Chapter 6: Level Annuities Payable Once per Time Unit

Solution 6.01

E Section 6.02, Annuity-Immediate

The equation of value at time n can be used to solve for n:

$$\begin{aligned}
1{,}000 s_{\overline{n}|0.05} &= 2{,}061.34 a_{\overline{10}|0.05} \\
1{,}000 \times \frac{1.05^n - 1}{0.05} &= 2{,}061.34 \times \frac{1 - 1.05^{-10}}{0.05} \\
20{,}000(1.05^n - 1) &= 2{,}061.34 \times 7.7217 \\
1.05^n - 1 &= 0.7959 \\
n \times \ln(1.05) &= \ln(1.7959) \\
n &= \mathbf{12}
\end{aligned}$$

The BA-II Plus can be used to solve the problem as follows:

10 [N] 5 [I/Y] 2,061.34 [PMT] [CPT] [PV]

Result is –15,917.12.

[FV] 0 [PV] 1,000 [PMT] [CPT] [N]

Solution is **12**.

Solution 6.02

B Section 6.02, Annuity-Immediate

The equation of value at time 0 can be used to find X:

$$\begin{aligned}
&2{,}000 a_{\overline{15}|0.08} = X a_{\overline{5}|0.08} \\
&2{,}000 \times \frac{1 - v^{15}}{0.08} = X \times \frac{1 - v^5}{0.08} \\
&2{,}000(1 - 1.08^{-15}) = X(1 - 1.08^{-5}) \\
&X = \mathbf{4{,}287.55}
\end{aligned}$$

The BA-II Plus can be used to solve the problem as follows:

15 [N] 8 [I/Y] 2,000 [PMT] [CPT] [PV]

Result is –17,118.95.

5 [N] [CPT] [PMT]

Solution is **4,287.55**.

Solution 6.03

A Section 6.02, Annuity-Immediate

The equation of value at time 15 can be used to find X:

$$\begin{aligned}
&X s_{\overline{5}|0.08} \times 1.08^{10} = 10{,}000 a_{\overline{30}|0.08} \\
&X \times \frac{1.08^5 - 1}{0.08} \times 1.08^{10} = 10{,}000 \times \frac{1 - 1.08^{-30}}{0.08} \\
&X(1.08^5 - 1) \times 1.08^{10} = 10{,}000(1 - 1.08^{-30}) \\
&X = \mathbf{8{,}888.51}
\end{aligned}$$

The BA-II Plus can be used to solve the problem as follows:

30 [*N*] 8 [*I*/*Y*] 10,000 [*PMT*] [CPT] [*PV*]

Result is –112,577.83.

[÷] 1.08 [y^x] 10 [=] [*FV*]

5 [*N*] 0 [*PV*] [CPT] [*PMT*]

Solution is **8,888.51**.

Solution 6.04

D Section 6.02, Annuity-Immediate

The purchase price of the new car is equal to the down payment plus the present value of the monthly payments:

$$X = 5,000 + 500a_{\overline{60}|\frac{0.07}{12}} = 5,000 + 500 \times \frac{1-\left(1+\frac{0.07}{12}\right)^{-60}}{\frac{0.07}{12}}$$

$$= 5,000 + 500 \times 50.5020 = \mathbf{30,251.00}$$

The BA-II Plus can be used to solve the problem as follows:

60 [*N*] 7 [÷] 12 [=] [*I*/*Y*] 500 [*PMT*] [CPT] [*PV*]

Result is –25,251.99.

[+/-] [+] 5,000 [=]

Solution is **30,251.00**.

Solution 6.05

B Section 6.02, Annuity-Immediate

The equation of value at time 0 can be used to solve for i:

$$1,700a_{\overline{5}|} = 1,000a_{\overline{10}|}$$

$$1.7 \times \frac{1-v^5}{i} = \frac{1-v^{10}}{i}$$

$$1.7 \times (1-v^5) = 1-v^{10}$$

$$1.7 \times (1-v^5) = (1-v^5)(1+v^5)$$

$$1.7 = 1+v^5$$

$$i = \mathbf{0.0739}$$

Solution 6.06

C Section 6.02, Annuity-Immediate

The equation of value at time 0 can be used to solve for i:

$$\frac{100}{i} = 114.16a_{\overline{20}|}$$

$$\frac{100}{i} = 114.16 \times \frac{1-v^{20}}{i}$$

$$0.8760 = 1-v^{20}$$

$$v = 0.9009$$

$$i = \mathbf{0.11}$$

Solution 6.07

B Section 6.02, Annuity-Immediate

The ratio of the given values can be used to solve for v^{3n} :

$$\frac{a_{\overline{6n}|}}{a_{\overline{3n}|}} = \frac{\frac{1-v^{6n}}{i}}{\frac{1-v^{3n}}{i}}$$

$$\frac{39.3119}{26.0000} = \frac{1-v^{6n}}{1-v^{3n}}$$

$$\frac{39.3119}{26.0000} = 1 + v^{3n}$$

$$v^{3n} = 0.5120$$

An annuity that pays for 9*n* units of time has the same present value as an annuity that pays for 3*n* units of time plus the present value of a deferred annuity that pays for 6*n* units of time:

$$a_{\overline{9n}|} = a_{\overline{3n}|} + v^{3n}a_{\overline{6n}|} = 26.0000 + 0.5120 \times 39.3119 = \mathbf{46.1275}$$

Solution 6.08

C Section 6.02, Annuity-Immediate

The monthly effective interest rate is:

$$\frac{0.22}{12} = 0.01833$$

Let’s use months as our unit of time. The extra payment occurs 5 months after the loan begins:

$$15{,}000 = 800a_{\overline{n}|} + 1{,}200v^5$$

$$15{,}000 = 800a_{\overline{n}|} + \frac{1{,}200}{1.01833^5}$$

$$13{,}904.1996 = 800a_{\overline{n}|}$$

$$17.3802 = a_{\overline{n}|}$$

$$17.3802 = \frac{1-1.01833^{-n}}{0.01833}$$

$$1.01833^{-n} = 0.6814$$

$$-n \times \ln(1.01833) = \ln(0.6814)$$

$$n = 21.1182$$

Therefore the last full payment occurs 21 months after the loan begins, which is **October 1, 2017**.

We can use the BA II Plus to find the value of n:

0.22 [÷] 12 [+] 1 [=] [STO] 1 [–] 1 [=] [×] 100 [=] [*I*/*Y*]

15,000 [–] 1,200 [÷] [RCL] 1 [y^x] 5 [=]

(Result is 13,904.1996)

[*PV*] 800 [+/-] [*PMT*]

[CPT] [*N*]

Result is 21.1182.

