Course FM Practice Exam 1 – Solutions

Solution 1

C First-Order Macaulay Approximation

The Macaulay duration is the modified duration multiplied by the one-period accumulation factor:

$$MacD = ModD \times (1+y) = 10.3 \times 1.08 = 11.124$$

The first-order Macaulay approximation of the new price is:

$$E_{Mac} \approx P(y) \times \left(\frac{1+y}{1+y+\Delta y}\right)^{MacD} = 1{,}213.50 \times \left(\frac{1.08}{1.073}\right)^{11.124} = 1{,}304.53$$

The first-order modified approximation of the new price is:

$$E_{Mod} \approx P(y) \times \left[1 - ModD \times \Delta y\right] = 1{,}213.50 \times \left[1 - 10.3 \times (-0.007)\right] = 1{,}300.99$$

The difference in the estimates is:

$$E_{Mac} - E_{Mod} = 1{,}304.53 - 1{,}300.99 = \mathbf{3.54}$$

Solution 2

C Interest rate swap

The notional amount is not needed to answer this question.

The zero-coupon bond prices are:

$$P(0,1) = 1.045^{-1} = 0.956938$$

$$P(0,2) = 1.048^{-2} = 0.910495$$

$$P(0,3) = 1.051^{-3} = 0.861374$$

The 1-year implied forward rates are:

$$r_0(0,1) = s_0 = 0.045$$

$$r_0(1,2) = \frac{1.048^2}{1.045} - 1 = 0.051009$$

$$r_0(2,3) = \frac{1.051^3}{1.048^2} - 1 = 0.057026$$

The fixed swap rate is:

$$\begin{aligned} R &= \frac{P(0,1)r_0(0,1) + P(0,2)r_0(1,2) + P(0,3)r_0(2,3)}{P(0,1) + P(0,2) + P(0,3)} \\ &= \frac{0.956938(0.045) + 0.910495(0.051009) + 0.861374(0.057026)}{0.956938 + 0.910495 + 0.861374} \\ &= \frac{0.138626}{2.728807} = 0.0508 \end{aligned}$$

A quicker way to determine the answer is:

$$R = \frac{1 - P(0,3)}{P(0,1) + P(0,2) + P(0,3)} = \frac{1 - 1.051^{-3}}{2.728807} = 0.0508$$

Solution 3

A Accumulated value of cash flows at annual effective interest rate

The equation of value for Sheryl's accumulated value at time 10 years is:

$$39,661.45 = 5,000(1+i)^{10} + 10,000(1+i)^5 + 15,000(1+i)^0$$

We let $x = (1+i)^5$, and we can solve the equation of value for x with the quadratic equation:

$$\begin{aligned} 0 &= 5,000x^2 + 10,000x - 24,661.45 \\ x &= \frac{-10,000 \pm \sqrt{(10,000)^2 - 4(5,000)(-24,661.45)}}{2(5,000)} \\ x &= \frac{-10,000 \pm 24,356.2928}{10,000} \end{aligned}$$

We discard the negative solution since it doesn't make sense with interest rates, and then we solve for i:

$$\begin{aligned} x &= 1.43563 \\ (1+i)^5 &= 1.43563 \\ 1+i &= 1.075 \\ i &= 0.075 \end{aligned}$$

Solution 4

B Varying perpetuity-immediate and level perpetuity-due

Bob's present value is:

$$50\ddot{a}_{\overline{\infty|}} = \frac{50}{i/(1+i)} = \frac{50(1+i)}{i}$$

Tom's payments are a compound-increasing perpetuity-immediate. The present value of which is:

$$a_{\overline{\infty}|j} = \lim_{n\to\infty} a_{\overline{n}|j} = \lim_{n\to\infty} \frac{1}{1+e}\left[\frac{1-(1+j)^{-n}}{j}\right] = \frac{1}{1+e}\left[\frac{1}{j}\right]$$

Since $j=(1+e)/(i-e)$, we have:

$$a_{\overline{\infty}|j} = \frac{1}{1+e}\left[\frac{1}{j}\right] = \frac{1}{1+e}\left[\frac{1+e}{i-e}\right] = \frac{1}{i-e}$$

Tom's present value is:

$$5a_{\overline{\infty}|j} = \frac{5}{i-0.025}$$

Since their present values are equal, we can set up the equation of value and solve for i:

$$\frac{50(1+i)}{i} = \frac{5}{i-0.025}$$
$$50(1+i)(i-0.025) = 5i$$
$$50(i-0.025+i^2-0.025i) = 5i$$
$$50i^2+43.75i-1.25 = 0$$
$$i = \frac{-43.75 \pm \sqrt{43.75^2-4(50)(-1.25)}}{2(50)}$$
$$i = \frac{2.76949}{100}$$
$$i = 0.0277$$

We discarded the negative solution since it didn't make sense for an interest rate.

Solution 5

A Loan repayment

The first 15 payments pay the principal down at a rate that is equal to 100% of the interest rate. Since the interest rate is 5%, the portion of the principal that is paid down by each of the first 15 payments is:

$$(2.00-1.00)\times 0.05 = 0.05$$

After 15 years, the original principal has been reduced by 5% for 15 years. The equation of value at the end of 15 years is:

$$250{,}000 \times 0.95^{15} = Xa_{\overline{15}|0.05}$$
$$115{,}822.8075 = X \times \frac{1-1.05^{-15}}{0.05}$$
$$115{,}822.8075 = 10.37966X$$
$$X = 11{,}158.63$$

The BA-II Plus can be used to answer this question:

250,000 [×] 0.95 [y^x] 15 [=] [*PV*]

15 [*N*] 5 [*I*/*Y*] [CPT] [*PMT*]

The result is –11,158.63, so $X = 11{,}158.63$.

Solution 6

A Callable bond price

Let's work in semiannual effective periods and define the bond variables first:

$$F = C = 1{,}000$$
$$n = 30 \times 2 = 60$$
$$i = 0.08/2 = 0.04$$
$$g = r = 0.06/2 = 0.03$$
$$\text{coupon} = 0.03 \times 1{,}000 = 30$$

Since $g < i$, the bond is a discount bond, and the minimum yield is determined from a call at the latest possible call date. (If the call price changes, then we would need to check the price at the latest date of each call price change.) The latest call date in this case is to assume that the bond will mature at time 30 years.

