \sqrt{dt} = \sigma \sqrt{h} = 0.12 \sqrt{\frac{1}{365}} = 0.00628
\]

Itô’s Lemma is the basis of the delta-gamma-theta approximation:

\[
dP = P_r dr + \frac{1}{2} P_{rr} (dr)^2 + P_t dt
\]

Since we are using only the delta-gamma portion of the approximation, we can set the final term to 0. Let’s write the delta-gamma approximation in its discrete form:

\[
P(0.08628,0,10) - P(0.08,0,10) = (0.00628) P_r + \frac{1}{2} (0.00628)^2 P_{rr}
\]
Delta is $P_r$, and gamma is $P_{rr}$. Both are found below:

\[
P(r,t,T) = A(t,T)e^{-B(t,T)r}
\]

\[
P_r = -B(t,T)P(r,t,T)
\]

\[
P_{rr} = [B(t,T)]^2 P(r,t,T)
\]

\[
B(0,10) = \frac{1 - e^{-0.3(10)}}{0.3} = 3.16738
\]

\[
P_{r}(0.08,0,10) = -3.16738 \times 0.60024 = -1.90119
\]

\[
P_{rr}(0.08,0,10) = 3.16738^2 \times 0.60024 = 6.02177
\]

The delta-gamma approximation of the new price can now be calculated:

\[
P(0.08628,0,10) = P(0.08,0,10) + 0.00628 \times P_r + \frac{1}{2} \times 0.00628^2 P_{rr}
\]

\[
= 0.60024 + 0.00628 \times (-1.90119) + \frac{1}{2} \times 0.00628^2 \times 6.02177
\]

\[
= 0.588417
\]

Since the bond matures for $1,000, its new price is:

\[
1,000 \times 0.588417 = 588.417
\]

**Solution 37**

A Chapter 24, Delta-Gamma-Approximation

The process describes the Cox-Ingersoll-Ross model with:

\[
a = 0.3, \quad b = 0.1, \quad \sigma = 0.4
\]

The Sharpe ratio is zero, so $\tilde{\phi}$ is zero:

\[
\phi(r,t) = 0 \quad \Rightarrow \quad \frac{\tilde{\phi} \sqrt{r}}{\sigma} = 0 \quad \Rightarrow \quad \tilde{\phi} = 0
\]

The short-term interest rate increases by one standard deviation, so under the CIR model it increases by:

\[
\sigma \sqrt{r} \times dt = \sigma \sqrt{r} \times h = 0.4 \sqrt{\frac{0.08}{365}} = 0.00592
\]

Itô’s Lemma is the basis of the delta-gamma-theta approximation:

\[
dP = P_r dr + \frac{1}{2} P_{rr} (dr)^2 + P_t dt
\]
Since we are using only the delta-gamma portion of the approximation, we can set the final term to 0. Let’s write the delta-gamma approximation in its discrete form:

\[
P(0.08592,0,10) - P(0.08,0,10) = (0.00592)P_r + \frac{1}{2}(0.00592)^2 P_{rr}
\]

We must calculate the value of \( \gamma \) in order to obtain \( B(0,10) \):

\[
\gamma = \sqrt{(\alpha - \phi)^2 + 2\sigma^2} = \sqrt{(0.3 - 0)^2 + 2 \times 0.4^2} = 0.64031
\]

\[
B(0,10) = \frac{2(e^{\gamma(T-t)} - 1)}{(\alpha - \phi + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}
\]

\[
= \frac{2\left(e^{0.64031 \times (10)} - 1\right)}{(0.3 - 0 + 0.64031)\left(e^{0.64031 \times (10)} - 1\right) + 2(0.64031)}
\]

\[
= \frac{1,205.45657}{568.03352} = 2.12216
\]

Now we can solve for delta and gamma:

\[
P(r,t,T) = A(t,T)e^{-B(t,T)r}
\]

\[
P_r = -B(t,T)P(r,t,T)
\]

\[
P_{rr} = [B(t,T)]^2 P(r,t,T)
\]

\[
P_r(0.08,0,10) = -2.12216 \times 0.50045 = -1.06203
\]

\[
P_{rr}(0.08,0,10) = 2.12216^2 \times 0.50045 = 2.25380
\]

The delta-gamma approximation of the new price can now be calculated:

\[
P(0.08592,0,10) = P(0.08,0,10) + (0.00592)P_r + \frac{1}{2}(0.00592)^2 P_{rr}
\]

\[
= 0.50045 + 0.00592 \times (-1.06203) + \frac{1}{2}(0.00592)^2 \times 2.25380 = 0.494200
\]

Since the bond matures for $1000, its new price is:

\[
1,000 \times 0.494200 = 494.20
\]
Solution 38

D Chapter 24, Binomial Interest Rate Model

This question is similar to Problem 24.7 at the end of the textbook chapter.

Since interest rates can move up or down by 300 basis points each period, the interest-rate tree is:

\[
\begin{array}{ccc}
0.17 & 0.14 & 0.11 \\
0.11 & 0.11 & 0.08 \\
0.08 & 0.05 \\
\end{array}
\]

The price of the 3-year zero-coupon bond is:

\[
P_0(0,3) = E^x \left[ e^{-\sum_{i=0}^{2} r_i} \right]
\]

\[
= 0.25 \times e^{-0.11+0.14+0.17} + 0.25 \times e^{-0.11+0.14+0.11} + 0.25 \times e^{-0.11+0.08+0.11} + 0.25 \times e^{-0.11+0.08+0.05} = 0.72054
\]

The yield of the 3-year zero-coupon bond is:

\[
yield = - \frac{\ln[P(0,T)]}{T} = - \frac{\ln(0.72054)}{3} = 0.10925
\]

Solution 39

E Chapter 24, Black-Derman-Toy Model

The tree of prices for the 3-year bond is:

\[
\begin{array}{ccc}
0.7717 & 0.6648 & 0.7351 \\
0.6086 & 0.8366 & 0.8858 \\
\end{array}
\]
The calculations to obtain these prices are:

\[ P(2,3,r_{uu}) = \frac{1}{1.2959} = 0.7717 \]
\[ P(2,3,r_{ud}) = P(2,3,r_{du}) = \frac{1}{1.1953} = 0.8366 \]
\[ P(2,3,r_{dd}) = \frac{1}{1.1289} = 0.8858 \]
\[ P(1,3,r_u) = \frac{0.5(0.7717 + 0.8366)}{1.2097} = 0.6648 \]
\[ P(1,3,r_d) = \frac{0.5(0.8366 + 0.8858)}{1.1716} = 0.7351 \]
\[ P(0,3) = \frac{0.5(0.6648 + 0.7351)}{1.15} = 0.6086 \]

We don’t need to calculate \( P(0,3) \) to answer this question, but we included it for completeness.

The formula for the one-year yield volatility for a \( T \)-year zero-coupon bond is:

\[ \text{Yield volatility} = 0.5 \times \ln \left[ \frac{P(1,T,r_u)^{-1/(T-1)} - 1}{P(1,T,r_d)^{-1/(T-1)} - 1} \right] \]

The yield volatility of the 3-year bond is:

\[ \text{Yield volatility} = 0.5 \times \ln \left[ \frac{0.6648^{-1/(3-1)} - 1}{0.7351^{-1/(3-1)} - 1} \right] = 0.1542 \]

Solution 40

A Chapter 24, Black-Derman-Toy Model

Although the cap payments are made at the end of each year, we will value them at the beginning of each year using the following formula:

\[ T\text{-year caplet payoff at time } (T - 1) = \frac{\text{Max}\left[0, R_{T-1} - K_R\right]}{1 + R_{T-1}} \times \text{Notional} \]

A 4-year cap consists of a 1-year caplet, a 2-year caplet, a 3-year caplet, and a 4-year caplet.

The payoff for the 1-year caplet is zero since 15% is less than 21%.