Solution 6.09

B Section 6.02, Annuity-Immediate

Choice **B** is the correct answer, because the expression in Choice B is not a valid expression for $a_{\overline{n|}}$. Let's consider each choice.

Choice A is valid:

$$\frac{v^n - 1}{(v-1)(1+i)} = \frac{1-v^n}{(1-v)(1+i)} = \frac{1-v^n}{d(1+i)} = \frac{1-v^n}{\frac{i}{1+i} \times (1+i)} = \frac{1-v^n}{i} = a_{\overline{n|}}$$

Choice B is not valid:

$$\frac{(1+i)^n - 1}{1-v} v^n = \frac{1-v^n}{d} = (1+i) \times \frac{1-v^n}{i} = (1+i) \times a_{\overline{n|}}$$

Choice C is valid:

$$s_{\overline{n|}} \times (1-iv)^n = s_{\overline{n|}} \times (1-d)^n = s_{\overline{n|}} \times v^n = \frac{(1+i)^n - 1}{i} \times v^n = \frac{1-v^n}{i} = a_{\overline{n|}}$$

Choice D is valid:

$$\left(\frac{1-v^n}{iv}\right)\left(\frac{1}{1+i}\right) = \left(\frac{1-v^n}{i}\right)(1+i)\left(\frac{1}{1+i}\right) = \frac{1-v^n}{i} = a_{\overline{n|}}$$

Choice E is valid:

$$\frac{1+v+v^2+\cdots+v^{n-1}}{1+i} = v \times \left(1+v+v^2+\cdots+v^{n-1}\right) = v+v^2+v^3+\cdots+v^n = a_{\overline{n|}}$$

Solution 6.10

B Section 6.02, Annuity-Immediate

Another way to describe the deposits is to say that deposits of 87 are made for 3*n* years and additional deposits of 87 are made during the first *n* years.

The equation of value at time 3*n* can be used to solve for *i*:

$$36,419.74 = 87s_{\overline{3n|}} + 87s_{\overline{n|}} \times (1+i)^{2n}$$

$$36,419.74 = 87 \times \frac{(1+i)^{3n} - 1}{i} + 87 \times \frac{(1+i)^n - 1}{i} \times (1+i)^{2n}$$

$$36,419.74i = 87 \times \left[(1+i)^{3n} - 1\right] + 87 \times \left[(1+i)^n - 1\right] \times (1+i)^{2n}$$

$$36,419.74i = 87 \times \left[27-1\right] + 87 \times \left[3-1\right] \times 9$$

$$i = \mathbf{0.1051}$$

Solution 6.11

C Section 6.02, Annuity-Immediate

The equation of value at the end of 17 years can be used to solve for *X*:

$$Xs_{\overline{7|}0.04} \times (1.08)^{10} = 8,000$$

$$X \times \frac{1.04^7 - 1}{0.04} = 3,705.5479$$

$$X \times 7.8993 = 3,705.5479$$

$$X = \mathbf{469.16}$$

The BA-II Plus can be used to solve the problem as follows:

8,000 [÷] 1.08 [y^x] 10 [=] [*FV*]

7 [*N*] 4 [*I*/*Y*] [CPT] [*PMT*]

Result is –469.16. Answer is **469.16**.

Solution 6.12

A Section 6.02, Annuity-Immediate

Setting the present values of the first two annuities equal to one another gives us an equation in terms of i:

$$\frac{1}{0.08} = a_{\overline{40|}i}$$

The BA-II Plus can be used to solve for i, and then n:

0.08 [1/*x*] [*PV*]

40 [*N*] 1 [+/-] [*PMT*] [CPT] [*I*/*Y*]

Result is 7.5677.

[–] 1 [=] [*I*/*Y*] [CPT] [*N*]

Answer is **27.04**.

Solution 6.13

C Section 6.03, Annuity-Due

The present value of the annuity-immediate is:

$$10 \times a_{\overline{10|}} = 10 \times \frac{1 - 1.08^{-10}}{0.08} = 67.1008$$

The annuity-due equation of value at time 0 can be used to solve for X:

$$X \times \ddot{a}_{\overline{12|}} = 67.1008$$

$$X \times \frac{1 - 1.08^{-12}}{\frac{0.08}{1.08}} = 67.1008$$

$$X \times 8.1390 = 67.1008$$

$$X = \mathbf{8.2444}$$

Alternatively, the BA-II Plus can be used to answer this question:

10 [*N*] 8 [*I*/*Y*] 10 [*PMT*] [CPT] [*PV*]

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

12 [*N*] [CPT] [*PMT*]

Answer is **8.2444**.

Solution 6.14

C Section 6.03, Annuity-Due

The value of the annuity at the end of 10 years is:

$$\ddot{s}_{\overline{10|}}$$

But the question asks for the value on the date of the last deposit, and the last deposit is made at time 9. Therefore the value at the end of 10 years must be discounted by back by one year:

$$\frac{\ddot{s}_{\overline{10}|}}{1+i} = s_{\overline{10}|} = \frac{1.04^{10}-1}{0.04} = \mathbf{12.0061}$$

Solution 6.15

B Section 6.03, Annuity-Due

This question is similar to Question 6.02, but the payments now occur at the beginning of each year instead of the end of each year.

The equation of value at time 0 can be used to find X:

$$2{,}000\ddot{a}_{\overline{15}|0.08} = X\ddot{a}_{\overline{5}|0.08}$$

$$2{,}000 \times \frac{1-v^{15}}{\frac{0.08}{1.08}} = X \times \frac{1-v^{5}}{\frac{0.08}{1.08}}$$

$$2{,}000(1-1.08^{-15}) = X(1-1.08^{-5})$$

$$X = \mathbf{4{,}287.55}$$

The BA-II Plus can be used to solve the problem. We can leave the calculator in the END mode, because the denominators in the equation of value cancel, so the answer is the same regardless of whether the annuities pay at the beginning or end of each year:

15 [*N*] 8 [*I*/*Y*] 2,000 [*PMT*] [CPT] [*PV*]

Result is –17,118.95.

5 [*N*] [CPT] [*PMT*]

Solution is **4,287.55**.

Solution 6.16

D Section 6.03, Annuity-Due

The annual effective interest rate is:

$$i = \frac{d}{1-d} = \frac{0.05}{1-0.05} = 0.05263$$

The equation of value at time 0 can be used to solve for X:

$$\frac{3}{0.05263} = \frac{X}{0.05}$$

$$X = \mathbf{2.85}$$

Solution 6.17

D Section 6.03, Annuity-Immediate and Annuity-Due

The correct answer is Choice **D**.