The appropriate annuity factor is:

$$a_{\overline{60}|4\%} = \frac{1-(1.04)^{-60}}{0.04} = 22.62349$$

The price of the bond that guarantees a minimum yield of at least an 8% nominal annual rate convertible semiannually is:

$$P = 30a_{\overline{60}|4\%} + 1{,}000(1.04)^{-60} = 773.7651$$

Let's check this answer using the BA-35 calculator by pressing [2nd][CMR], 1,000 [FV], 30 [PMT], 4 [%i], 60 [N], and [CPT][PV], and the result is 773.7651. Using the BA II Plus, we press [2nd][CLR TVM], –1,000 [FV], –30 [PMT], 4 [I/Y], 60 [N], and [CPT][PV], and the result is the same.

We can verify that this is the price to guarantee the required minimum return by calculating the prices if the bond had been called at other dates. If the prices at other possible call dates are higher, then the return would be lower. The table below illustrates the prices at other chosen call dates:

N	Price
40	802.0723
41	800.0695
…	…
58	775.7043
59	774.7157

Since all the other prices are higher than $773.7651, they would provide a lower return than the required return if the bond is not called early.

Solution 7

C Nominal interest rate and force of interest

The accumulated value of Peter's account after 6.5 years is:

$$X\left(1+\frac{i^{(4)}}{4}\right)^{4\times 6.5}$$

The accumulated value in David's account after 6.5 years is:

$$X\left(e^{0.07\times 6.5}\right)$$

Since their accumulated values are equal at this time, we set up the equation of value and solve for the annual nominal rate of interest convertible quarterly:

$$X\left(1+\frac{i^{(4)}}{4}\right)^{26}=Xe^{0.455}$$

$$\left(1+\frac{i^{(4)}}{4}\right)^{26}=1.576173$$

$$1+\frac{i^{(4)}}{4}=1.01765$$

$$i^{(4)}=0.07062$$

Alternatively, it is quicker if we note that both X and 6.5 years are not necessary to work this problem. We then have:

$$\left(1+\frac{i^{(4)}}{4}\right)^4 = e^{0.07}$$
$$i^{(4)} = 0.0706$$

Solution 8

E Bond coupon

Let's work in semiannual effective periods and define the bond variables first:

$$F = 1{,}000$$
$$C = 1{,}050$$
$$n = 30 \times 2 = 60$$
$$i = 0.08/2 = 0.04$$
$$a_{\overline{60}|4\%} = \frac{1-(1.04)^{-60}}{0.04} = 22.62349$$

We set up the equation of value for the price of the bond and we can solve it for X, the coupon amount:

$$948.19 = Xa_{\overline{60}|4\%} + 1{,}050(1.04)^{-60}$$
$$948.19 = 22.62349X + 99.81342$$
$$X = 37.50$$

The semiannual effective coupon rate is $37.50/1{,}000 = 3.75\%$, and the annual coupon rate convertible semiannually is twice this amount: $3.75\% \times 2 = 7.5\%$.

Alternatively, we can use a financial calculator to help us determine the answer. Using the BA 35, we press [2nd][CMR], 948.19 [PV], 4 [%I], 1,050 [FV], 60 [N], [CPT][PMT], and the result is 37.50. Using the BA II Plus, we press [2nd][CLR TVM], –948.19 [PV], 4 [I/Y], 1,050 [FV], 60 [N], [CPT][PMT], and we get the same result.

Solution 9

C Macaulay duration of stock with level dividends

The formula for Macaulay duration is:

$$MacD = \frac{\sum tCF_t\left(1+\frac{y}{m}\right)^{-mt}}{\sum CF_t\left(1+\frac{y}{m}\right)^{-mt}}$$

The denominator in this case is just the present value of a level perpetuity-immediate, which is given by:

$$1v+1v^2+1v^3+\cdots=v\left[1+v+v^2+\cdots\right]=v\left[\frac{1-v^{\infty}}{1-v}\right]=\frac{v}{1-v}=\frac{1/(1+i)}{1-1/(1+i)}=\frac{1+i}{i(1+i)}=\frac{1}{i}$$

The numerator is the present value of an increasing perpetuity-immediate:

$$(Ia)_{\overline{\infty|}}=\lim_{n\to\infty}(Ia)_{\overline{n|}}=\lim_{n\to\infty}\frac{\ddot{a}_{\overline{n|}}-nv^n}{i}=\frac{\ddot{a}_{\overline{\infty|}}-0}{i}=\frac{1/d}{i}=\frac{1+i}{i^2}$$

We then have:

$$MacD=\frac{(1+i)/i^2}{1/i}=\frac{1+i}{i}$$

We are given than the Macaulay duration is 21.0, so we can solve the equation for i:

$$\begin{aligned}\frac{1+i}{i}&=21.0\\ 21.0i&=1+i\\ 20.0i&=1\\ i&=0.05\end{aligned}$$

Solution 10

C Loan drop payment

If we accumulate the initial loan balance to time 1, then we can treat the loan as a loan with the first payment occurring one year later:

$$\frac{1{,}000{,}000}{(1-0.04)^1}=1{,}041{,}666.667$$

The annual effective interest rate is:

$$\frac{0.04}{1-0.04}=0.04167$$

We can solve the time-0 equation of value for n:

$$1{,}041{,}666.667 = 100{,}000 a_{\overline{n}|}$$

$$1{,}041{,}666.667 = 100{,}000 \times \frac{1 - 1.04167^{-n}}{0.04167}$$

$$0.43403 = 1 - 1.04167^{-n}$$

$$0.56597 = 1.04167^{-n}$$

$$\ln(0.56597) = -n \ln(1.04167)$$

$$n = -\frac{\ln(0.56597)}{\ln(1.04167)}$$

$$n = 13.9437$$

Therefore, there are 13 payments of 100,000 and a final drop payment at time 14:

$$1{,}041{,}666.667 = 100{,}000 a_{\overline{13}|} + DropPmt \times (1 - 0.04)^{14}$$

$$1{,}041{,}666.667 = 100{,}000 \times \frac{1 - 1.04167^{-13}}{0.04167} + DropPmt \times 0.96^{14}$$

$$DropPmt = \frac{1{,}041{,}666.667 - 100{,}000(9.88317)}{0.96^{14}}$$

$$DropPmt = 94{,}479.31$$

We can use the BA II Plus to answer this question:

1,000,000 [÷] 0.96 [=] [+/-] [*PV*]

0.04 [÷] 0.96 [×] 100 [=] [*I*/*Y*] 100,000 [*PMT*]

[CPT] [*N*]

The result is 13.9437.