The payoff for the 2-year caplet is also zero since 20.96% and 17.16% are both less than 21%.
The payoff for the 3-year caplet is positive only when the short-term interest rate increases to 29.59%:

\[ R_2 = 0.2959 \Rightarrow \frac{0.2959 - 0.21}{1.2959} \times 100 = 6.6286 \]

The payoff for the 4-year caplet is positive only when the short-term rate increases to 25.03%:

\[ R_3 = 0.2503 \Rightarrow \frac{0.2503 - 0.21}{1.2503} \times 100 = 3.2232 \]

The possible payments from the cap are illustrated in the tree below:

The value of the 3-year caplet is:

\[ \text{Value of 3-year caplet} = (0.5)^2 \times \frac{6.6286}{1.15 \times 1.2096} = 1.1913 \]

The value of the 4-year caplet is:

\[ \text{Value of the 4-year caplet} = (0.5)^3 \times \frac{3.2232}{1.15 \times 1.2096 \times 1.2959} = 0.2235 \]

The value of the 4-year cap is:

\[ (1\text{-year caplet}) + (2\text{-year caplet}) + (3\text{-year caplet}) + (4\text{-year caplet}) = 0.000 + 0.000 + 1.1913 + 0.2235 = 1.4148 \]

**Solution 41**

C  Chapter 24, Cox-Ingersoll-Ross Model

We begin with the Sharpe ratio and parameterize it for the CIR model:

\[ \phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)} \]
\[ \bar{\phi} = \frac{\alpha(r,t,T) - r}{\sigma} \]
\[ \bar{\phi} r = \frac{\alpha(r,t,T) - r}{B(t,T)} \]
We use the value of $\alpha(0.08, 0.3)$ provided in the question:

\[
\bar{\phi} \times 0.08 = \frac{0.105324 - 0.08}{B(0, 3)}
\]

\[
B(0, 3) \bar{\phi} = 0.31655
\]

Making use of the fact that $B(0, 3) = B(2, 5)$, we have:

\[
\bar{\phi} \times 0.09 = \frac{\alpha(0.09, 2, 5) - 0.09}{B(2, 5)}
\]

\[
\alpha(0.09, 2, 5) = 0.09 + B(2, 5) \bar{\phi}(0.09)
\]

\[
= 0.09 + 0.31655(0.09)
\]

\[
= 0.1184895
\]

**Solution 42**

**D** Chapter 24, Black Formula

The option expires in 1 year, so $T = 1$. The underlying bond matures 1 year after the option expires, so $s = 1$. The bond forward price is:

\[
F = P_0(T, T+s) = \frac{P(0, T+s)}{P(0, T)} = \frac{0.8799}{0.9555} = 0.92088
\]

The volatility of the forward price is:

\[
\sigma = 0.07
\]

We have:

\[
d_1 = \frac{\ln \left( \frac{F}{K} \right) + 0.5 \sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{0.92088}{0.9111} \right) + 0.5(0.07)^2(1)}{0.07 \sqrt{1}} = 0.18752
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = 0.18752 - 0.07 \sqrt{1} = 0.11752
\]

\[
N(d_1) = N(0.18752) = 0.57437
\]

\[
N(d_2) = N(0.11752) = 0.54678
\]

The Black formula for the call price is:

\[
C = P(0, T) \left[ F \times N(d_1) - K \times N(d_2) \right]
\]

\[
= 0.9555 \left[ 0.92088 \times 0.57437 - 0.9111 \times 0.54678 \right]
\]

\[
= 0.02939
\]
Solution 43

D Chapter 24, Interest Rate Cap

The tree of interest rates is:

\[ 13.991\% \]
\[ 12.038\% \]
\[ 7.000\% \]
\[ 10.328\% \]
\[ 10.055\% \]
\[ 7.623\% \]

Although the cap payments are made at the end of each year, we will follow the textbook’s convention of valuing them at the beginning of each year using the following formula:

\[
T\text{-year caplet payoff at time } (T-1) = \frac{\text{Max}[0, R_{T-1} - K]}{1 + R_{T-1}} \times \text{Notional}
\]

The interest rate cap consists of a 1-year caplet, a 2-year caplet, and a 3-year caplet.

The payoff for the 1-year caplet is zero since 7% is less than 11.0%.

The payoff for the 2-year caplet is positive only when the short-term interest rate increases to 12.038%, since 10.055% is less than 11%:

\[
R_1 = 0.12038 \Rightarrow \frac{0.12038 - 0.11}{1.12038} \times 100 = 0.9265
\]

The payoff for the 3-year caplet is positive only when the short-term rate increases to 13.991%, since the other short-term interest rates at that time are less than 11%:

\[
R_2 = 0.13991 \Rightarrow \frac{0.13991 - 0.11}{1.13991} \times 100 = 2.6239
\]

The payoff table is illustrated in the tree below:

\[
\begin{array}{cccc}
2.6239 \\
0.9265 \\
0.0000 \\
0.0000 \\
0.0000 \\
0.0000 \\
\end{array}
\]

The value of the 2-year caplet is:

\[
\text{Value of 2-year caplet} = 0.5 \times \frac{0.9265}{1.07} = 0.4329
\]

The value of the 3-year caplet is:

\[
\text{Value of the 3-year caplet} = (0.5)^2 \times \frac{2.6239}{1.07 \times 1.12038} = 0.5472
\]
The value of the 3-year cap is:

\[
(\text{1-year caplet}) + (\text{2-year caplet}) + (\text{3-year caplet}) \\
= 0.0000 + 0.4329 + 0.5472 = 0.9801
\]

Solution 44

B Chapter 24, Vasicek Model

In the Vasicek Model, we have:

\[
P(r, t, T) = A(t, T)e^{-B(t, T)r}
\]

We use the following two facts about the Vasicek model:

- \(A(t, T)\) and \(B(t, T)\) do not depend on \(r\).
- \(A(t, T) = A(0, T - t)\) and \(B(t, T) = B(0, T - t)\). This implies:
  \[
  A(0, 3) = A(2, 5) = A(3, 6) \\
  B(0, 3) = B(2, 5) = B(3, 6)
  \]

Since \(A(0, 3) = A(2, 5)\), we have two equations and two unknowns:

\[
A(0, 3)e^{-B(0, 3)(0.08)} = 0.8782 \\
A(0, 3)e^{-B(0, 3)(0.10)} = 0.8395
\]

Dividing the second equation into the first equation allows us to find \(B(0, 3)\):

\[
e^{-B(0, 3)(0.08) + B(0, 3)(0.10)} = \frac{0.8782}{0.8395}
\]

\[
B(0, 3)(0.02) = \ln \left( \frac{0.8782}{0.8395} \right)
\]

\[
B(0, 3) = 2.25339
\]

We can now solve for the value of \(A(0, 3)\):

\[
A(0, 3)e^{-B(0, 3)(0.08)} = 0.8782 \\
A(0, 3) = 0.8782e^{B(0, 3)(0.08)} \\
A(0, 3) = 0.8782e^{2.25339(0.08)} \\
A(0, 3) = 1.05168
\]

We can now solve for \(r^*\):

\[
A(0, 3)e^{-B(0, 3)(r^*)} = 0.8586 \\
1.05168e^{-2.25339r^*} = 0.8586 \\
-2.25339r^* = \ln \left( \frac{0.8586}{1.05168} \right)
\]

\[
r^* = 0.0900
\]
Solution 45

B  Chapter 24, Vasicek Model

The process describes the Vasicek model with:

\[ a = 0.2 \]
\[ b = 0.1 \]
\[ \sigma = 0.02 \]

The Sharpe ratio is:

\[ \phi = 0.05 \]