Statement I is true because the present value of the perpetuity-due is equal to X plus the present value of the perpetuity-immediate:

$$X + \frac{X}{i} > \frac{X}{i}$$

Statement II is true because the present value of the perpetuity immediate is equal to the present value of the annuity-immediate plus the present value of payments that would have continued after the annuity immediate expires. A convenient way to show this mathematically is to begin with the fact that the complement of the discount factor is less than one:

$$1 > 1 - v^n$$

$$\frac{X}{i} > \frac{X}{i}(1 - v^n)$$

$$\frac{X}{i} > Xa_{\overline{n|}}$$

Statement III is false, because if the interest rate is high enough and/or annuity's term is long enough, then the present value of the annuity-due can be more than the present value of the perpetuity-immediate. As an example, suppose that the annuity-due payments end in 100 years and the annual effective interest rate is 50%:

$$\ddot{a}_{\overline{n|}} = X\frac{1 - v^n}{i}(1 + i) = X\frac{1 - 1.5^{-100}}{0.5}(1.5) = 3X$$

$$PV(\text{Perpetuity-immediate}) = \frac{X}{i} = 2X$$

Solution 6.18

C Section 6.03, Level Annuities

The accumulated value of the annuity-immediate at time $(n + 1)$ is equal to the accumulated value of an annuity-due:

$$s_{\overline{n|}} \times (1 + i) = 30.7725$$

$$\ddot{s}_{\overline{n|}} = 30.7725$$

$$\frac{(1 + i)^n - 1}{d} = 30.7725$$

$$\frac{3.7975 - 1}{d} = 30.7725$$

$$d = 0.0909$$

We can use d to find the value of n:

$$(1 + i)^n = 3.7975$$

$$(1 - d)^{-n} = 3.7975$$

$$(1 - 0.0909)^{-n} = 3.7975$$

$$-n \times \ln(0.9091) = \ln(3.7975)$$

$$n = \mathbf{14.00}$$

Solution 6.19

D Section 6.03, Level Annuities

The parents make 18 contributions of X. On the son's 19[th] birthday, the equation of value is:

$$X\left[1.04^{18} + 1.04^{17} + \cdots + 1.04\right] = 40,000\left[1 + v + v^2 + v^3\right]$$

$$X\sum_{k=1}^{18} 1.04^k = 40,000\left[1 + v + v^2 + v^3\right]$$

Only Choices A and D show 18 contributions, so the correct answer must be Choice A or Choice D.

The right side of the equation in both Choices A and D shows the value of the withdrawals at time 19. The left side of Choice A shows the value of the contributions at time 0, so the equation of value is not correct. The left side of Choice D shows the value of the contributions at time 19, so the equation of value is valid, and Choice **D** is the correct answer.

Solution 6.20

A Section 6.03, Annuity-Due

The amount needed to fund Justin's retirement on his 65^{th} birthday is:

$$\frac{4,000}{7.25} \times 1,000 = 551,724.1379$$

We can solve for the monthly contributions into the fund:

$$X\ddot{s}_{\overline{25\times12}|5\%/12} = 551,724.1379$$

$$X\ddot{s}_{\overline{300}|0.4167\%} = 551,724.1379$$

$$X \times \frac{1.004167^{300} - 1}{\frac{0.004167}{1.004167}} = 551,724.1379$$

$$X \times 597.9910 = 551,724.1379$$

$$X = \mathbf{922.6295}$$

We can use the BA II Plus to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

4,000 [÷] 7.25 [×] 1,000 [=] [*FV*]

300 [*N*] 5 [÷] 12 [=] [*I*/*Y*]

[CPT] [*PMT*]

Answer is **922.6295**.

Solution 6.21

D Section 6.03, Annuity-Due

The effective 2-year interest rate is found below:

$$\left(1 + \frac{i^{(1/2)}}{1/2}\right)^{1/2} = \left(1 + \frac{i^{(2)}}{2}\right)^2$$

$$\left(1 + \frac{i^{(1/2)}}{1/2}\right)^{1/2} = \left(1 + \frac{0.06}{2}\right)^2$$

$$\frac{i^{(1/2)}}{1/2} = 0.1255$$

Let's use 2 years as our unit of time:

$$20\ddot{a}_{\overline{10}|0.1255} = 20 \times \frac{1 - 1.1255^{-10}}{\frac{0.1255}{1.1255}} = 20 \times 6.2185 = \mathbf{124.37}$$

Alternatively, the BA-II Plus can be used to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

1.03 [y^x] 4 [=] [–] 1 [×] [=] 100 [=] [*I*/*Y*]

10 [*N*] 20 [*PMT*] [CPT] [*PV*]
Result is –124.37. Answer is **124.37**.

Solution 6.22

E Section 6.03, Annuity-Due

The accumulated value is:

$$\begin{aligned}
&1.07^{15}+1.07^{14}+1.07^{13}+1.07^{12}+1.07^{11}\\
&\quad+2\left(1.07^{10}+1.07^{9}+1.07^{8}+1.07^{7}+1.07^{6}\right)\\
&\quad+3\left(1.07^{5}+1.07^{4}+1.07^{3}+1.07^{2}+1.07^{1}\right)\\
&=\left(1.07^{15}+1.07^{14}+\cdots+1.07^{1}\right)\\
&\quad+\left(1.07^{10}+1.07^{9}+\cdots+1.07^{1}\right)\\
&\quad+\left(1.07^{5}+1.07^{4}+\cdots+1.07^{1}\right)\\
&=\ddot{s}_{\overline{15}|}+\ddot{s}_{\overline{10}|}+\ddot{s}_{\overline{5}|}=\frac{1,07^{15}-1}{\frac{0.07}{1.07}}+\frac{1,07^{10}-1}{\frac{0.07}{1.07}}+\frac{1,07^{5}-1}{\frac{0.07}{1.07}}\\
&=26.8881+14.7836+6.1533=\mathbf{47.8249}
\end{aligned}$$

Alternatively, the BA-II Plus can be used to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
15 [*N*] 7 [*I*/*Y*] 1 [*PMT*] [CPT] [*FV*] [STO] 1
10 [*N*] [CPT] [*FV*] [STO] 2
5 [*N*] [CPT] [*FV*] [+] [RCL] 2 [+] [RCL] 1 [=]
Result is –47.8249. Answer is **47.8249**.