13 [*N*] [CPT] [*FV*] [÷] 0.96 [=]

The answer is 94,479.31.

Solution 11

D *P*thly annuity-due present value

Liz's annuity payments occur today, and at times 2 years, 4 years, 6 years, and so on, until at time 18 years. There are 10 payments in all. Since these payments occur every other year, let's work in two-year effective periods. If we let i denote the two-year effective interest rate, the equation of value is:

$$62{,}787.98 = 10{,}000 \ddot{a}_{\overline{10}|} = 10{,}000 \frac{1 - (1+i)^{-10}}{i/(1+i)}$$

We could use trial and error to determine the value of i that satisfies the above equation. It is a valid approach during the exam if it saves time, but we can also use a calculator to quickly determine the answer.

Using the BA 35, we press [2nd][CMR], [2nd][BGN], 10,000 [PMT], 10 [N], 68,752.84 [PV], [CPT][%i], and the result is 9.50%. Using the BA II Plus, we press [2nd][CLR TVM], [2nd][BGN] [2nd][SET] [2nd][QUIT], 10,000 [PMT], 10 [N], –68,752.84 [PV], [CPT][I/Y] and we get the same result.

We need to be careful, since this rate of 9.5% is the two-year effective interest rate. We need to convert this rate to the annual nominal interest rate convertible semiannually:

$$\frac{i^{(2)}}{2} = \left(1 + \frac{i^{(1/2)}}{1/2}\right)^{1/2} - 1 = (1.095)^{0.5} - 1 = 0.02295$$

$$i^{(2)} = 0.04590$$

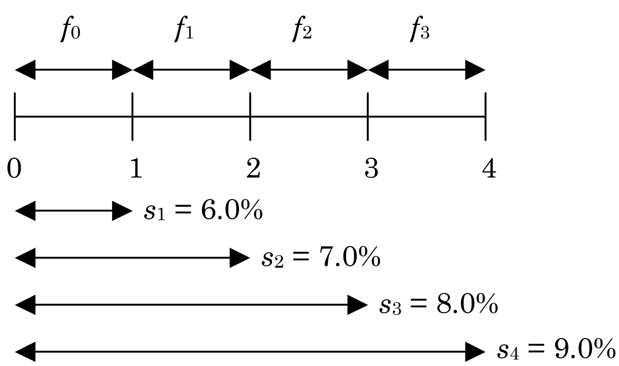
Solution 12

E Forward interest rate

The answer can be quickly determined by the following relationship:

$$f_3 = \frac{(1+s_4)^4}{(1+s_3)^3} - 1 = \frac{1.09^4}{1.08^3} - 1 = 0.12055$$

Alternatively, since forward rates can be a little confusing, it may help to draw a timeline. The timeline below shows the various spot and forward rates and where they are placed on the timeline.



The one-year forward rate covering the span of the fourth year from time 3 years to time 4 years is f_3. To determine this forward rate, we need to calculate f_1 and f_2 first:

$$f_0 = s_1 = 6.0\%$$

$$f_1 = \frac{1.07^2}{1.06} - 1 = 0.08009$$

$$f_2 = \frac{1.08^3}{(1.06)(1.008009)} - 1 = 0.10028$$

$$f_3 = \frac{1.09^4}{(1.06)(1.08009)(1.10028)} - 1 = 0.12055$$

Solution 13

B Dollar-weighted interest rate

The fund balance of $1,050 at the start of the year is accumulated with 12 months of interest. The withdrawal on May 1 is accumulated with 8 months of interest and the withdrawal on June 15 is accumulated with 6.5 months of interest. There are deposits at the end of every month from January 31 to December 31, and each of these deposits are accumulated to the end of the year. The fund value at the end of the year is $1,160.

We let i be the annual effective interest rate, and the equation of value is:

$$1{,}050(1+i)^{12/12} + 90(1+i)^{11/12} + 90(1+i)^{10/12} + \cdots + 90(1+i)^{1/12} + 90(1+i)^{0/12}$$
$$-600(1+i)^{8/12} - 400(1+i)^{6.5/12} = 1{,}160$$

Since the fund activity occurs during a 12-month period, we can use the simple interest approximation to simplify the above equation:

$$1{,}050(1+\frac{12}{12}i) + [90(1+\frac{11}{12}i) + 90(1+\frac{10}{12}i) + \cdots + 90(1+\frac{1}{12}i) + 90(1+\frac{0}{12}i)]$$
$$-600(1+\frac{8}{12}i) - 400(1+\frac{6.5}{12}i) = 1{,}160$$

Recognizing that $1+2+\cdots+n = (n)(n+1)/2$, the part in the brackets above becomes:

$$90 \times 12 + \frac{90}{12}i(11+10+\cdots+1) = 90 \times 12 + \frac{90}{12}i(66) = 1{,}080 + 495i$$

We can now solve the full equation for i, the annual effective interest rate:

$$1{,}050 + 1{,}050i + 1{,}080 + 495i - 600 - 400i - 400 - 216.6667i = 1{,}160$$
$$928.3333i = 30$$
$$i = 0.0323$$

Lastly, we solve for the nominal annual rate compounded monthly:

$$i^{(12)} = 12\left[(1.0323)^{1/12} - 1\right] = 0.0318$$

Solution 14

A Loan: amount of principal in a payment

The loan payment P is the sum of the interest component and the principal component:

$$P = 2{,}058.08 + 297.79 = 2{,}355.87$$

We can determine the balance at time 12 years from the interest component of the 13th annual payment:

$$I_{13} = B_{12} \times i$$
$$B_{12} = \frac{2{,}058.08}{0.09} = 22{,}867.55556$$

The balance at time 12 years is also equal to the present value of the remaining future loan payments. There are $n-12$ remaining loan payments, so another equation for the loan balance at time 12 years is:

$$B_{12} = 2{,}355.87 a_{\overline{n-12}|9\%} = 22{,}867.55556$$

Using the BA 35 calculator, we press [2nd][CMR], 22,867.55556 [PV], 9 [%i], 2,355.87 [PMT], [CPT][N], and the result is 24.00014. Using the BA II Plus, we press [2nd][CLR TVM], 22,867.55556 [PV], 9 [I/Y], –2,355.87 [PMT], [CPT][N] and we get the same result.

