Chapter 7: Level Annuities Payable More than Once per Time Unit

7.01 Annuity-Immediate Payable \( m \)thly

It is common for recurring payments to be expressed as a rate per unit of time, even though the payments can be paid more frequently than that unit of time. For example, an annuity of $12,000 per year could be paid at a rate of $1,000 per month. In this example, the unit of time is one year and the payment period, which is the period of time between payments, is one month.

Consider an annuity-immediate that pays 1 per time unit, in increments of \( 1/m \) at the end of each period, where each period is of length \( 1/m \).

If each period is one month, then \( m \) is equal to 12.

As shown in the top row in the figure below, the annuity-immediate pays at a rate of 1 per unit of time:

\[
\begin{array}{ccccccccc}
\text{Payment} & \frac{1}{m} & \cdots & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\
\text{Time} & 0 & \frac{1}{m} & \cdots & 1 & \cdots & 2 & \cdots & (n-1) & \cdots & n \\
\text{Period} & 0 & 1 & \cdots & m & \cdots & 2m & \cdots & (n-1)m & \cdots & nm
\end{array}
\]

If the effective annual interest rate is \( i \), then the discount factor for a period of length \( 1/m \) is:

\[
\frac{1}{v^m} = \left(\frac{1}{1+i}\right)^m = \left(\frac{1}{1 + \frac{i(m)}{m}}\right)^m = \frac{1}{1 + \frac{i(m)}{m}}
\]

The annuity-immediate pays for \( nm \) periods of length \( 1/m \), and the effective interest rate for a period of length \( 1/m \) is \( i(m)/m \), so the present value of the annuity-immediate can be found by setting the unit of time equal to the length of a period:

\[
P_{V0} = \frac{1}{m} a_{nm;;;;;;;;m} = \frac{1}{m} v^m + \frac{1}{m} v^{2m} + \cdots + \frac{1}{m} v^{nm-1} + \frac{1}{m} v^{nm} = \frac{1}{m} \left[ \frac{1}{1 - v^m} \right] = \frac{1}{m} \left[ \frac{1 - \left(\frac{1}{v^m}\right)^{nm+1}}{1 - \left(\frac{1}{v^m}\right)} \right]
\]

\[
= \frac{1}{m} \left( \frac{1 + \frac{i(m)}{m}}{1 + \frac{i(m)}{m}} \right) \left[ \frac{1}{v^m} - \frac{nm+1}{m} \right] = \frac{1}{m} \left[ \frac{1}{v^m} - \frac{nm}{m} \right] = \frac{1}{m} \left[ \frac{1 - \left(\frac{1}{v^m}\right)^{nm}}{1 - \left(\frac{1}{v^m}\right)} \right]
\]
Since \( \left( \frac{1}{v^m} \right)^{nm} = v^n \), the final expression above can be written as:

\[
\frac{1}{m} \left[ \frac{1 - v^n}{i(m)} \right]
\]

If we simplify, we have:

\[
\frac{1}{m} \left[ \frac{1 - v^n}{i(m)} \right] = \frac{1 - v^n}{i(m)}
\]

This annuity is described as an annuity-immediate payable \( m \)thly, and the notation for its present value is:

\[
a_{ni}^{(m)} = \frac{1}{i(m)}
\]

Including the effective interest rate per time unit, as measured by the number of units inside the right-angle bracket, is optional, but the expression above can also be written as:

\[
a_{ni}^{(m)} = \frac{1 - v^n}{i(m)}
\]

Including the effective interest rate per time unit is also optional for the following annuity notation introduced in this chapter: \( a_{si}^{(m)} \), \( s_{si}^{(m)} \), and \( s_{ni}^{(m)} \).

The notation for the accumulated value of the annuity-immediate at time \( n \) is \( s_{ni}^{(m)} \), and it is equal to the accumulated value of the present value:

\[
A_{n} = s_{ni}^{(m)} = (1 + i)^n \times \frac{1 - v^n}{i(m)} = \frac{(1 + i)^n - 1}{i(m)}
\]

Alternatively, the accumulated value can be found by noting that there are \( nm \) periods, each of which have an effective interest rate of \( \frac{i(m)}{m} \):

\[
\frac{1}{m} s_{nm}^{(m)} \frac{i(m)}{m} = \frac{1}{m} \left( 1 + \frac{i(m)}{m} \right)^{nm} - 1 = \frac{1}{m} \times \frac{(1 + i)^n - 1}{i(m)}
\]

### Annuity-Immediate Payable \( m \)thly

The present value at time 0 and the accumulated value at time \( n \) of an annuity-immediate that makes payments of 1 per time unit, payable \( m \)thly, for \( n \) units of time are:

\[
a_{ni}^{(m)} = \frac{1}{m} a_{nm}^{(m)} \frac{i(m)}{m} = \frac{1 - v^n}{i(m)}
\]

\[
s_{ni}^{(m)} = \frac{1}{m} s_{nm}^{(m)} \frac{i(m)}{m} = \frac{(1 + i)^n - 1}{i(m)}
\]

#### Example 7.01

The nominal annual interest rate compounded monthly is 12%. Find the present value of an annuity-immediate that makes monthly payments at a rate of $36 per year for 10 years.