We can use these values to find the price of the 10-year zero-coupon bond:

\[ \bar{r} = b + \frac{\phi \sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.1 + 0.05 \left( \frac{0.02}{0.2} \right) - 0.5 \left( \frac{0.02^2}{0.2^2} \right) = 0.1000 \]

\[ B(t, T) = \frac{1 - e^{-\bar{r}(T-t)}}{a} = \frac{1 - e^{-0.2(10)}}{0.2} = 4.32332 \]

\[ A(t, T) = e^{-\bar{r} \cdot [B(t, T) - (T-t)]} = e^{-0.1000(4.32332-10) - \frac{4.32332^2 \times 0.02^2}{4 \times 0.2}} = 0.56157 \]

\[ P(r, t, T) = A(t, T)e^{-B(t, T)r} = 0.56157e^{-4.32332 \times 0.08} = 0.39737 \]

Since the bond matures for $100, its price is:

\[ 100 \times 0.39737 = 39.737 \]

Solution 46

C  Chapter 24, Vasicek Model

The process describes the Vasicek model with:

\[ a = 0.15 \]
\[ b = 0.12 \]
\[ \sigma = 0.03 \]
\[ \phi = 0.07 \]

Since the current value of the short-term interest rate is 8%:

\[ a(r) = a \times (b - r) = 0.15 \times (0.12 - 0.08) = 0.006 \]
\[ \sigma(r) = \sigma = 0.03 \]

We can use the following relationship to find the value of theta, \( P_t \):

\[ rP = \frac{1}{2} [\sigma(r)]^2 P_{rr} + [a(r) + \sigma(r)\phi(r, t)]P_r + P_t \]
We can use the formula for the price of a zero-coupon bond to find the first and second derivative with respect to the short-term interest rate:

\[ P(r,t,T) = A(t,T)e^{-B(t,T)r} \]
\[ P_r = -B(t,T)A(t,T)e^{-B(t,T)r} = -B(t,T)P(r,t,T) \]
\[ P_{rr} = [B(t,T)]^2 A(t,T)e^{-B(t,T)r} = [B(t,T)]^2 P(r,t,T) \]

The value of \( B(0,10) \) is:

\[ B(0,10) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.15(10-0)}}{0.15} = 5.17913 \]

We have:

\[ P(0.08,0,10) = 0.36636 \]
\[ P_r = -B(t,T)P(r,t,T) = -5.17913 \times 0.36636 = -1.89743 \]
\[ P_{rr} = [B(t,T)]^2 P(r,t,T) = 5.17913^2 \times 0.36636 = 9.82702 \]

We can now find theta:

\[ rP = \frac{1}{2} [\sigma(r)]^2 P_{rr} + [\sigma(r) + \sigma(r)\phi(r,t)]P_r + P_t \]
\[ 0.08(0.36636) = \frac{1}{2} [0.03]^2 (9.82702) + [0.006 + (0.03)(0.07)](-1.89743) + P_t \]
\[ P_t = 0.0403 \]

Solution 47

A Chapter 24, Risk-Neutral Interest Rate Models

Zachary is using the Vasicek model:

\[ dr = a(b - r)dt + \sigma dZ(t) \]
\[ dr = 0 \times (0.10 - r)dt + 0.05dZ(t) \]
\[ dr = 0.05dZ(t) \]

Although Zachary’s model resembles Charlie’s model, Zachary’s model uses a non-zero Sharpe ratio:

\[ \phi = 0.20 \]

Therefore, Zachary’s model is not risk-neutral.

Charlie’s model is a risk-neutral version of the Vasicek model, so Charlie uses a Sharpe ratio of zero:

\[ \phi = 0.00 \]

Therefore, Charlie’s model is different from Zachary’s model.
The risk-neutral version of Zachary’s model is:

\[
\begin{align*}
    dr &= [a(r) + \sigma(r) \phi(r, t)]dt + \sigma(r)d\tilde{Z}(t) \\
    &= [a(b - r) + \sigma \phi]dt + \sigma d\tilde{Z}(t) \\
    &= [0 \times (0.10 - r) + 0.05 \times 0.20]dt + 0.05d\tilde{Z}(t) \\
    &= 0.01dt + 0.05d\tilde{Z}(t)
\end{align*}
\]

This risk-neutral process is the same as the risk-neutral process used by Ann. Therefore, Choice (A) is the correct answer.

\section*{Solution 48}

A Chapter 24, Delta-Gamma-Theta Approximation

The interest rate process describes the Vasicek model. In this model, we have:

\[
\begin{align*}
    P(r, t, T) &= A(t, T)e^{-B(t, T)r} \\
    P_r(r, t, T) &= -B(t, T)A(t, T)e^{-B(t, T)r} \\
    P_{rr}(r, t, T) &= [B(t, T)]^2 A(t, T)e^{-B(t, T)r}
\end{align*}
\]

We can calculate the value of \( B(t, T) \):

\[
B(0,5) = \frac{1 - e^{-a(T-t)}}{a} = \frac{1 - e^{-0.25(5-0)}}{0.25} = 2.85398
\]

We are given \( P_t \) in the question, and we can calculate \( P_r \) and \( P_{rr} \):

\[
\begin{align*}
    P_r(0.07, 0, 5) &= -B(0, 5)A(0, 5)e^{-B(0, 5)r} = -2.85398 \times 0.5961 = -1.70126 \\
    P_{rr}(0.07, 0, 5) &= [B(0, 5)]^2 A(0, 5)e^{-B(0, 5)r} = 2.85398^2 \times 0.5961 = 4.85536 \\
    P_t(0.07, 0, 5) &= 0.0685
\end{align*}
\]

A one standard deviation movement in the short-term interest rate over one day is:

\[
\sigma \sqrt{dt} = \sigma \sqrt{h} = 0.10 \times \sqrt{\frac{1}{365}} = 0.0052342
\]

We can use the delta-gamma-theta approximation to estimate the resulting change in the price:

\[
\begin{align*}
    dP &= P_r dr + \frac{1}{2} P_{rr} (dr)^2 + P_t dt \\
    &\approx -1.70126(0.0052342) + \frac{1}{2} (4.85536)(0.0052342)^2 + 0.0685 \left( \frac{1}{365} \right) = -0.008651
\end{align*}
\]

The new price is:

\[
0.5961 - 0.008651 = 0.5874
\]
Solution 49

C Chapter 24, Black Model

We need the volatility of a forward that matures at time 2 and has a bond maturing at time 4 as its underlying asset:

\[
\sigma = \sqrt{\frac{\text{Var} \left\{ \ln \left[ P_t(T, T + s) \right] \right\}}{t}} = \sqrt{\frac{\text{Var} \left\{ \ln \left[ P_t(2, 4) \right] \right\}}{t}} = \sqrt{\frac{0.0289}{t}} = 0.17
\]

The bond forward price is:

\[
F = P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)} = \frac{P(0, 4)}{P(0, 2)} = \frac{0.6956}{0.8573} = 0.81138
\]

The values of \(d_1\) and \(d_2\) are:

\[
d_1 = \ln \left( \frac{F}{K} \right) + 0.5\sigma^2 T = \frac{\ln \left( \frac{0.81138}{0.92} \right) + 0.5(0.17)^2(2)}{0.17\sqrt{2}} = -0.40235
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = -0.40235 - 0.17\sqrt{2} = -0.64277
\]

We have:

\[
N(d_1) = N(-0.40235) = 0.34371
\]

\[
N(d_2) = N(-0.64277) = 0.26019
\]

The Black formula for the call price is:

\[
C = P(0, T) \left[ F \times N(d_1) - K \times N(d_2) \right]
\]

\[
= 0.8573 \left[ 0.81138 \times 0.34371 - 0.92 \times 0.26019 \right]
\]

\[
= 0.03387
\]