Since $n-12=24$, we know $n=36$, i.e., the loan initially had 36 annual payments.

To calculate the amount of principal in the 23rd payment, we need to know the balance at time 22 years. At time 22 years, there are $36-22=14$ payments remaining, so the appropriate annuity factor is:

$$a_{\overline{14}|9\%} = \frac{1-(1.09)^{-14}}{0.09} = 7.78615$$

The balance at time 22 years is then:

$$B_{22} = 2{,}355.87 a_{\overline{14}|9\%} = 18{,}343.15812$$

Now we can determine the amount of interest and principal in the 23rd payment:

$$I_{23} = iB_{22} = 0.09 \times 18{,}343.15812 = 1{,}650.88423$$
$$P_{23} = P - I_{23} = 2{,}355.87 - 1{,}650.88423 = 704.98577$$

Solution 15

A Annuity-immediate present value

With information from annuity 2, we can determine the annual effective interest rate. Using the BA 35, we press [2nd][CMR], 475.54 [FV], 5 [N], –81.06 [PMT], [CPT][%i], and the result is 7.9993, or 8.0%. Using the BA II Plus or the BA II Plus Professional, we press [2nd][CLR TVM], 475.54 [FV], 5 [N], –81.06 [PMT], [CPT][I/Y] and we get the same result.

Since the accumulated value of annuity 2 at time 5 years is \$475.54, the present value at time 0 of annuity 2 is:

$$PV = (1.08)^{-5}(475.54) = 323.64453$$

Annuity 1 has five annual payments from time 6 years to time 10 years. If we use an annuity-immediate present value factor to value these payments, its value is determined at time 5 years (1 year before the first payment), so we need to discount the time 5 present value back 5 years to determine the time 0 present value.

Since the present value of annuity 1 is twice the present value of annuity 2, we set up the equation of value for annuity 1 and solve for the unknown payment X:

$$\begin{aligned}2(323.64453) &= X(1.08)^{-5}a_{\overline{5}|8\%}\\ 647.28907 &= X(1.08)^{-5}\left[\frac{1-(1.08)^{-5}}{0.08}\right]\\ 2.71737X &= 647.28907\\ X &= 238.20\end{aligned}$$

Solution 16

B Redington immunization

We don't need to know the amount of the liability payment to answer this question.

The duration of the asset portfolio must be equal to duration of the liability. Let w be the percentage of the asset portfolio that is invested in the asset that pays at time 4:

$$\begin{aligned}&4w + 8(1-w) = 6\\ &-4w = -2\\ &w = \frac{1}{2}\end{aligned}$$

Since the present value of the asset portfolio is equal to the present value of the liability, the present values of the asset cash flows at time 0 are:

$$PV_X = \frac{1}{2}\times PV_L$$

$$PV_Y = \frac{1}{2}\times PV_L$$

The amounts of the cash flows are found by accumulating their present values:

$$\frac{X}{Y} = \frac{PV_X \times 1.03^4}{PV_Y \times 1.03^8} = \frac{\frac{1}{2}\times PV_L \times 1.03^4}{\frac{1}{2}\times PV_L \times 1.03^8} = \frac{1}{1.03^4} = 0.89$$

Alternatively, we can answer this question as follows.

The present values of the assets and liability are:

$$P_A = X(1+i)^{-4} + Y(1+i)^{-8}$$
$$P_L = 50{,}000(1+i)^{-6}$$

The derivatives of the present value of the assets and liability are:

$$P_A' = -4X(1+i)^{-5} - 8Y(1+i)^{-9}$$
$$P_L' = -6(50{,}000)(1+i)^{-7}$$

Valuing the above equations when the annual effective interest rate is 3%, we have:

$$P_A = 0.88849X + 0.78941Y$$
$$P_L = 41{,}874.2128$$
$$P_A' = -3.45044X - 6.13133Y$$
$$P_L' = -243{,}927.4534$$

We set the present values equal to each other and the derivatives equal to each other and we have two equations with two unknown variables:

$$0.88849X + 0.78941Y = 41{,}874.2128$$
$$-3.45044X - 6.13133Y = -243{,}927.4534$$

We multiply the second equation by the factor 0.88849/3.45044:

$$-0.88849X - 1.57882Y = -62{,}811.3193$$

Subtracting this result from the first equation, we have:

$$0X - 0.78941Y = -20{,}937.1064$$
$$Y = 26{,}522.50$$

With this value for Y, we can now solve for X:

$$X = \frac{41{,}874.2128 - 0.78941(26{,}522.50)}{0.88849} = 23{,}564.90$$

The ratio is then:

$$\frac{X}{Y} = \frac{23{,}564.90}{26{,}522.50} = 0.89$$

Solution 17

E Bond yield: reinvestment of coupon payments

Let's work in semiannual effective periods and define the bond variables first:

$$F = 1{,}000$$
$$C = 1{,}050$$
$$n = 15 \times 2 = 30$$
$$i = 0.08/2 = 0.04$$
$$r = 0.07/2 = 0.035$$
$$\text{coupon} = 0.035 \times 1{,}000 = 35$$
$$a_{\overline{30}|4\%} = \frac{1-(1.04)^{-30}}{0.04} = 17.29203$$

The price of the bond is:

$$P = 35a_{\overline{30}|4\%} + 1{,}050(1.04)^{-30} = 928.95577$$

Alternatively, using the BA 35, we press [2nd][CMR], 1,050 [FV], 30 [N], 35 [PMT], 4 [%i], [CPT][PV], and the result is 928.95577. Using the BA II Plus, we press [2nd][CLR TVM], –1,050 [FV], 30 [N], –35 [PMT], 4 [I/Y], [CPT][PV], and we get the same result.