Solution 6.23

D Section 6.03, Annuity-Due

On June 1, 2050, the amount in Fred's account is:

$$\begin{aligned}
&5,000\times 1.09^{2050-2025}+29,687.69\times 1.09^{2050-2040}\\
&=5,000\times 1.09^{25}+29,687.69\times 1.09^{10}=113,396.9622
\end{aligned}$$

The equation of value as of June 1, 2050 is:

$$2,000\ddot{s}_{\overline{n}|}\times 1.07^{25-n}=113,396.9622$$

Let’s discount both sides for 25 years to convert the equation above into an equation of value for Ethel as of June 1, 2025:

$$\frac{2,000\ddot{s}_{\overline{n|}} \times 1.07^{25-n}}{1.07^{25}} = \frac{113,396.9622}{1.07^{25}}$$

$$2,000\ddot{s}_{\overline{n|}} \times 1.07^{-n} = 20,893.2970$$

$$2,000\ddot{a}_{\overline{n|}} = 20,893.2970$$

$$2,000 \times \frac{1-1.07^{-n}}{\frac{0.07}{1.07}} = 20,893.2970$$

$$1.07^{-n} = 0.3166$$

$$-n \times \ln(1.07) = \ln(0.3166)$$

$$n = \mathbf{17}$$

Alternatively, the BA-II Plus can be used to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

5,000 [×] 1.09 [y^x] 25 [=] [+] 29,687.69 [×] 1.09 [y^x] 10 [=]

[÷] 1.07 [y^x] 25 [=] [*PV*]

7 [*I*/*Y*] 2,000 [+/-] [*PMT*]

[CPT] [*N*]

Result is **17**.

Solution 6.24

B Section 6.03, Annuity-Due

At the end of 15 years, the value in the fund is:

$$500\ddot{s}_{\overline{15|}0.12} = 500 \times \frac{1.12^{15}-1}{0.12} \times 1.12 = 500 \times 41.7533 = 20,876.6402$$

The effective 6-month interest rate is:

$$1.12^{0.5} - 1 = 0.05830$$

Let *n* be the number of 6-month periods that the fund can support withdrawals of 2,000:

$$2,000\ddot{a}_{\overline{n|}0.0583} = 20,876.64$$

$$\frac{1-1.0583^{-n}}{0.0583} \times 1.0583 = 10.4383$$

$$1 - 1.0583^{-n} = 0.5750$$

$$n = 15.1021$$

The 15th payment of 2,000 is made at the beginning of the 15th 6-month period, which is the same as the end of the 14th period, so the 15th payment is made at the end of 7 years.

Six months after the 15th payment of 2,000 is made, the balance in the fund is:

$$20,876.6402(1.0583)^{15} - 2,000\ddot{s}_{\overline{15|}0.0583}$$

$$= 48,842.2629 - 2,000 \times \frac{1.0583^{15}-1}{0.0583} \times 1.0583$$

$$= 48,842.2629 - 2,000 \times 24.3165$$

$$= \mathbf{209.33}$$


We can use the BA II Plus to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
15 [*N*] 12 [*I*/*Y*] 500 [*PMT*] [CPT] [*FV*]
(Result is –20,876.6402)
[*PV*]
1.12 [y^x] 0.5 [=] [–] 1 [=] [×] 100 [=] [*I*/*Y*]
2,000 [*PMT*] 0 [*FV*] [CPT] [*N*]
(Result is 15.1021)
15 [*N*] [CPT] [*FV*]
Answer is **209.33**.

Solution 6.25

B Section 6.03, Level Annuities

The first tuition payment is due at the beginning of the 18th year, which is at the end of 17 years. The payment of X is made at the end of 18 years. The equation of value at the end of 17 years is:

$$\left[700\ddot{s}_{\overline{18}|} + X\right]v = 7,000\left[1.04^{17} + 1.04^{18}v\right]$$

$$700s_{\overline{18}|} + Xv = 7,000\left[1.04^{17} + 1.04^{18}v\right]$$

$$700 \times \frac{1.08^{18} - 1}{0.08} + \frac{X}{1.08} = 7,000 \times 3.8237$$

$$700 \times 37.4502 + \frac{X}{1.08} = 26,765.5957$$

$$26,215.1706 + \frac{X}{1.08} = 26,765.5957$$

$$X = \mathbf{594.46}$$


We can use the BA II Plus to answer this question:

18 [*N*] 8 [*I*/*Y*] 700 [*PMT*] [CPT] [*FV*]
1.04 [y^x] 17 [+] 1.04 [y^x] 18 [÷] 1.08 [=] [×] 7,000
[+] [RCL] [*FV*] [=] [×] 1.08
Answer is **594.46**.

Solution 6.26

B Section 6.04, Deferred Annuities

Since Amy receives the first n payments and Beth receives the next m payments, the difference between the present values of their payments is:

$$X\ddot{a}_{\overline{n}|} - Xv^n\ddot{a}_{\overline{m}|} = X\left[\ddot{a}_{\overline{n}|} - v^n\ddot{a}_{\overline{m}|}\right] = X\left[\ddot{a}_{\overline{n}|} - v^{n-1} \times v \times (1+i)a_{\overline{m}|}\right]$$

$$= X\left[\ddot{a}_{\overline{n}|} - v^{n-1}a_{\overline{m}|}\right]$$

This matches Choice **B**.

Solution 6.27

D Section 6.04, Deferred Perpetuities

Aaron's share of the present value of the perpetuity is 30%:

$$Xa_{\overline{n}|} = 0.3 \times \frac{X}{i}$$

$$1 - v^n = 0.3$$

$$v^n = 0.7$$

Charlie's share of the present value of the perpetuity is K:

$$\frac{X}{i} \times v^{3n} = K \times \frac{X}{i}$$

$$0.7^3 = K$$

$$K = \mathbf{0.343}$$

Solution 6.28

C Section 6.04, Deferred Annuities

The annual effective interest rate is:

$$i = \left(1 + \tfrac{0.07}{12}\right)^{12} - 1 = 0.07229$$

The first payment from the perpetuity is made in 7 years. At the end of 6 years, the perpetuity is a perpetuity-immediate. The present value of the perpetuity-immediate is:

$$PV_0 = v^6 \times \frac{500}{0.07229} = (1.07229)^{-6} \times \frac{500}{0.07229} = \mathbf{4,550.06}$$

Solution 6.29

B Section 6.04, Deferred Annuities

Since the first two sets of cash flows have the same present value at time 0, they must also have the same present value at time 1:

Time 0: $10,000v = 1,200a_{\overline{12}|}$

Time 1: $10,000 = 1,200\ddot{a}_{\overline{12}|}$

We can use the BA II Plus to obtain the annual effective interest rate:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

12 [*N*] –10,000 [*PV*] 1,200 [*PMT*] [CPT] [*I/Y*]

Result is 7.4503

The annual effective interest rate is 7.4503%.

The third set of cash flows consists of 11 payments beginning at time 20. The equation of value at time 0 for the first and third sets of cash flows is below:

$$10,000v = v^{20}X\ddot{a}_{\overline{11}|}$$

$$10,000 = v^{19}X\ddot{a}_{\overline{11}|}$$

We can use the BA II Plus to find X. Continuing from the calculation of the interest rate above, and leaving the calculator in the BGN mode, we have:

11 [*N*] 1 [*PMT*] [CPT] [*PV*]

[÷] ([RCL] [*I/Y*] [÷] 100 [+] 1) [y^x] 19 [=] [1/*x*] [×] 10,000 [=]

Result is –4,970.90. Answer is **4,970.90**.