Solution 50

E Chapter 24, Cox-Ingersoll-Ross Model

We begin with the Sharpe ratio and parameterize it for the Cox-Ingersoll-Ross model:

\[
\phi(r, t) = \frac{\alpha(r, t, T) - r}{q(r, t, T)}
\]

\[
\phi \frac{\sqrt{r}}{\sigma} = \frac{\alpha(r, t, T) - r}{B(t, T)\sigma(r)}
\]

\[
\phi r = \frac{\alpha(r, t, T) - r}{B(t, T)}
\]
We can substitute 7.0% for $\alpha(0.05,0,10)$:

$$\bar{\phi} \times 0.05 = \frac{0.07 - 0.05}{B(0,10)}$$

$$0.07 = 0.05 + B(0,10)\bar{\phi}(0.05)$$

$$B(0,10)\bar{\phi} = 0.40$$

Making use of the fact that $B(0,10) = B(5,15)$, we have:

$$\bar{\phi} \times 0.06 = \frac{\alpha(0.06,5,15) - 0.06}{B(5,15)}$$

$$\alpha(0.06,5,15) = 0.06 + B(5,15)\bar{\phi}(0.06)$$

$$= 0.06 + 0.40(0.06)$$

$$= 0.084$$

Solution 51

C Chapter 24, Vasicek Model

The Vasicek model of short-term interest rates is:

$$dr = \alpha(b - r)dt + \sigma dZ$$

Therefore, we can determine the value of $\alpha$:

$$dr = 0.35(b - r)dt + \sigma dZ \quad \Rightarrow \quad \alpha = 0.35$$

In the Vasicek model, the Sharpe ratio is constant:

$$\phi(r,t) = \phi$$

Therefore, for any $r$, $t$, and $T$, we have:

$$\phi = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

Since the Sharpe ratio is constant:

$$\frac{\alpha(0.06,0.5) - 0.06}{q(0.06,0.5)} = \frac{\alpha(0.03,2.8) - 0.03}{q(0.03,2.8)}$$

We now make use of the following formula for $q(r,t,T)$:

$$q(r,t,T) = B(t,T)\sigma(r) = B(t,T)\sigma$$
Substituting this expression for $q(r,t,T)$ into the preceding equation allows us to solve for $\alpha(0.03, 2, 8)$:

\[
\frac{\alpha(0.06, 0.5) - 0.06}{q(0.06, 0.5)} = \frac{\alpha(0.03, 2, 8) - 0.03}{q(0.03, 2, 8)}
\]

\[
\frac{\alpha(0.06, 0.5) - 0.06}{B(0.5) \sigma} = \frac{\alpha(0.03, 2, 8) - 0.03}{B(2, 8) \sigma}
\]

\[\frac{0.073 - 0.06}{0.35} = \frac{\alpha(0.03, 2, 8) - 0.03}{0.35}
\]

\[\frac{2.3606}{0.013} = \frac{\alpha(0.03, 2, 8) - 0.03}{2.5073}
\]

\[\alpha(0.03, 2, 8) = 0.0438
\]

**Solution 52**

**D** Chapter 24, Interest Rate Derivative

The realistic process for the short rate follows:

\[dr = a(r)dt + \sigma(r)dZ \quad \text{where:} \quad a(r) = 0.04 - 0.25r \quad \& \quad \sigma(r) = 0.4
\]

The risk-neutral process follows:

\[dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}
\]

We can use the coefficient of the first term of the risk-neutral process to solve for the Sharpe ratio, $\phi(r,t)$:

\[0.10 - 0.25r = a(r) + \sigma(r)\phi(r,t)
\]

\[0.10 - 0.25r = 0.04 - 0.25r + 0.4\phi(r,t)
\]

\[\phi(r,t) = 0.15
\]

The derivative, like all interest-rate dependent assets, must have a Sharpe ratio of 0.15. Let’s rearrange the differential equation for $g$, so that we can more easily observe its Sharpe ratio:

\[\frac{dg}{g} = (r + 0.09)dt - \beta dZ \quad \Rightarrow \quad \phi(r,t) = \frac{(r + 0.09) - r}{\beta} = \frac{0.09}{\beta}
\]

Since the Sharpe ratio is 0.15:

\[0.15 = \frac{0.09}{\beta}
\]

\[\beta = 0.6
\]
Solution 53

Chapter 24, Interest Rate Derivative

Let’s rearrange the differential equation for $g$, so that we can more easily observe its Sharpe ratio:

$$\frac{dg}{g} = (r + 0.02)dt - 0.08dZ$$

$$\Rightarrow \phi(r, t) = \frac{(r + 0.02) - r}{0.08} = 0.25$$

The realistic process for the short rate follows:

$$dr = a(r)dt + \sigma(r)dZ$$

where:

$$a(r) = 0.04 - 0.25r \quad \& \quad \sigma(r) = 0.4$$

The risk-neutral process follows:

$$dr = [a(r) + \sigma(r)\phi(r, t)]dt + \sigma(r)d\tilde{Z}$$

We can now find the coefficient of the first term of the risk-neutral process:

$$\mu(r) = a(r) + \sigma(r)\phi(r, t)$$

$$= 0.04 - 0.25r + 0.4 \times 0.25$$

$$= 0.14 - 0.25r$$

Solution 54

Chapter 24, Black-Derman-Toy Model

Each node is $e^{2\sigma \sqrt{h}}$ times the one below it. This means that 38.75% is $e^{4\sigma_2}$ times as large as 10.08%:

$$0.3875 = 0.1008e^{4\sigma_2}$$

$$\sigma_2 = \ln \left( \frac{0.3875}{0.1008} \right) \times \frac{1}{4} = 0.33664$$

The interest rate for the node above 10.08% is:

$$0.1008e^{2\sigma_2} = 0.1008e^{2 \times 0.33664} = 0.1976$$

The tree of prices for the 3-year bond is:

<table>
<thead>
<tr>
<th>Year</th>
<th>0.7207</th>
<th>0.6567</th>
<th>?</th>
<th>0.8350</th>
<th>0.7669</th>
<th>0.9084</th>
</tr>
</thead>
</table>

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The calculations to obtain these prices are:

\[ P(2,3, r_{uu}) = \frac{1}{1.3875} = 0.7207 \]
\[ P(2,3, r_{ud}) = P(2,3, r_{du}) = \frac{1}{1.1976} = 0.8350 \]
\[ P(2,3, r_{dd}) = \frac{1}{1.1008} = 0.9084 \]
\[ P(1,3, r_u) = \frac{0.5(0.7207 + 0.8350)}{1.1845} = 0.6567 \]
\[ P(1,3, r_d) = \frac{0.5(0.8350 + 0.9084)}{1.1367} = 0.7669 \]

We cannot calculate the current price of the bond, \( P(0,3) \) because we do not know the value of \( r_0 \), but we do not need \( P(0,3) \) to answer this question.

The formula for the one-year yield volatility for a \( T \)-year zero-coupon bond is:

\[
\text{Yield volatility} = 0.5 \times \ln \left[ \frac{P(1,T, r_u)^{-1/(T-1)} - 1}{P(1,T, r_d)^{-1/(T-1)} - 1} \right]
\]

The yield volatility of the 3-year bond is:

\[
\text{Yield volatility} = 0.5 \times \ln \left[ \frac{0.6567^{-1/(3-1)} - 1}{0.7669^{-1/(3-1)} - 1} \right] = 0.2501
\]

Solution 55

D Chapter 24, Black-Derman-Toy Model

The value of \( r_0 \) is the same as the yield on the 1-year zero-coupon bond, and the yield volatility of the 2-year bond is \( \sigma_1 \):

\[
\frac{1}{1 + r_0} = 0.9346 \quad \Rightarrow \quad r_0 = 7\%
\]
\[
\sigma_1 = 9\%
\]

The first part of the interest rate tree is:

\[
\begin{align*}
r_u &= R_t e^{2\sigma_1} \\
r_d &= R_t e^{0.18} \\
\frac{1}{1 + r_0} &= 0.9346 \quad \Rightarrow \quad r_0 = 7\% \\
r_d &= R_t \quad \Rightarrow \quad r_d
\end{align*}
\]
We need to determine the value of $r_d$. The tree must correctly price the 2-year zero-coupon bond, so:

$$P(0,2) = \frac{1}{1.07} (0.5) \left( \frac{1}{1 + r_d e^{0.18}} + \frac{1}{1 + r_d} \right)$$

$$0.8264 = 0.9346(0.5) \left( \frac{1}{1 + r_d e^{0.18}} + \frac{1}{1 + r_d} \right)$$

The quickest way to answer this question is to use trial and error. Let’s try Choice C first:

$$0.9346(0.5) \left( \frac{1}{1 + (0.10)e^{0.18}} + \frac{1}{1 + 0.10} \right) = 0.8422$$

Since the 0.8422 is greater than 0.8264, let’s try a higher short rate. Let’s try Choice D:

$$0.9346(0.5) \left( \frac{1}{1 + (0.1193)e^{0.18}} + \frac{1}{1 + 0.1193} \right) = 0.8264$$

Choice D is the correct answer.