George reinvests each of the 30 coupons at a semiannual effective interest rate of $0.09/2 = 0.045$. The accumulated value of these coupons at time 15 years is:

$$35s_{\overline{30}|4.5\%} = 35\left[\frac{(1.045)^{30}-1}{0.045}\right] = 2{,}135.24744$$

To determine his annual effective yield, we recognize that George paid \$928.95577 at time 0, and it grew to \$1,050 + \$2,135.24744 = 3,185.24744 at time 15 years. We set up this equation of value and solve for *i*, the annual effective interest rate:

$$928.95577(1+i)^{15} = 3{,}185.24744$$
$$(1+i)^{15} = 3.42885$$
$$1+i = 1.08562$$
$$i = 0.08562$$

Solution 18

B Loan interest rates

The equation of value is:

$$10{,}000 = X\left[a_{\overline{10}|4\%} + v^{10}a_{\overline{5}|6\%}\right]$$

Determining the required values, we have:

$$a_{\overline{10}|4\%} = \frac{1-1.04^{-10}}{0.04} = 8.11090$$
$$v^{10} = 1.04^{-10} = 0.67556$$
$$a_{\overline{5}|6\%} = \frac{1-1.06^{-5}}{0.06} = 4.21236$$

Solving for *X*, we have:

$$\begin{aligned} X &= \frac{10,000}{\left(8.11090 + 0.67556 \times 4.21236\right)} \\ &= \frac{10,000}{10.95662} \\ &= 912.69041 \end{aligned}$$

Solution 19

D Bond investment income

The book value at time 12 is the present value of the bond's future coupon payments:

$$\begin{aligned} BV_{12} &= 1,000v^{20-12} + (0.04)1,000a_{\overline{20-12}|5\%} \\ &= 1,000(1.05)^{-8} + 40\frac{1-1.05^{-8}}{0.05} \\ &= 676.83936 + 40(6.46321) \\ &= 935.36787 \end{aligned}$$

The investment income earned during the 13th year, i.e., the interest portion of the 13th coupon is:

$$935.36787(0.05) = 46.76839$$

The BA II Plus can be used to find the interest portion of the 13th coupon:

20 [N] 5 [I/Y] 40 [PMT] 1,000 [FV] [CPT][PV]

The result is –875.37790, so the price of the bond at time 0 is 875.38.

To determine the investment income:

[2ND][AMORT] 13 [ENTER] ↓ 13 [ENTER] ↓

By continuing to hit the down arrow key, we observe these values:

BAL = –942.13627, so $BV_{13} = 942.14$

PRN = –6.76839, so $DA_{13} = 6.77$

INT = 46.76839, so $InvInc_{13} = 46.77$

Solution 20

B Annuity-due accumulated value

If John had made all of the annual deposits from 1/1/78 to 12/31/07, he would have made $2007 - 1978 + 1 = 30$ deposits. The deposits were supposed to occur on January 1 of each year from 1/1/78 to 1/1/07. John missed deposits 16 through 21. The 16th deposit was supposed to occur on 1/1/93, since the first deposit occurred on 1/1/78. Likewise, the 21st deposit was supposed to occur on 1/1/98. Thus, John made the first 15 deposits (from 1/1/78 to 1/1/92), missed the next 6 deposits (from 1/1/93 to 1/1/98), and made the last 9 deposits (from 1/1/99 to 1/1/07).

The accumulated value of the first 15 deposits at time 16 years (1/1/93) is:

$$10{,}000\ddot{s}_{\overline{15}|4.5\%} = 10{,}000\left(\frac{(1.045)^{15} - 1}{0.045/1.045}\right) = 217{,}193.3673$$

Since we need the accumulated value as of 12/31/07 (or 1/1/08), we need to accumulate this accumulated value for another $2008 - 1993 = 15$ years. The accumulated value of the first 15 deposits at time 12/31/07 is:

$$217{,}193.3673(1.045)^{15} = 420{,}330.5105$$

The accumulated value of the last 9 deposits at 12/31/07 (i.e., 9 years after 1/1/99) is:

$$10{,}000\ddot{s}_{\overline{9}|4.5\%} = 10{,}000\left(\frac{(1.045)^{9} - 1}{0.045/1.045}\right) = 112{,}882.0937$$

The total accumulated value of these two pieces at time 1/1/08 is:

$$420{,}330.5106 + 112{,}882.0937 = 533{,}212.6043$$

Alternatively, we can approach this problem from another angle. We can assume that John made all 30 payments. The accumulated value of the 30 annual payments at 12/31/07 is:

$$10{,}000\ddot{s}_{\overline{30}|4.5\%} = 10{,}000\left(\frac{(1.045)^{30} - 1}{0.045/1.045}\right) = 637{,}523.8779$$

The accumulated value of the 6 deposits that were missed, valued at 1/1/99 is:

$$10{,}000\ddot{s}_{\overline{6}|4.5\%} = 10{,}000\left(\frac{(1.045)^{6} - 1}{0.045/1.045}\right) = 70{,}191.5179$$

We need to accumulate the 1/1/99 accumulated value of the missing deposits to 12/31/07. There are $2007-1999+1=9$ years from 1/1/99 to 12/31/07, so the 12/31/07 accumulated value of the missed deposits is:

$$70{,}191.5179(1.045)^9 = 104{,}311.2736$$

Now we can subtract the accumulated value at 12/31/07 of the missing deposits from the accumulated value at 12/31/07 of the deposits assuming that none were missed. The resulting accumulated value is:

$$637{,}523.8779 - 104{,}311.2736 = 533{,}212.6043$$

Solution 21

C Macaulay duration of bond

The formula for Macaulay duration is:

$$MacD = \frac{\sum tCF_t\left(1+\frac{y}{m}\right)^{-mt}}{\sum CF_t\left(1+\frac{y}{m}\right)^{-mt}}$$

For this bond with coupon payments of \$$X$ at the end of each year, the Macaulay duration is:

$$\frac{X(1)(1.09)^{-1} + X(2)(1.09)^{-2} + X(3)(1.09)^{-3} + \cdots + X(10)(1.09)^{-10} + 400(10)(1.09)^{-10}}{X(1.09)^{-1} + X(1.09)^{-2} + X(3)(1.09)^{-3} + \cdots + X(1.09)^{-10} + 400(1.09)^{-10}}$$

This can be written as:

$$7.466 = \frac{X(Ia)_{\overline{10|}9\%} + 400(10)(1.09)^{-10}}{Xa_{\overline{10|}9\%} + 400(1.09)^{-10}}$$