Solution 6.30

E Section 6.04, Level Annuity

The present value of the perpetuity-due is equal to the present value of a perpetuity-immediate plus a payment of 20,000 at time 0:

$$\begin{aligned}
&20{,}000 + 20{,}000a_{\overline{8}|0.09} + \frac{1}{1.09^8} \times \frac{20{,}000}{0.11} \\
&\quad = 20{,}000 + 20{,}000 \times \frac{1-1.09^{-8}}{0.09} + 91{,}248.4145 \\
&\quad = 20{,}000 + 20{,}000 \times 5.5348 + 91{,}248.4145 \\
&\quad = 221{,}944.7968
\end{aligned}$$

The equation of value at time 0 can be used to find X:

$$221{,}944.7968 = X\ddot{a}_{\overline{8}|0.09} + \frac{X}{1.09^8}\ddot{a}_{\overline{22}|0.11}$$

$$221{,}944.7968 = X\left[\frac{1-1.09^{-8}}{\frac{0.09}{1.09}} + \frac{1}{1.09^8} \times \frac{1-1.11^{-22}}{\frac{0.11}{1.11}}\right]$$

$$221{,}944.7968 = X\left[6.0330 + 4.5545\right]$$

$$X = \mathbf{20{,}963.06}$$

We can use the BA II Plus to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

8 [*N*] 9 [*I*/*Y*] 1 [*PMT*] [CPT] [*PV*]

(Result is –6.0330) [+/-] [STO] 1

22 [*N*] 11 [*I*/*Y*] 1 [*PMT*] [CPT] [*PV*] [÷] 1.09 [y^x] 8 [=]

(Result is –4.5545) [+/-] [STO] 2

9 [*N*] 9 [*I*/*Y*] 20,000 [*PMT*] [CPT] [*PV*] [+/-]

[+] 20,000 [÷] 0.11 [÷] 1.09 [y^x] 8 [=]

(Result is 221,944.7968) [÷] [(] [RCL] 1 [+] [RCL] 2 [)] [=]

Answer is **20,963.06**.

Chapter 7: Level Annuities, Payable More than Once per Time Unit

Solution 7.01

D Section 7.01, Annuity-Immediate

The annual interest rate compounded quarterly is:

$$i^{(4)} = \left(1.08^{0.25} - 1\right) \times 4 = 0.07771$$

The annual rate of payment is 200 per year, and the present value is:

$$200 \times a_{\overline{10}|}^{(4)} = 200 \times \frac{1 - v^{10}}{i^{(4)}} = 200 \times \frac{1 - 1.08^{-10}}{0.07771} = 200 \times 6.9082 = \mathbf{1,381.63}$$

Alternatively, we can use the quarterly effective interest rate along with the quarterly rate of payment. The quarterly effective interest rate is:

$$\frac{i^{(4)}}{4} = 1.08^{0.25} - 1 = 0.01943$$

The quarterly rate of payment is 50 per quarter, and the present value is:

$$50 \times a_{\overline{40}|0.01943} = 50 \times \frac{1 - 1.01943^{-40}}{0.01943} = 50 \times \frac{1 - 1.08^{-10}}{0.01943} = 50 \times 27.6326$$
$$= \mathbf{1,381.63}$$

The BA-II Plus can be used to solve the problem as follows:

1.08 [y^x] 0.25 [–] 1 [=] [×] 100 [=] [*I*/*Y*]

40 [*N*] 50 [*PMT*] [CPT] [*PV*]

Result is –1,381.63. Answer is **1,381.63**.

Solution 7.02

D Section 7.01, Perpetuity-Immediate

Let's use one quarter (i.e., 3 months) as our unit of time. If the first payment were at the end of one unit of time, then the present value would be:

$$\frac{45}{0.03} = 1,500$$

Accumulating this amount by one month, we have:

$$1,500 \times 1.03^{1/3} = \mathbf{1,514.85}$$

Alternatively, we can find the present value as follows:

$$\frac{45}{1.03^{2/3}} + \frac{45}{1.03^{5/3}} + \frac{45}{1.03^{8/3}} + \cdots = 1.03^{1/3}\left[\frac{45}{1.03} + \frac{45}{1.03^2} + \frac{45}{1.03^3} + \cdots\right]$$
$$= 1.03^{1/3}\left[\frac{45}{0.03}\right] = \mathbf{1,514.85}$$

Solution 7.03

A   Section 7.01, Annuity-Immediate

We can use the BA II Plus to answer this question:

48 [*N*] 15,000 [+/-] [*PV*] 600 [*PMT*] [CPT] [*I*/*Y*]

(Result is 3.0577)

[÷] 100 [+] 1 [=] [y^x] 3 [1/*x*] [=] [–] 1 [=] [×] 12 [=]

Answer is **0.1211**.

Solution 7.04

D   Section 7.01, Annuity-Immediate

The monthly effective interest rate used to find the present values of David's and Rebecca's annuities is:

$$\frac{i^{(12)}}{12} = 1.08^{1/12} - 1 = 0.006434$$

Let X be the monthly annuity payment received by Rebecca. We equate the present values of David's and Rebecca's annuities and solve for X:

$$1,000 \times a_{\overline{10}|0.08} = X \times a_{\overline{120}|0.006434}$$

$$1,000 \times \frac{1-1.08^{-10}}{0.08} = X \times \frac{1-1.08^{-10}}{0.006434}$$

$$1,000 \times 6.7101 = X \times 83.4324$$

$$6,710.0814 = X \times 83.4324$$

$$X = 80.4254$$

An annual effective rate of 10% is equivalent to the following monthly effective interest rate:

$$\frac{i^{(12)}}{12} = 1.10^{1/12} - 1 = 0.007974$$

Rebecca's payments are accumulated at an annual effective interest rate of 10%:

$$80.4254 \times s_{\overline{120}|0.007974} = 80.4254 \times \frac{1.10^{10}-1}{0.007974} = 80.4254 \times 199.8639$$
$$= 16,074.1259$$

David's payments are accumulated at an annual effective interest rate of 9%:

$$1,000 \times s_{\overline{10}|0.09} = 1,000 \times \frac{1.09^{10}-1}{0.09} = 1,000 \times 15.1929 = 15,192.9297$$

The difference between the accumulated value of Rebecca's payments and the accumulated value of David's payments is:

$$16,074.1259 - 15,192.9297 = \mathbf{881.20}$$

The BA-II Plus can be used to solve the problem as follows:

10 [*N*] 8 [*I*/*Y*] 1,000 [*PMT*] [CPT] [*PV*]

1.08 [y^x] 12 [1/*x*] [–] 1 [=] [×] 100 [=] [*I*/*Y*]

120 [*N*] [CPT] [*PMT*]

Result is 80.4254.