Solution 56

E  Chapters 20 and 24, CIR Model

Let’s find the differential form of each choice.

Choice A does not contain a random variable, so its differential form is easy to find:

$$dr(t) = \left[ -abr(0)e^{-at} + a\sigma e^{-at} \right] dt$$

Choice B is easily recognized as an arithmetic Brownian motion:

$$dr(t) = bdt + \sigma dZ(t)$$

Choice C is easily recognized as a geometric Brownian motion:

$$dr(t) = br(t)dt + \sigma r(t)dZ(t)$$

Choice D is (perhaps not so easily) recognized as an Ornstein-Uhlenbeck process:

$$dr(t) = a \times [b - r(t)] dt + \sigma dZ(t)$$

Choice D therefore describes the Vasicek Model.

By the process of elimination, we see that Choice E must be the correct answer. But for the sake of thoroughness, let’s find the differential of Choice E as well.
The first two terms of Choice E do not contain random variables, so their differentials are easy to find. The third term will be more difficult:

\[ r(t) = r(0)e^{-at} + b\left(1 - e^{-at}\right) + \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)}dZ(s) \]
\[ dr(t) = -ar(0)e^{-at} dt + abe^{-at} dt + d \left[ \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)}dZ(s) \right] \]

The third term has a function of \( t \) in the integral. We can pull the \( t \)-dependent portion out of the integral, so that we are finding the differential of a product. We then use the following version of the product rule to find the differential:

\[ d[U(t)V(t)] = dU(t)V(t) + U(t)dV(t) \]

The differential of the third term is:

\[ d \left[ \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)}dZ(s) \right] = d \left[ \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)}dZ(s) \right] \]
\[ = -a\sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)}dZ(s) + \sigma e^{-at} e^{at} \sqrt{r(t)}dZ(t) \]
\[ = -a\sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)}dZ(s) + \sigma \sqrt{r(t)}dZ(t) \]

Putting all three terms together, we have:

\[ dr(t) = -ar(0)e^{-at} dt + abe^{-at} dt - a\sigma e^{-at} dt \int_0^t e^{as} \sqrt{r(s)}dZ(s) + \sigma \sqrt{r(t)}dZ(t) \]
\[ = -a \left[ r(0)e^{-at} - be^{-at} + \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)}dZ(s) \right] dt + \sigma \sqrt{r(t)}dZ(t) \]
\[ = -a \left[ r(0)e^{-at} + b - be^{-at} + \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)}dZ(s) \right] dt + \sigma \sqrt{r(t)}dZ(t) \]
\[ = -a \left[ r(0)e^{-at} + b - be^{-at} + \sigma e^{-at} \int_0^t e^{as} \sqrt{r(s)}dZ(s) \right] dt + abdt + \sigma \sqrt{r(t)}dZ(t) \]
\[ = -a \left[ r(0)e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{a(s-t)} \sqrt{r(s)}dZ(s) \right] dt + abdt + \sigma \sqrt{r(t)}dZ(t) \]

The portion in the brackets above is equal to the expression for \( r(t) \) provided in the Choice E:

\[ dr(t) = -ar(t) dt + abdt + \sigma \sqrt{r(t)}dZ(t) \]
\[ = abdt - ar(t) dt + \sigma \sqrt{r(t)}dZ(t) \]
\[ = a[b - r(t)] dt + \sigma \sqrt{r(t)}dZ(t) \]
Solution 57

C  Chapter 24, Delta-Gamma Approximation for Bonds

In the model described in the question, we have:

\[ P(r,t,T) = A(t,T) e^{-B(t,T)r} \]
\[ P_r(r,t,T) = -B(t,T)A(t,T)e^{-B(t,T)r} = -B(t,T)P(r,t,T) \]
\[ P_{rr}(r,t,T) = [B(t,T)]^2 A(t,T)e^{-B(t,T)r} = [B(t,T)]^2 P(r,t,T) \]

The delta-gamma-theta approximation is:

\[ P[r(t + h), t + h, T] - P[r(t), t, T] = [r(t + h) - r(t)] P_r + 0.5 [r(t + h) - r(t)]^2 P_{rr} + P_t h \]

This question asks us to use only the delta and gamma portions of the approximation, so we make the following adjustments to the formula above:

- The theta term, \( P_t h \), is removed from the expression above.
- We replace \( r(t) \) with 0.06.
- We replace \( r(t + h) \) with 0.02.

The delta-gamma approximation is therefore:

\[ P_{est}(0.02,0,2) - P(0.06,0,2) \approx \left[-0.04 \right] P_r(0.06,0,2) + 0.5 \left[-0.04 \right]^2 P_{rr}(0.06,0,2) \]
\[ P_{est}(0.02,0,2) \approx P(0.06,0,2) + \left[-0.04 \right] P_r(0.06,0,2) + 0.5 \left[-0.04 \right]^2 P_{rr}(0.06,0,2) \]
\[ P_{est}(0.02,0,2) \approx P(0.06,0,2) \left[1 + 0.04 \times 3 + 0.5 \times (-0.04)^2 \times 3^2 \right] \]
\[ P_{est}(0.02,0,2) \approx P(0.06,0,2) \times 1.1272 \]

Therefore, we have:

\[ \frac{P_{est}(0.02,0,2)}{P(0.06,0,2)} \approx \frac{P(0.06,0,2) \times 1.1272}{P(0.06,0,2)} = 1.1272 \]

Solution 58

A  Chapters 10 & 24, Risk-Neutral Probability

Let’s use \( P(0,2) \) to denote the price of a 2-year zero-coupon bond that matures for $1.

We can make use of put-call parity:

\[ C(69) + 69 \times P(0,2) = S_0 + P(69) \]
\[ P(69) - C(69) = 69 \times P(0,2) - 75 \]
We can use the stock prices to determine the risk-neutral probability that the up state of the world occurs:

\[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{(1 + r_0)^h - d}{u - d} = \frac{(1.06)^1 - 50/75}{100/75 - 50/75} = \frac{0.3933}{0.6667} = 0.59 \]

If the up state occurs, then the zero-coupon bond will have a value of \( \frac{1}{1.075} \) at time 1, and if the down state occurs, then the zero-coupon bond will have a value of \( \frac{1}{1.045} \) at time 1.