Determining the required values, we have:

$$a_{\overline{10|}9\%} = \frac{1-(1.09)^{-10}}{0.09} = 6.41766$$

$$\ddot{a}_{\overline{10|}9\%} = (1.09)(6.41766) = 6.99525$$

$$(Ia)_{\overline{10|}9\%} = \frac{6.99525 - 10(1.09)^{-10}}{0.09} = 30.79043$$

Plugging these values into the above equation, we can solve for X:

$$7.466 = \frac{X(30.79043)+1,689.64323}{X(6.41766)+168.96432}$$

$$47.91423X + 1,261.48763 = 30.79043X + 1,689.64323$$

$$17.12380X = 428.15559$$

$$X = 25.00$$

Solution 22

B Compound decreasing and increasing annuity

The present value of a compound increasing annuity-immediate is:

$$\frac{1}{1+e}a_{\overline{n}|j} = \frac{1}{1+e}\left[\frac{1-(1+j)^{-n}}{j}\right] \text{ where } j = \frac{i-e}{1+e}$$

Since we have monthly payments and increases that occur monthly, let's work in monthly periods. The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = (1.125)^{1/12} - 1 = 0.009864$$

Let's split the annuity into two parts: the compound decreasing part during the first two years and the compound increasing part during the last two years. The monthly payments for the compound decreasing part start at time 1 month. The first payment is \$5,000. There are 24 payments for the decreasing part and the monthly rate of decrease is 1%.

The monthly interest rate for the first two years is:

$$j = \frac{0.009864-(-0.01)}{1+(-0.01)} = 0.020064$$

The present value at time 0 of the decreasing part is:

$$5,000\frac{1}{1+(-0.01)}\left[\frac{1-(1.020064)^{-24}}{0.020064}\right] = 95,455.41662$$

The monthly payments for the compound increasing part start at time 25 months. The first payment for the increasing part is $5,000(0.99)^{23}(1.005)$. There are also 24 payments for the increasing part and the monthly rate of increase is 0.5%. The monthly interest rate for the second two years is:

$$j = \frac{0.009864-0.005}{1.005} = 0.004839$$

The present value at time 24 months of the increasing part is:

$$5{,}000(0.99)^{23}(1.005)\frac{1}{1+(0.005)}\left[\frac{1-(1.004839)^{-24}}{0.004839}\right]=89{,}706.74436$$

The present value of the decreasing part must be accumulated for five years and the present value of the increasing part must be accumulated for three years to determine the accumulated value of the entire series of payments at time five years:

$$95{,}455.41662(1.125)^{5}+89{,}706.74436(1.125)^{3}=299{,}740.7459$$

Solution 23

E Pricing a bond using spot rates

The bond pays annual coupons of $0.08\times 1{,}000=80$. The present value of the bond is:

$$PV=\frac{80}{1.06}+\frac{80}{1.07^{2}}+\frac{80}{1.08^{3}}+\frac{1{,}080}{1.09^{4}}=973.95260$$

The annual effective yield can be quickly determined using a financial calculator. Using the BA 35, we press [2nd][CMR], 1,000 [FV], 4 [N], 80 [PMT], 973.9526 [PV], [CPT][%i], and the result is 8.80048. Using the BA II Plus, we press [2nd][CLR TVM], 1,000 [FV], 4 [N], 80 [PMT], –973.9526 [PV], [CPT][%i], and we get the same result of 8.8%.

Solution 24

D Decreasing annuity-immediate present value

The payments start at $1,050 at time 1 year and decrease by $15 each year. There are 30 payments, so the last payment of $615 occurs at time 30 years. The decreases are $15 and there are 30 payments, so let's subtract $15\times 30=450$ from the first payment of $1,050, and we're left with $600. If we subtract $600 from each of the 30 payments, we are left with a payment stream that starts at $450 at time 1 and decreases by $15 each year to $15 at time 30.

The present value of a level series of 30 payments of $600, we have:

$$600a_{\overline{30}|6.5\%}=600\frac{1-1.065^{-30}}{0.065}=600(13.05868)=7{,}835.20554$$

The present value of the decreasing payment stream is:

$$15(Da)_{\overline{30}|6.5\%}=15\frac{30-a_{\overline{30}|6.5\%}}{0.065}=15\frac{30-13.05868}{0.065}=3{,}909.53633$$

Putting the two present values back together, we have:

$$PV=7{,}835.20554+3{,}909.53633=11{,}744.7419$$

Solution 25

A Loan balance

We are given that the outstanding loan balance at the end of the 9th year is \$1,355.22. So the final loan payment must be this balance times the quantity of one plus the effective interest rate, since the final payment must pay off this balance plus the interest from the last period on this balance. The level payment is then:

$$P = 1{,}355.22 \times 1.075 = 1{,}456.8615$$

The balance at the end of the third year is the present value of the remaining seven payments:

$$B_3 = 1{,}456.8615 a_{\overline{7}|7.5\%} = 1{,}456.8615\left(\frac{1-(1.075)^{-7}}{0.075}\right) = 7{,}716.41$$

Alternatively, since this is a 10-year loan, the final loan payment must contain exactly this amount as the principal amount to pay off the loan, so:

$$P_{10} = 1{,}355.22$$

We can develop a relationship between successive principal amounts. We recall that the outstanding loan balance at any time t is the present value of the remaining loan payments:

$$B_t = Pa_{\overline{n-t}|i} = P\frac{1-v^{n-t}}{i}$$

The amount of interest in loan payment t is the periodic effective interest rate times the prior loan balance, and the amount of principal in loan payment t is the loan payment minus the amount of interest in loan payment t:

$$I_t = iB_{t-1} = iP\frac{1-v^{n-t+1}}{i} = P\left(1-v^{n-t+1}\right)$$
$$P_t = P - I_t = P - P\left(1-v^{n-t+1}\right) = Pv^{n-t+1}$$