1.10 [y^x] 12 [1/*x*] [–] 1 [=] [×] 100 [=] [*I*/*Y*]

0 [*PV*] [CPT] [*FV*] [+/-] [STO] 1

10 [*N*] 9 [*I*/*Y*] 1,000 [*PMT*] [CPT] [*FV*]

[+] [RCL] 1 [=]

Answer is **881.20**.

Solution 7.05

C Section 7.01, Annuity-Immediate

The monthly effective interest rate for the first 22 months is:

$$\frac{0.09}{12} = 0.0075$$

The accumulated value of the loan after 22 months minus the accumulated value of the payments is:

$$30,000(1.0075)^{22} - 700s_{\overline{22|}0.0075} = 35,360.02 - 700 \times \frac{1.0075^{22} - 1}{0.0075}$$

$$= 35,360.02 - 700 \times 23.8223 = 18,684.4093$$

The new monthly effective interest rate used to refinance the loan is:

$$\frac{0.06}{12} = 0.005$$

The equation of value for the refinanced loan after 22 months is:

$$18,684.4093 = Xa_{\overline{30|}0.005}$$

$$18,684.4093 = X \times \frac{1 - 1.005^{-30}}{0.005}$$

$$18,684.4093 = 27.7941X$$

$$X = \mathbf{672.24}$$

We can use the BA II Plus to answer this question:

22 [*N*] 9 [÷] 12 [=] [*I*/*Y*] 30,000 [*PV*] 700 [+/-] [*PMT*] [CPT] [*FV*]

(Result is −18,684.4093)

[*PV*] 30 [*N*] 6 [÷] 12 [=] [*I*/*Y*] 0 [*FV*] [CPT] [*PMT*]

Answer is **672.24**.

Solution 7.06

E Section 7.01, Annuity-Immediate

The monthly interest rate for the first 60 months is:

$$\frac{0.072}{12} = 0.006$$

The accumulated value of the loan after 60 months minus the accumulated value of the payments is:

$$294,584.81(1.006)^{60} - 2,000s_{\overline{60|}0.006} = 421,783.1173 - 2,000 \times \frac{1.006^{60} - 1}{0.006}$$

$$= 421,783.1173 - 2,000 \times 71.9647 = 277,853.6466$$

The new interest rate is to refinance the loan is:

$$\frac{0.036}{12} = 0.003$$

The equation of value for the refinanced loan after 60 months is:

$$277,853.6466 = 2,000a_{\overline{n}|0.003}$$

$$277,853.6466 = 2,000 \times \frac{1 - 1.003^{-n}}{0.003}$$

$$0.5832 = 1.003^{-n}$$

$$\ln(0.5832) = -n \times \ln(1.003)$$

$$n = \mathbf{180}$$

We can use the BA II Plus to answer this question:

60 [N] 7.2 [÷] 12 [=] [I/Y] 294,584.81 [PV] 2,000 [+/-] [PMT]
[CPT] [FV]
(Result is –277,853.6466)
[PV] 3.6 [÷] 12 [=] [I/Y] 2,000 [PMT] 0 [FV] [CPT] [N]
Answer is **180**.

Solution 7.07

B Section 7.01, Perpetuity-Immediate

The present value of the first annuity can be used to find the effective 4-year interest rate:

$$\frac{15}{\frac{i^{(1/4)}}{1/4}} = 37.50$$

$$\frac{i^{(1/4)}}{1/4} = 0.40$$

The 4-month effective interest rate is found below:

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 + \frac{i^{(p)}}{p}\right)^p$$

$$(1.40)^{1/4} = \left(1 + \frac{i^{(3)}}{3}\right)^3$$

$$\frac{i^{(3)}}{3} = \left(1.40^{1/4}\right)^{1/3} - 1 = 0.02844$$

The present value of the second perpetuity is:

$$\frac{1}{0.02844} = \mathbf{35.17}$$

Solution 7.08

B Section 7.01, Level Annuity

The monthly effective interest rate is:

$$\frac{0.054}{12} = 0.0045$$

August 1 of the year (y + 5), is 67 months after January 1 of the year y. During these 67 months, there are 22 quarterly payments of 1,000. The equation of value at time 0 is:

$$X+\left[\frac{1,000}{1.0045^{3}}+\frac{1,000}{1.0045^{6}}+\cdots+\frac{1,000}{1.0045^{66}}\right]=\frac{1.7X}{1.0045^{67}}$$

$$X+\sum_{k=1}^{22}\frac{1,000}{1.0045^{3k}}=\frac{1.7X}{1.0045^{67}}$$

The answer is Choice **B**.

Solution 7.09

E Section 7.01, Level Annuity

The monthly effective interest rate is:

$$\frac{0.054}{12}=0.0045$$

The quarterly effective interest rate is:

$$1.0045^{3}-1=0.01356$$

August 1 of the year (y + 5) is 67 months after January 1 of the year y. During these 67 months, there are 22 quarterly payments of 1,000.

The equation of value at time 0 is:

$$X+\left[\frac{1,000}{1.0045^{3}}+\frac{1,000}{1.0045^{6}}+\cdots+\frac{1,000}{1.0045^{66}}\right]=\frac{1.7X}{1.0045^{67}}$$

$$X+\sum_{k=1}^{22}\frac{1,000}{1.0045^{3k}}=\frac{1.7X}{1.0045^{67}}$$

$$X+1,000\times a_{\overline{22}|0.01356}=\frac{1.7X}{1.0045^{67}}$$

$$1,000\times\frac{1-1.01356^{-22}}{0.01356}=\left(\frac{1.7}{1.0045^{67}}-1\right)X$$

$$18,911.8629=0.2584X$$

$$X=\mathbf{73,201.33}$$

Solution 7.10

E Section 7.02, Annuity-Due

The annual discount rate compounded quarterly is found below:

$$\left(1-\frac{d^{(4)}}{4}\right)^{-4}=1.08$$

$$d^{(4)}=0.07623$$

The annual rate of payment is 200 per year, and the present value is:

$$200\times\ddot{a}^{(4)}_{\overline{10}|}=200\times\frac{1-v^{10}}{d^{(4)}}=200\times\frac{1-1.08^{-10}}{0.07623}=200\times7.0424=\mathbf{1,408.47}$$

Alternatively, we can use the quarterly effective interest rate along with the quarterly rate of payment. The quarterly effective interest rate is:

$$\frac{i^{(4)}}{4}=1.08^{0.25}-1=0.01943$$

The quarterly rate of payment is 50 per quarter, and the present value is:

$$50 \times \ddot{a}_{\overline{40}|0.01943} = 50 \times \frac{1 - 1.01943^{-40}}{0.01943} \times 1.01943 = 50 \times \frac{1 - 1.08^{-10}}{0.01943} \times 1.01943$$
$$= 50 \times 28.1694 = \mathbf{1,408.47}$$

The BA-II Plus can be used to solve the problem as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

1.08 [y^x] 0.25 [–] 1 [=] [×] 100 [=] [*I/Y*]

40 [*N*] 50 [*PMT*] [CPT] [*PV*]

Result is –1,408.47. Answer is **1,408.47**.