The time 0 value is found using the risk-neutral probabilities and the risk-free rate at time 0:

\[ P(0,2) = \frac{1}{1.06} \left[ 0.59 \times \frac{1}{1.075} + 0.41 \times \frac{1}{1.045} \right] = 0.88791 \]

We can now use the equation for put-call parity described above to find the solution:

\[ P(69) - C(69) = 69 \times P(0,2) - 75 = 69 \times 0.88791 - 75 = -13.7344 \]

**Solution 59**

**D** Chapter 24, Theta in the Cox-Ingersoll-Ross Model

The process describes the Cox-Ingersoll-Ross-Model with:

\[ a(r) = a \times (b - r) = 0.09(0.11 - r) \]
\[ \sigma(r) = \sigma \sqrt{r} = 0.22 \sqrt{r} \]

The price of a 5-year bond is:

\[ P(r, t, T) = A(t, T)e^{-B(t,T)r} = 0.9054e^{-3.4854 \times 0.06} = 0.7345 \]

The formula for the price of a bond must satisfy the following partial differential equation:

\[ rP = \frac{1}{2} [\sigma(r)]^2 P_{rr} + [a(r) + \sigma(r) \phi(r, t)] P_t + P_t \]

Delta and gamma for the bond are:

\[ P(r, t, T) = A(t, T)e^{-B(t,T)r} \]
\[ \text{Delta:} \quad P_r = -B(t,T)P(r, t, T) = -3.4854 \times 0.7345 = -2.5602 \]
\[ \text{Gamma:} \quad P_{rr} = [B(t,T)]^2 P(r, t, T) = (3.4854)^2 \times 0.7345 = 8.9233 \]
We can now solve for theta:

\[ rP = \frac{1}{2} \left[ \sigma(r)^2 P_{rr} + [a(r) + \sigma(r)\phi(r,t)]P_r + P_t \right] \]

\[ 0.06 \times 0.7345 \]

\[ = \frac{1}{2} \left( 0.22\sqrt{0.06} \right)^2 \times 8.9233 + \left[ 0.09(0.11 - 0.06) + 0.22\sqrt{0.06 \times (0.00)} \right](-2.5602) + P_t \]

\[ P_t = 0.0426 \]

**Solution 60**

**B** Chapter 24, Caplet in Black-Derman-Toy Model

In each column of rates, each rate is greater than the rate below it by a factor of:

\[ e^{2\sigma_i \sqrt{h}} \]

Therefore, the missing rate in the third column is:

\[ 0.2290e^{-2\sigma_i \sqrt{h}} = 0.2290 \times \frac{0.2290}{0.3001} = 0.1747 \]

The missing rate in the fourth column is:

\[ 0.3582e^{-2\sigma_i \sqrt{h}} = 0.3582 \times \frac{0.1451}{0.1962} = 0.2649 \]

The tree of short-term rates is:

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 1</th>
<th>Time 2</th>
<th>Time 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.6478</td>
<td>0.0000</td>
<td>5.1314</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The caplet pays off only if the interest rate at the end of the third year is greater than 20%. The payoff table is:
The payments have been converted to their equivalents payable at the end of 3 years. The calculations for the tree above are shown below:

\[
\frac{100 \times (0.3582 - 0.20)}{1.3582} = 11.6478 \\
\frac{100 \times (0.2649 - 0.20)}{1.2649} = 5.1314
\]

We work recursively to calculate the value of the caplet. For example, if the interest rate increases at the end of the first year and at the end of the second year, then its value at the end of the second year is:

\[
\frac{0.5 \times 11.6478 + 0.5 \times 5.1314}{1.3001} = 6.4530
\]

The completed tree is:

```
Time 0       Time 1       Time 2       Time 3

11.6478      6.4530      3.5245      5.1314
1.9204      2.0876      0.8923      0.0000
0.0000      0.0000      0.0000      0.0000
```

The value of the year-4 caplet is $1.9204.

**Solution 61**

C Chapter 24, Cox-Ingersoll-Ross Model

The CIR model is:

\[
dr = a(b - r)dt + \sigma \sqrt{r} dZ
\]

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

- \( a = 0.12 \)
- \( b = \frac{0.0096}{0.12} = 0.08 \)
- \( \sigma = 0.10 \)

We can define \( c \) in terms of \( \bar{\phi} \):

\[
\phi(r, t) = \frac{\bar{\phi} \sqrt{r}}{\sigma} \quad \& \quad \phi(r, t) = c \sqrt{r} \quad \Rightarrow \quad c = \frac{\bar{\phi}}{\sigma}
\]

In the CIR model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

\[
\bar{r} = \frac{2ab}{(a - \bar{\phi} + \gamma)} = \lim_{T \to \infty} \left[ \frac{-\ln[P(r, t, T)]}{T - t} \right]
\]
From the information provided in part (ii) of the question, we know that \( \bar{r} \) is equal to 0.0725. Therefore:

\[
\frac{2ab}{(a-\bar{\phi}+\gamma)} = 0.0725
\]

\[
2 \times 0.12 \times 0.08 = 0.0725
\]

\[
(0.12-\bar{\phi}+\gamma) = 0.0192 = 0.0725(0.12-\bar{\phi}+\gamma)
\]

\[
0.26483 = 0.12-\bar{\phi}+\gamma
\]

\[
\gamma = \bar{\phi} + 0.14483
\]

We can substitute this value into the formula for \( \gamma \):

\[
\gamma = \sqrt{(a-\bar{\phi})^2 + 2\sigma^2}
\]

\[
\bar{\phi} + 0.14483 = \sqrt{(0.12-\bar{\phi})^2 + 2(0.1)^2}
\]

\[
\bar{\phi}^2 + 0.28966\bar{\phi} + 0.020975 = (0.12-\bar{\phi})^2 + 2(0.1)^2
\]

\[
\bar{\phi}^2 + 0.28966\bar{\phi} + 0.020975 = (0.0144 - 0.24\bar{\phi} + \bar{\phi}^2) + 0.02
\]

\[
0.52966\bar{\phi} = 0.013425
\]

\[
\bar{\phi} = 0.025347
\]

The value of \( c \) is:

\[
c = \frac{\phi}{\sigma} = \frac{0.025347}{0.10} = 0.25347
\]

Solution 62

A Chapter 24, Vasicek Model

The Vasicek model is:

\[
dr = a(b-r)dt + \sigma dZ
\]

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

\[
a = 0.30 \quad b = \frac{0.03}{0.3} = 0.10 \quad \sigma = 0.12
\]

In the Vasicek model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

\[
\bar{r} = b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = \lim_{T \to \infty} \left[ -\ln[P(r,t,T)] \right]
\]

\[
\frac{-\ln[T-t]}{T-t}
\]
From the information provided the question, we know that $r$ is equal to 0.04. Therefore:

$$b + \phi \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = 0.04$$

$$0.1 + \phi \frac{0.12}{0.30} - 0.5 \frac{0.12^2}{0.30^2} = 0.04$$

$$\phi = 0.05$$

Solution 63

D  Chapter 24, Binomial Interest Rate Model

A 1-year bond in 2 years is a 3-year bond now. The tree of prices for the 3-year bond is:

<table>
<thead>
<tr>
<th></th>
<th>0.8353</th>
<th>1.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7278</td>
<td>0.8869</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.6658</td>
<td>0.9048</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.8421</td>
<td>0.9418</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

There is no need to calculate the first two columns of the tree to answer this question, but they are shown above for illustrative purposes.