So if we divide successive loan payments, we see that a principal amount in a loan payment is $(1 + i)$ times the principal amount in the prior loan payment:

$$\frac{P_{t+1}}{P_t} = \frac{Pv^{n-t}}{Pv^{n-t+1}} = \frac{1}{v} = 1 + i$$

The initial loan amount L is the sum of the principal amounts in each loan payment. In this case, we have:

$$L = P_1 + P_2 + \cdots + P_{10}$$

We have just determined an iterative relationship between the principal amounts in each loan payment, so we can restate the above relationship as:

$$\begin{aligned}L &= 1{,}355.22v^9 + 1{,}355.22v^8 + \cdots + 1{,}355.22 \\ &= 1{,}355.22[1 + v + v^2 + \cdots + v^9] \\ &= 1{,}355.22\left[\frac{1-v^{10}}{1-v}\right] \\ &= 1{,}355.22\left[\frac{1-(1.075)^{-10}}{1-(1.075)^{-1}}\right] \\ &= 10{,}000.0153\end{aligned}$$

So the loan balance at the end of the third year is the initial loan amount less the principal amounts in the first three loan payments:

$$\begin{aligned}B_3 &= L - P_1 - P_2 - P_3 \\ &= 10{,}000.0153 - 1{,}355.22(1.075)^{-9} - 1{,}355.22(1.075)^{-8} - 1{,}355.22(1.075)^{-7} \\ &= 7{,}716.41\end{aligned}$$

Solution 26

B Net present value

The net present values for projects X and Y are:

$$NPV_X = -1{,}000 + 700(1+i)^{-5} + 700(1+i)^{-10}$$

$$NPV_Y = -800(1+i)^{-5} + 1{,}016.31(1+i)^{-10}$$

Since their net present values are equal, we set up the equation of value. We let $x = (1+i)^{-5}$ and solve for x using the quadratic equation:

$$-1{,}000 + 700(1+i)^{-5} + 700(1+i)^{-10} = -800(1+i)^{-5} + 1{,}016.31(1+i)^{-10}$$

$$316.31x^2 - 1{,}500x + 1{,}000 = 0$$

$$x = \frac{1{,}500 \pm \sqrt{(-1{,}500)^2 - 4(316.31)(1{,}000)}}{2(316.31)}$$

$$x = 0.80246 \text{ or } 3.93973$$

We can now solve for the unknown annual effective interest rate:

$$\begin{aligned}&(1+i)^{-5} = 0.80246 && \textit{or} && (1+i)^{-5} = 3.93973 \\ &1+i = 1.0450 && && 1+i = 0.76016 \\ &i = 0.045 && && i = -0.23984\end{aligned}$$

Since the negative interest rate doesn't make sense, the answer is 4.5%.

Solution 27

A Surplus

The present value of the liability is:

$$PV_L = \frac{5,000,000}{1.06^{15}} = 2,086,325.304$$

The company needs to buy a bond that will mature for the liability amount of $5,000,000 at time 15 years, assuming the interest rate does not change. Since the coupon rate of the 15-year bond is equal to its yield, the 15-year bond is priced at par. The bond's face amount should therefore also be $2,086,325.304. To verify this, we have:

$$\text{bond } F = 2,086,325.304$$

$$\text{annual coupon} = 0.06 \times 2,086,325.304 = 125,179.5182$$

This bond will exactly match the liability in 15 years, since the bond's face amount plus the reinvested coupons will add to $5,000,000 at time 15 years, assuming the interest rate does not change:

$$\begin{aligned}2,086,325.304 + 125,179.51852 s_{\overline{15}|6\%} &= 2,086,325.304 + 125,179.5182 \frac{\left(1.06^{15} - 1\right)}{0.06} \\ &= 5,000,000.00\end{aligned}$$

The interest rate changes on 12/31/10, which is exactly 3 years after 12/31/07. The new interest rate of 5.5% remains in effect for the remaining 12 years until 12/31/22. The coupons are reinvested for 3 years at 6.0%, and then they are reinvested for the remaining 12 years at 5.5%. The accumulated value of the coupons is then:

$$\begin{aligned}AV(\text{coupons})_{15} &= 125,179.5182 s_{\overline{3}|6\%} (1.055)^{12} + 125,179.5182 s_{\overline{12}|5.5\%} \\ &= 125,179.5182 \left(\frac{1.06^3 - 1}{0.06}\right)(1.055)^{12} + 125,179.5182 \left(\frac{1.055^{12} - 1}{0.055}\right) \\ &= 2,808,812.43\end{aligned}$$

Combined with the face amount, the bond's value at 12/31/22 is then:

$$2,808,812.43 + 2,086,325.30 = 4,895,137.73$$

The insurance company's liability at 12/31/22 is still $5,000,000, so the insurance company's profit at this time is:

$$4,895,137.73 - 5,000,000 = -104,862.27$$

Solution 28

B Level annuity-due accumulated value factor

Expression A is true since:

$$\ddot{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{d} = \frac{(1+i)^n - 1}{iv} \text{ since } d = iv$$

Expression B is false since:

$$\ddot{s}_{\overline{n|}} = (1+i)^n \ddot{a}_{\overline{n|}} = (1+i)^n \left[(1+i)a_{\overline{n|}}\right] = (1+i)^{n+1} a_{\overline{n|}}$$

Expression C is true since:

$$\ddot{s}_{\overline{n|}} = (1+i)^n \ddot{a}_{\overline{n|}} = (1+i)^n \left[\frac{1-v^n}{d}\right] = (1+i)^n \left[\frac{1-v^n}{1-v}\right] \text{ since } d = 1-v$$

Expression D is true since:

$$\ddot{s}_{\overline{n|}} = (1+i)^n + (1+i)^{n-1} + \cdots + (1+i)^2 + (1+i)^1$$

Expression E is true since:

$$s_{\overline{n|}} = (1+i)^{n-1} + \cdots + 1$$

$$s_{\overline{n+1|}} = (1+i)^n + \cdots + 1$$

$$\ddot{s}_{\overline{n|}} = (1+i)^n + \cdots + (1+i)$$

so

$$\ddot{s}_{\overline{n|}} = s_{\overline{n+1|}} - 1$$

Solution 29

C Callable bond price

Working with what we have, we can quickly determine the bond's annual coupon payment $\$C$. Then we calculate the price that results in an annual effective yield of 8.34%. There is no need to determine P.

To determine $\$C$, we assume the bond is called at the end of the 15th year. Using the BA 35, we press [2nd][CMR], 1,025 [FV], 15 [N], 8 [%i], 793.89 [PV], [CPT][PMT], and the result is 55.00. Using the BA II Plus, we press [2nd][CLR TVM], 1,025 [FV], 15 [N], 8 [I/Y], –793.89 [PV], [CPT][PMT], and we get the same result.