Solution 7.11

C Section 7.02, Perpetuity-Due

Let's use one quarter (i.e., 3 months) as our unit of time. The present value of the perpetuity-immediate is:

$$\frac{45}{0.03} = 1,500$$

The monthly effective interest rate is:

$$1.03^{1/3} - 1 = 0.009902$$

We now use one month as our unit of time and set the present value of the perpetuity-immediate equal to the present value of the perpetuity-due:

$$1,500 = X\left(\frac{1}{0.009902} + 1\right)$$
$$X = \mathbf{14.71}$$

Alternatively, we can use the monthly effective discount rate:

$$1,500 = \frac{X}{\frac{d^{(4)}}{4}}$$

$$1,500 = \frac{X}{\frac{0.009902}{1.009902}}$$
$$X = \mathbf{14.71}$$

Solution 7.12

E Section 7.02, Annuity-Due

Let *P* be the purchase price and *j* be the monthly effective interest rate. The equation of value at time 0 is:

$$P = \frac{P}{11}\ddot{a}_{\overline{12}|j}$$

$$1 = \frac{1}{11} \times \ddot{a}_{\overline{12}|j}$$

We can use the BA-II Plus calculator to solve for the monthly effective interest rate. Once we have the monthly effective interest rate, we convert it into the annual effective interest rate:

[2nd] [[BGN] [2nd] [SET] [2nd] [QUIT]
12 [*N*] 1 [*PV*] 11 [1/*x*] [+/-] [*PMT*] [CPT] [*I*/*Y*]
(Result is 1.6231)
[÷] 100 [+] 1 [=] [y^x] 12 [=] [–] 1 [=]
Answer is **0.2131**.

Solution 7.13

D Section 7.02, Level Annuities

The equation of value at the end of 11 years is:

$$\left[100\ddot{s}_{\overline{36}|j}(1+i)^6+200\ddot{s}_{\overline{36}|j}(1+i)^3+300\ddot{s}_{\overline{36}|j}\right](1+i)^2=30{,}000$$

The equation can be rearranged to match Choice **D**:

$$\left[\ddot{s}_{\overline{36}|j}(1+i)^6+2\ddot{s}_{\overline{36}|j}(1+i)^3+3\ddot{s}_{\overline{36}|j}\right](1+i)^2=300$$

$$\ddot{s}_{\overline{36}|j}(1+i)^2\left[(1+i)^6+2(1+i)^3+3\right]=300$$

Solution 7.14

B Section 7.02, Annuity-Due

The monthly effective interest rate used to find the present value of Minnie's annuity-immediate is:

$$\frac{i^{(12)}}{12}=1.08^{1/12}-1=0.006434$$

Let X be the monthly annuity payment received by Minnie. We equate the present values of Harry and Minnie and solve for X:

$$1{,}000\times\ddot{a}_{\overline{10}|0.08}=X\times\ddot{a}_{\overline{120}|0.006434}$$

$$1{,}000\times\frac{1-1.08^{-10}}{0.08}\times1.08=X\times\frac{1-1.08^{-10}}{0.006434}\times1.006434$$

$$1{,}000\times7.2469=X\times$$

$$7{,}246.8879=X\times83.9692$$

$$X=86.3041$$

An annual effective rate of 10% is equivalent to the following monthly effective interest rate:

$$\frac{i^{(12)}}{12}=1.10^{1/12}-1=0.007974$$

Minnie's payments are accumulated at an annual effective interest rate of 10%:

$$\begin{aligned}86.3041\times\ddot{s}_{\overline{120}|0.007974}&=86.3041\times\frac{1.10^{10}-1}{0.007974}\times1.007974\\&=86.3041\times201.4576\\&=17{,}386.6215\end{aligned}$$

Harry's payments are accumulated at an annual effective interest rate of 9%:

$$1,000 \times \ddot{s}_{\overline{10}|0.09} = 1,000 \times \frac{1.09^{10} - 1}{0.09} \times 1.09 = 1,000 \times 16.5603 = 16,560.2934$$

The difference between the accumulated value of Minnie's payments and the accumulated value of Harry's payments is:

$$17,386.6215 - 16,560.2934 = \mathbf{826.33}$$

The BA-II Plus can be used to solve the problem as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

10 [*N*] 8 [*I/Y*] 1,000 [*PMT*] [CPT] [*PV*]

1.08 [y^x] 12 [1/*x*] [–] 1 [=] [×] 100 [=] [*I/Y*]

120 [*N*] [CPT] [*PMT*]

Result is 86.3041.

1.10 [y^x] 12 [1/*x*] [–] 1 [=] [×] 100 [=] [*I/Y*]

0 [*PV*] [CPT] [*FV*] [+/-] [STO] 1

10 [*N*] 9 [*I/Y*] 1,000 [*PMT*] [CPT] [*FV*]

[+] [RCL] 1 [=]

Answer is **826.33**.

Solution 7.15

A Section 7.02, Annuity-Due

The monthly effective interest rate is:

$$\frac{0.07}{12} = 0.005833$$

To have 2,000 of monthly income beginning on her 70th birthday, the woman needs the following lump sum on her 70th birthday:

$$\frac{2,000}{9.45} \times 1,000 = 211,640.2116$$

Her contributions must accumulate to 211,640.2116:

$$X\ddot{s}_{\overline{33 \times 12}|0.005833} = 211,640.2116$$

$$X \times \frac{1.005833^{396} - 1}{0.005833} \times 1.005833 = 211,640.2116$$

$$1,553.0706X = 211,640.2116$$

$$X = \mathbf{136.27}$$

We can use the BA II Plus to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

2,000 [÷] 9.45 [×] 1,000 [=] [*FV*]

33 [×] 12 [=] [*N*] 7 [÷] 12 [=] [*I/Y*]

[CPT] [*PMT*]

Result is –136.27. Answer is **136.27**.