The third column is easily obtained by discounting a 1-year bond at the possible interest rates at time 2:

$$e^{-0.18} = 0.8353$$

$$e^{-0.12} = 0.8869$$

$$e^{-0.10} = 0.9048$$

$$e^{-0.06} = 0.9418$$

The put option pays off only if the time-2 price falls below $0.90. This occurs at 2 nodes, and the payoffs are:

- up-up: $V_2 = 0.90 - 0.8353 = 0.0647$
- up-down: $V_2 = 0.90 - 0.8869 = 0.0131$

The price of the put option is:

$$V_0 = E^{\Delta t} \left[ V_2 \times e^{-\sum_{i=0}^{2} \eta_i} \right] = \frac{(0.8)^2 \cdot 0.0647}{e^{(0.12+0.15)}} + \frac{(0.8)(0.2) \cdot 0.0131}{e^{(0.12+0.15)}} = 0.0332$$
Solution 64

B Chapter 24, Black-Derman-Toy Model

The ratio of the short rate to the rate below it is constant within a column, so:

\[
\frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}} \implies \frac{0.45}{0.20} = \frac{r_{ud}}{r_{ud}} \implies r_{ud} = 0.30
\]

The value of a 2-year bond is:

\[
P(0,2) = 0.5 \left[ \left( \frac{1}{1 + r_0} \right) \left( \frac{1}{1 + r_u} \right) + \left( \frac{1}{1 + r_0} \right) \left( \frac{1}{1 + r_d} \right) \right]
\]

\[
= 0.5 \frac{1}{1 + r_0} \left[ \frac{1}{1.40} + \frac{1}{1.25} \right] = \frac{0.75714}{1 + r_0}
\]

The value of a 3-year bond is:

\[
P(0,3) = 0.25 \left[ \left( \frac{1}{1 + r_0} \right) \left( \frac{1}{1 + r_u} \right) \left( \frac{1}{1 + r_{uu}} \right) + \left( \frac{1}{1 + r_0} \right) \left( \frac{1}{1 + r_u} \right) \left( \frac{1}{1 + r_{ud}} \right) + \left( \frac{1}{1 + r_0} \right) \left( \frac{1}{1 + r_d} \right) \left( \frac{1}{1 + r_{dd}} \right) \right]
\]

\[
= 0.25 \frac{1}{1 + r_0} \left[ \frac{1}{1.40} \left( \frac{1}{1.45} \right) + \frac{1}{1.40} \left( \frac{1}{1.30} \right) + \frac{1}{1.25} \left( \frac{1}{1.30} \right) + \frac{1}{1.25} \left( \frac{1}{1.20} \right) \right]
\]

\[
= \frac{0.58103}{1 + r_0}
\]

The forward price is:

\[
F_{0,2} [P(2,3)] = P_0(2,3) = \frac{P(0,3)}{P(0,2)} = \frac{0.58103}{\frac{0.75714}{1 + r_0}} = 0.76740
\]

Multiplying by 1,000, we have:

\[
1,000 \times F_{0,2} [P(2,3)] = 1,000 \times 0.76740 = 767.40
\]

Solution 65

C Chapter 24, Risk-Neutral Version of the Vasicek Model

The process for the short rate follows the Vasicek Model. The true process can be rewritten in the familiar form:

\[
dr = 0.08(0.10 - r)dt + 0.04dZ \quad \implies \quad a = 0.08, \ b = 0.10, \ \sigma = 0.04
\]
The risk-neutral process can be written as:
\[
dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z} = [a(b - r) + \sigma \phi]dt + \sigma d\tilde{Z}
\]
\[
= [0.08(0.10 - r) + 0.04\phi]dt + 0.04d\tilde{Z}
\]
We can use set the bracketed expression above equal to the bracketed expression in (ii):
\[
0.08(0.10 - r) + 0.04\phi = 0.0128 - 0.08r
\]
\[
0.008 + 0.04\phi - 0.08r = 0.0128 - 0.08r
\]
\[
\phi = 0.12
\]
The parameter \(B(2,6)\) is:
\[
B(t, T) = \frac{1 - e^{-a(T-t)}}{a}
\]
\[
B(2, 6) = \frac{1 - e^{-0.08(6-2)}}{0.08} = 3.4231
\]
We can use the Sharpe ratio to determine \(\alpha(0.04, 2, 6)\):
\[
\phi(r, t) = \frac{\alpha(r, t, T) - r}{q(r, t, T)}
\]
\[
\phi = \frac{\alpha(r, t, T) - r}{B(t, T)\sigma}
\]
\[
\alpha(r, t, T) = r + B(t, T)\sigma\phi
\]
\[
\alpha(0.04, 2, 6) = 0.04 + 3.4231 \times 0.04 \times 0.12 = 0.05643
\]

**Solution 66**

**B** Chapter 24, Risk-Neutral Version of the Vasicek Model

From (i), we observe that:
\[
a(r) = x - y \times r(t)
\]
\[
\sigma(r) = \sigma = 0.04
\]
The risk-neutral version of the Vasicek model is:
\[
dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z} = [x - y \times r(t) + 0.04\phi]dt + 0.04d\tilde{Z}
\]
Setting the above expression in brackets equal to the expression in brackets from (ii), we have:
\[
x - y \times r(t) + 0.04\phi = x + 0.0048 - y \times r(t)
\]
\[
x + 0.04\phi = x + 0.0048
\]
\[
0.04\phi = 0.0048
\]
\[
\phi = 0.12
\]
The parameter $B(3,7)$ is:

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$B(3,7) = \frac{1 - e^{-a(7-3)}}{a} = \frac{1 - e^{-4y}}{y} = \bar{a}_{17} = 3.42314$$

We can use the Sharpe ratio to determine $\alpha(0.06,3,7)$:

$$\phi(r,t) = \alpha(r,t,T) - r$$

$$\phi = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma}$$

$$\alpha(r,t,T) = r + B(t,T)\sigma\phi$$

$$\alpha(0.06,3,7) = 0.06 + 3.42314 \times 0.04 \times 0.12 = 0.07643$$

Solution 67

C Chapter 12, Risk-Neutral Version of the Vasicek Model

The process for the short rate follows the Vasicek Model. The true process can be rewritten in the familiar form:

$$dr = 0.08(0.10 - r)dt + 0.04dZ \quad \Rightarrow \quad a = 0.08, \ b = 0.10, \ \sigma = 0.04$$

We can use (iii) to obtain the Sharpe ratio:

$$\bar{Z}(t) = Z(t) - \phi t$$

$$\bar{Z}(4) = Z(4) - 4\phi$$

$$0.02 = 0.22 - 4\phi$$

$$\phi = 0.05$$

The parameter $B(4,8)$ is:

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$B(4,8) = \frac{1 - e^{-a(8-4)}}{a} = \frac{1 - e^{-0.08 \times 4}}{0.08} = 3.42314$$

We can use the Sharpe ratio to determine $\alpha(0.07,4,8)$:

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$\phi = \frac{\alpha(r,t,T) - r}{B(t,T)\sigma}$$

$$\alpha(r,t,T) = r + B(t,T)\sigma\phi$$

$$\alpha(0.07,4,8) = 0.07 + 3.42314 \times 0.04 \times 0.05 = 0.0768$$
Solution 68

E Chapter 24, Cox-Ingersoll-Ross Model

The CIR model is:

\[ dr = a(b - r)dt + \sigma \sqrt{r} dZ \]

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

\[ a = 0.14 \quad b = \frac{0.014}{0.14} = 0.10 \quad \sigma = 0.12 \]

We can define \( c \) in terms of \( \phi \):

\[ \phi(r,t) = \frac{\sqrt{r}}{\sigma} \quad \& \quad \phi(r,t) = c \sqrt{r} \quad \Rightarrow \quad c = \frac{\phi}{\sigma} \]

In the CIR model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

\[ \bar{r} = \frac{2ab}{(a - \phi + \gamma)} = \lim_{T \to \infty} \left[ -\ln[P(r,t,T)] \right] \]  

\[ \bar{r} = \frac{2ab}{(a - \phi + \gamma)} = \frac{2 \times 0.14 \times 0.10}{(0.14 - \phi + \gamma)} = 0.14 \]

\[ 0.028 = 0.14(0.14 - \phi + \gamma) \]

\[ 0.2 = 0.14 - \phi + \gamma \]

\[ \gamma = \phi + 0.06 \]

We can substitute this value into the formula for \( \gamma \):

\[ \gamma = \sqrt{(a - \phi)^2 + 2\sigma^2} \]

\[ \phi + 0.06 = \sqrt{(0.14 - \phi)^2 + 2(0.12)^2} \]

\[ \phi^2 + 0.12\phi + 0.0036 = (0.14 - \phi)^2 + 2(0.12)^2 \]

\[ \phi^2 + 0.12\phi + 0.0036 = (0.0196 - 0.28\phi + \phi^2) + 0.0288 \]

\[ 0.4\phi = 0.0448 \]

\[ \phi = 0.112 \]

The value of \( c \) is:

\[ c = \frac{\phi}{\sigma} = \frac{0.112}{0.12} = 0.9333 \]
Solution 69

Chapter 24, Vasicek Model

The Vasicek model allows negative yields.

