Chapter 7: Level Annuities Payable More than Once per Time Unit

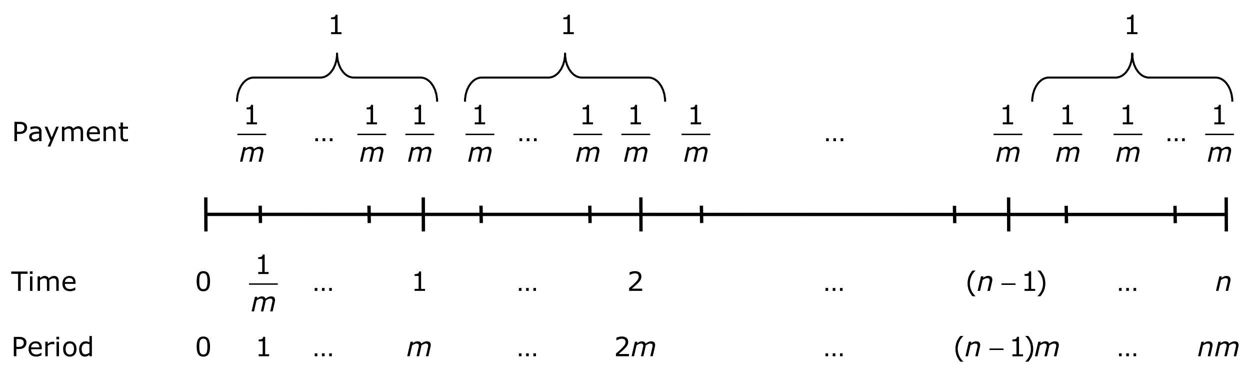
7.01 Annuity-Immediate Payable m^{th}ly

It is common for recurring payments to be expressed as a rate per unit of time, even though the payments can be paid more frequently than that unit of time. For example, an annuity of $12,000 per year could be paid at a rate of $1,000 per month. In this example, the unit of time is one year and the **payment period**, which is the period of time between payments, is one month.

Consider an annuity-immediate that pays 1 per time unit, in increments of $1/m$ at the end of each period, where each period is of length $1/m$.

If each period is one month, then m is equal to 12.

As shown in the top row in the figure below, the annuity-immediate pays at a rate of 1 per unit of time:



If the effective annual interest rate is i, then the discount factor for a period of length $1/m$ is:

$$v^{\frac{1}{m}} = \left(\frac{1}{1+i}\right)^{\frac{1}{m}} = \left(\frac{1}{\left(1+\frac{i^{(m)}}{m}\right)^{m}}\right)^{\frac{1}{m}} = \frac{1}{1+\frac{i^{(m)}}{m}}$$

The annuity-immediate pays for nm periods of length $1/m$, and the effective interest rate for a period of length $1/m$ is $i^{(m)}/m$, so the present value of the annuity-immediate can be found by setting the unit of time equal to the length of a period:

$$PV_0 = \frac{1}{m}a_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{\frac{nm-1}{m}} + \frac{1}{m}v^{\frac{nm}{m}} = \frac{1}{m}\left[\frac{v^{\frac{1}{m}} - v^{\frac{nm+1}{m}}}{1-v^{\frac{1}{m}}}\right]$$

$$= \frac{1}{m}\left(\frac{1+\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}\right)\left[\frac{v^{\frac{1}{m}} - v^{\frac{nm+1}{m}}}{1-v^{\frac{1}{m}}}\right] = \frac{1}{m}\left[\frac{1-v^{\frac{nm}{m}}}{\frac{i^{(m)}}{m}}\right] = \frac{1}{m}\left[\frac{1-\left(v^{\frac{1}{m}}\right)^{nm}}{\frac{i^{(m)}}{m}}\right]$$

Since $\left(v^{\frac{1}{m}}\right)^{nm} = v^n$, the final expression above can be written as:

$$\frac{1}{m}\left[\frac{1-v^n}{\frac{i^{(m)}}{m}}\right]$$

If we simplify, we have:

$$\frac{1}{m}\left[\frac{1-v^n}{\frac{i^{(m)}}{m}}\right] = \frac{1-v^n}{i^{(m)}}$$

This annuity is described as an annuity-immediate payable m^{th}ly, and the notation for its present value is:

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}}$$

Including the effective interest rate per time unit, as measured by the number of units inside the right-angle bracket, is optional, but the expression above can also be written as:

$$a_{\overline{n}|i}^{(m)} = \frac{1-v^n}{i^{(m)}}$$

Including the effective interest rate per time unit is also optional for the following annuity notation introduced in this chapter: $\ddot{a}_{\overline{n}|i}^{(m)}$, $s_{\overline{n}|i}^{(m)}$, and $\ddot{s}_{\overline{n}|i}^{(m)}$.

The notation for the accumulated value of the annuity-immediate at time n is $s_{\overline{n}|}^{(m)}$, and it is equal to the accumulated value of the present value:

$$AV_n = s_{\overline{n}|}^{(m)} = (1+i)^n \times \frac{1-v^n}{i^{(m)}} = \frac{(1+i)^n - 1}{i^{(m)}}$$

Alternatively, the accumulated value can be found by noting that there are nm periods, each of which have an effective interest rate of $\frac{i^{(m)}}{m}$:

$$\frac{1}{m} s_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{1}{m} \frac{\left(1+\frac{i^{(m)}}{m}\right)^{nm} - 1}{\frac{i^{(m)}}{m}} = \frac{1}{m} \times \frac{(1+i)^n - 1}{\frac{i^{(m)}}{m}}$$

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The present value at time 0 and the accumulated value at time n of an annuity-immediate that makes payments of 1 per time unit, payable m^{th}ly, for n units of time are:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m} a_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{1-v^n}{i^{(m)}}$$

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} s_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{(1+i)^n - 1}{i^{(m)}}$$

Example 7.01

The nominal annual interest rate compounded monthly is 12%. Find the present value of an annuity-immediate that makes monthly payments at a rate of \$36 per year for 10 years.