The investor actually held the bond for 20 years, when it was called for $1,025, and the investor's actual annual effective yield was 8.34%. The price that results in an annual effective yield of 8.34% can now be determined.

Using the BA 35, we press [2nd][CMR], 1,025 [FV], 20 [N], 55 [PMT], 8.34 [%i], [CPT][PV], and the result is 733.1175. Using the BA II Plus, we press [2nd][CLR TVM], −1,025 [FV], 20 [N], –55 [PMT], 8.34 [I/Y], [CPT][PV], and we get the same result. Thus, we conclude that the investor paid $733.12.

Solution 30

D Monthly annuity-immediate present value

Since Tina's retirement payments occur monthly, let's work in monthly periods. We set up the equation of value for the present value of these benefits, and we will be able to determine the annual effective interest rate. Tina will receive 240 monthly payments at the end of each month for 20 years.

$$4{,}000a_{\overline{240}|} = 587{,}938.54$$

A financial calculator can quickly determine the interest rate. Since we're working in months, the result will be a monthly effective interest rate, which we can convert to an annual effective interest rate.

Using the BA 35, we press [2nd][CMR], 240 [N], 4,000 [PMT], 587,938.54 [PV], [CPT][%i], and the result is 0.44717. Using the BA II Plus, we press [2nd][CLR TVM], 240 [N], 4,000 [PMT], –587,938.54 [PV], [CPT][%i], and we get the same result.

The annual effective interest rate is:

$$i = (1.0047717)^{12} - 1 = 0.0550$$

The present value of the zero-coupon bond is then:

$$X = \frac{1{,}000{,}000}{1.055^{18}} = 381{,}465.904$$

Solution 31

B Amortizing swap

The forward rats are:

$$f_{0,1} = 0.025$$

$$f_{1,2} = \frac{1.03^2}{1.025} - 1 = 0.03502$$

$$f_{2,3} = \frac{1.035^2}{1.03^2} - 1 = 0.04507$$

The swap rate is:

$$\frac{500{,}000(0.025)(1.025)^{-1} + 850{,}000(0.03502)(1.03)^{-2} + 1{,}100{,}000(0.04507)(1.035)^{-3}}{500{,}000(1.025)^{-1} + 850{,}000(1.03)^{-2} + 1{,}100{,}000(1.035)^{-3}}$$
$$= 0.03725$$

Solution 32

C Immunization

Statement I is not true since Redington (i.e., classical) immunization only protects a portfolio against small changes in interest rates.

Statement II is not true since the duration of the assets must be established so that it approximately matches the duration of the liabilities in order to meet the second immunization condition.

Statement III is true since the convexity of the assets must be greater than the convexity of the liabilities in order to meet the third immunization condition of Redington immunization.

Only statement III is true, so choice C is the correct answer.

Solution 33

D Dollar-weighted and time-weighted interest rates

Let's make a table of the given information, where the fund values are valued immediately before the next cash flow occurs.

t	F_t	c_t
0	100	0
0.5	90	–20
0.75	105	40
1.0	X	0

The equation for the time-weighted interest rate, i, is:

$$(1+i)^1 = \left(\frac{90}{100}\right)\left(\frac{105}{90-20}\right)\left(\frac{X}{105+40}\right)$$
$$1+i = 0.009310X$$

We are given that the time-weighted rate, i, equals the dollar-weighted rate, j, plus 0.097. Substituting this into the above equation, we have:

$$1+(j+0.097) = 0.009310X$$
$$j = 0.009310X - 1.097$$

The equation for the dollar-weighted interest rate, j, is:

$$X = 100(1+j)^1 - 20(1+j)^{0.5} + 40(1+j)^{0.25}$$

Since all of the cash flows occur within a year, we can use the simple interest approximation on the above equation. We have:

$$\begin{aligned} X &= 100(1+1j) - 20(1+0.5j) + 40(1+0.25j) \\ &= 100 + 100j - 20 - 10j + 40 + 10j \\ &= 120 + 100j \end{aligned}$$

Substituting our earlier result for j into this equation, we can solve for X:

$$\begin{aligned} X &= 120 + 100(0.009310X - 1.097) \\ 0.068966X &= 10.3 \\ X &= 149.35 \end{aligned}$$

Solution 34

A Immunization

The present value of the liability is:

$$P_L = 25{,}000a_{\overline{10}|5\%} = 25{,}000\frac{1-(1.05)^{-10}}{0.05} = 25{,}000\left(7.72173\right) = 193{,}043.3732$$

The asset portfolio has the same present value. The duration of the liability is:

$$\frac{25{,}000(1)v + 25{,}000(2)v^2 + \cdots + 25{,}000(10)v^{10}}{193{,}043.3732} = \frac{25{,}000\left(Ia\right)_{\overline{10}|5\%}}{193{,}043.3732}$$

We have:

$$\ddot{a}_{\overline{10}|} = (1+i)a_{\overline{10}|} = 1.05\left(7.72173\right) = 8.10782$$

$$\left(Ia\right)_{\overline{10}|5\%} = \frac{8.10782 - 10\left(1.05\right)^{-10}}{0.05} = 39.37378$$

The duration of the liability is then:

$$\frac{25{,}000\left(39{,}37378\right)}{193{,}043.3732} = 5.09909$$

The asset portfolio has the same duration. The duration of the 4-year zero-coupon bond is 4 and the duration of the 8 year zero-coupon bond is 8. If we let X equal the percentage of the asset portfolio to invest in the four-year zero-coupon bond, we can set up the equation of value and solve for X:

$$\begin{aligned} 4X + 8(1-X) &= 5.09909 \\ 4X + 8 - 8X &= 5.09909 \\ X &= 0.7252 \end{aligned}$$

The amount invested in the four-year bonds is:

$$0.7252 \times 193{,}043.3732 = 140{,}000.6039$$

Solution 35

D Quoted T-bill rate

The current price of the Canadian T-bill is:

$$\text{Canadian Quoted Rate} = \frac{365}{90} \times \frac{75}{P}$$

$$P = \frac{365}{90} \times \frac{75}{0.06176} = 4{,}924.9784$$

The annual effective rate is:

$$i = \left(\frac{5{,}000}{4{,}924.9784} \right)^{365/90} - 1 = 0.0632$$