The Vasicek model is:

\[ dr = a(b - r)dt + \sigma dZ \]

From the partial differential equation provided in the question, we can obtain the following parameters for the CIR model:

\[ a = 0.20 \quad b = \frac{0.01}{0.2} = 0.05 \quad \sigma = 0.14 \]

In the Vasicek model, as the maturity of a zero-coupon bond approaches infinity, its yield approaches:

\[ \tau = b + \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = \lim_{T \to \infty} \left[ -\ln[P(r, t, T)] \right] \]

From the information provided the question, we know that \( \tau \) is equal to -0.083.

Therefore:

\[ b + \frac{\sigma}{a} - 0.5 \frac{\sigma^2}{a^2} = -0.083 \]

\[ 0.05 + \frac{0.14}{0.20} - 0.5 \frac{0.14^2}{0.20^2} = -0.083 \]

\[ \phi = 0.16 \]

Solution 70

Chapter 24, Rendleman-Bartter Model

In the Rendleman-Bartter Model, the short-term rate follows geometric Brownian motion.

From the Chapter 18 Review Note, the payoff of a call option can be found as:

\[ E[\text{Call Payoff}] = S_t e^{(\alpha - \delta)(T - t)} N(d_1) - K N(d_2) \]
We can consider the derivative to be 1,000 call options with the short rate replacing the stock price and 0.10 replacing the strike price. Although we have $-0.25$ for the volatility parameter in the geometric Brownian motion, we use the absolute value of $-0.25$ in the formulas for $\hat{d}_1$ and $\hat{d}_2$, because the $\sigma$ in the formulas is the standard deviation of the stock’s return, and it must therefore be positive:

$$\hat{d}_1 = \frac{\ln\left(\frac{r(t)}{K}\right) + (\alpha - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = \frac{\ln\left(\frac{0.09}{0.10}\right) + (0.10 + 0.5 \times 0.25^2) \times 0.25}{0.25\sqrt{0.25}} = -0.58038$$

$$\hat{d}_2 = \frac{\ln\left(\frac{r(t)}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = \frac{\ln\left(\frac{0.09}{0.10}\right) + (0.10 - 0.5 \times 0.25^2) \times 0.25}{0.25\sqrt{0.25}} = -0.70538$$

The values of $N(\hat{d}_1)$ and $N(\hat{d}_2)$ are:

$$N(\hat{d}_1) = N(-0.58038) = 0.28083$$
$$N(\hat{d}_2) = N(-0.70538) = 0.24029$$

The expected value of the payoff is:

$$E[\text{Payoff}] = 1,000 \times \left[ S_te^{(\alpha - \delta)(T - t)}N(\hat{d}_1) - K \times N(\hat{d}_2) \right]$$
$$= 1,000 \times \left[ 0.09e^{0.10(0.25)} \times 0.28083 - 0.10 \times 0.24029 \right] = 1.8855$$

Solution 71

A Chapter 24, Black Model

The bond forward price is:

$$F = P_0(T, T + s) = \frac{P(0, T + s)}{P(0, T)} = \frac{P(0, 2)}{P(0, 1)} = \frac{0.8417}{0.9259} = 0.90906$$

The values of $d_1$ and $d_2$ are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{0.90906}{0.90009}\right) + 0.5(0.10)^2(1)}{0.10\sqrt{1}} = 0.14918$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.14918 - 0.10\sqrt{1} = 0.04918$$

We have:

$$N(-d_1) = N(-0.14918) = 0.44071$$
$$N(-d_2) = N(-0.04918) = 0.48039$$
The Black formula for the put price is:

\[ P = P(0,T)\left[K \times N(-d_2) - F \times N(-d_1)\right] \]

\[ = 0.9259\left[0.90009 \times 0.48039 - 0.90906 \times 0.44071\right] \]

\[ = 0.029408 \]

**Solution 72**

**E Chapter 24, Black Model**

The price of the caplet is:

\[ \text{Caplet price} = (1.111) \times (\text{Put Option Price}) \]

The put option:

- has a zero-coupon bond that expires at time 2 as its underlying asset
- expires at time 1
- has a strike price of:

\[ K = \frac{1}{1 + K_R} = \frac{1}{1.111} = 0.90009 \]

The bond forward price is:

\[ F = P_0(T,T + s) = \frac{P(0,T + s)}{P(0,T)} = \frac{P(0,2)}{P(0,1)} = \frac{0.8417}{0.9259} = 0.90906 \]

The volatility of the bond forward price is:

\[ \sigma = \sqrt{\frac{\text{Var}\left\{\ln P_t(T,T + s)\right\}}{t}} = \sqrt{\frac{\text{Var}\left\{\ln P_t(1,2)\right\}}{t}} = \sqrt{\frac{\text{Var}\left\{\ln \left[\frac{P(t,2)}{P(t,1)}\right]\right\}}{t}} \]

\[ = \sqrt{\frac{0.01t}{t}} = \sqrt{0.01} = 0.1 \]

The values of \(d_1\) and \(d_2\) are:

\[ d_1 = \frac{\ln \left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln \left(\frac{0.90906}{0.90009}\right) + 0.5(0.10)^2}{0.10\sqrt{1}} = 0.14918 \]

\[ d_2 = d_1 - \sigma \sqrt{T} = 0.14918 - 0.10\sqrt{1} = 0.04918 \]

We have:

\[ N(-d_1) = N(-0.14918) = 0.44071 \]

\[ N(-d_2) = N(-0.04918) = 0.48039 \]
The Black formula for the put price is:

\[
P = P(0, T) \left[ K \times N(-d_2) - F \times N(-d_1) \right]
\]

= 0.9259\left[0.90009 \times 0.48039 - 0.90906 \times 0.44071\right]

= 0.029408

The price of the caplet is:

\[
\text{Caplet price} = (1.111) \times (\text{Put Option Price}) = 1.111 \times 0.029408 = 0.03267
\]

**Solution 73**

D Chapter 24, Cox-Ingersoll-Ross Model

In the CIR Model, we have:

\[
P(r, t, T) = A(t, T)e^{-B(t, T)r}
\]

In the CIR model, we have:

\[
A(t, T) = A(0, T - t) \quad \text{and} \quad B(t, T) = B(0, T - t).
\]

This implies that we can define \( A \) and \( B \) as shown below:

\[
A = A(0, 8) = A(1, 9) = A(3, 11)
\]

\[
B = B(0, 8) = B(1, 9) = B(3, 11)
\]

We can obtain the ratio of the second price to the first price:

\[
\begin{align*}
P(r, 0, 8) &= 0.4005 = Ae^{-Br} \\
P(1.25r, 1, 9) &= 0.3581 = Ae^{-1.25Br}
\end{align*}
\]

\[
\Rightarrow e^{-0.25Br} = \frac{0.3581}{0.4005}
\]

The ratio obtained above can be used to find the bond price needed to answer the question:

\[
\begin{align*}
P(1.5r, 3, 11) &= Ae^{-1.5Br} = Ae^{-1.25r} \times e^{-0.25r} = 0.3581 \times \frac{0.3581}{0.4005} = 0.3202
\end{align*}
\]
Appendix B.1 & C – Solutions

Solution 1

E Textbook Appendix C, Convex Positions

If a position has positive gamma, then it is convex with respect to the stock’s price. If it has negative gamma, then it is not convex.

Call options and put options have positive values for gamma. The gamma of the stock is zero. Therefore, writing a call and writing a put results in a negative gamma, so Edward has established a position that is not convex.
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