Solution 7.16

B Section 7.02, Annuity-Due

Let j be the effective interest rate for an interval of 5 years:

$$j = (1+i)^5 - 1$$

There are 8 5-year intervals in 40 years, and there are 4 5-year intervals in 20 years. Therefore, the accumulated value at the end of 8 intervals is equal to 4 times the accumulated value at the end of 4 intervals:

$$250\ddot{s}_{\overline{8}|j} = 4 \times 250\ddot{s}_{\overline{4}|j}$$

$$\frac{(1+j)^8 - 1}{j\ /\ (j+1)} = 4 \times \frac{(1+j)^4 - 1}{j\ /\ (j+1)}$$

$$\frac{(1+j)^8 - 1}{(1+j)^4 - 1} = 4$$

$$(1+j)^4 + 1 = 4$$

$$(1+j)^4 = 3$$

$$j = 0.3161$$

The accumulated amount at the end of 40 years is:

$$X = 250\ddot{s}_{\overline{8}|0.3161} = 250 \times \frac{1.3161^8 - 1}{0.3161} \times 1.3161 = \mathbf{8,327.63}$$

The BA-II Plus can be used as follows:

3 [y^x] 0.25 [=] [–] 1 [=] [×] 100 [=] [*I*/*Y*]

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

8 [*N*] 250 [*PMT*] [CPT] [*FV*]

Result is –8,327.63. Solution is **8,327.63**.

Solution 7.17

B Section 7.02, Perpetuities

We can use the price of the first perpetuity to find the value of i:

$$6.74 = \frac{1}{(1+i)^3 - 1} + 1$$

$$i = 0.0550$$

The annual effective interest rate used to value the second annuity is:

$$i + 0.01 = 0.0550 + 0.01 = 0.0650$$

The second perpetuity begins at the end of 1 year, so it can be valued as a perpetuity immediate accumulated for 3 years:

$$6.74 = \frac{X}{1.0650^4 - 1} \times 1.0650^3$$

$$X = \mathbf{1.5982}$$

Solution 7.18

C Section 7.02, Level Annuity

The accumulated value at the end of 6 years is:

$$X\left(1.05^6 + 1.05^4 + 1.05^2\right) = 3.6581X$$

The accumulated value at the end of 10 years is:

$$\left(3.6581X + X\right)\times 1.07^{4} + X \times 1.07^{2} = 7.2507X$$

Let j be the 2-year effective rate. The equation of value to be solved is:

$$X\ddot{s}_{\overline{5}|j} = 7.2507X$$

$$\ddot{s}_{\overline{5}|j} = 7.2507$$

We use the BA II Plus to obtain the annual effective yield:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
5 [*N*] 1 [*PMT*] –7.2507 [*FV*] [CPT] [*I*/*Y*]
(Result is 12.6560)
[÷] 100 [+] 1 [=] [y^x] 0.5 [=] –1 [=]
Answer is **0.0614**.

Solution 7.19

D Section 7.02, Annuity-Due

The monthly effective interest rate is:

$$1.07^{1/12} - 1 = 0.005654$$

The withdrawals of $35,000 are made at times 17, 18, 19, and 20.

The equation of value at time 0 can be used to find X:

$$X\ddot{a}_{\overline{20\times 12}|0.005654} = \frac{35,000}{1.07^{17}}\ddot{a}_{\overline{4}|0.07}$$

$$X \times \frac{1-1.005654^{-240}}{0.05654}\times 1.05654 = 11,080.1037 \times \frac{1-1.07^{-4}}{0.07}\times 1.07$$

$$X \times 131.8986 = 11,080.1037 \times 3.6243$$

$$X = \mathbf{304.46}$$

We can use the BA II Plus to answer this question:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
1.07 [y^x] 12 [1/*x*] [=] [–] 1 [=] [×] 100 [=] [*I*/*Y*]
20 [×] 12 [=] [*N*] 1 [*PMT*] [CPT] [*PV*]
(Result is –131.8986) [STO] 1
4 [*N*] 7 [*I*/*Y*] 35,000 [*PMT*] [CPT] [*PV*]
(Result is –126,851.0616) [÷] 1.07 [y^x] 17 [=]
[÷] [RCL] 1 [=]
Answer is **304.46**.

Solution 7.20

D Section 7.03, Level Annuities, Payable Continuously

The present value of the perpetuity is:

$$\lim_{n\to\infty}\left[50 \times \bar{a}_{\overline{n}|}\right] = \lim_{n\to\infty}\left[50 \times \frac{1-v^{n}}{r}\right] = 50 \times \frac{1-0}{r} = \frac{50}{0.07} = \mathbf{714.29}$$

Solution 7.21

E Section 7.03, Level Annuities, Payable Continuously

The accumulated value of the annuity is:

$$50 \times \bar{s}_{\overline{n}|} = 50 \times \frac{(1+i)^n - 1}{r} = 50 \times \frac{e^{10 \times 0.07} - 1}{0.07} = 50 \times 14.4822 = \mathbf{724.11}$$

Solution 7.22

D Section 7.03, Level Annuities, Payable Continuously

The present value of the continuously payable annuity is:

$$\bar{a}_{\overline{10}|} = 8.1277$$

$$\frac{1 - e^{-10\delta}}{\delta} = 8.1277$$

The derivative is:

$$\frac{d\bar{a}_{\overline{10}|}}{d\delta} = -37.735$$

$$\frac{d}{d\delta}\left(\frac{1 - e^{-10\delta}}{\delta}\right) = -37.735$$

$$\frac{10e^{-10\delta}\delta - (1 - e^{-10\delta})}{\delta^2} = -37.735$$

We can use the present value of the annuity to find an expression for $e^{-10\delta}$:

$$\frac{1 - e^{-10\delta}}{\delta} = 8.1277$$

$$e^{-10\delta} = 1 - 8.1277\delta$$

We can now solve for δ:

$$\frac{10e^{-10\delta}\delta - (1 - e^{-10\delta})}{\delta^2} = -37.735$$

$$\frac{10(1 - 8.1277\delta)\delta - (1 - e^{-10\delta})}{\delta^2} = -37.735$$

$$\frac{10(1 - 8.1277\delta)\delta}{\delta^2} - \frac{(1 - e^{-10\delta})}{\delta^2} = -37.735$$

$$10(1 - 8.1277\delta) - \frac{(1 - e^{-10\delta})}{\delta} = -37.735\delta$$

$$10(1 - 8.1277\delta) - 8.1277 = -37.735\delta$$

$$1.8723 = 43.542\delta$$

$$\delta = \mathbf{0.04300}$$

Solution 7.23

A   Section 7.03, Level Annuities

We can find the present value of an $(n+1)$–year annuity-immediate:

$$\ddot{a}_{\overline{n+2}|} = a_{\overline{n+1}|} + 1$$

$$13.0685 = a_{\overline{n+1}|} + 1$$

$$a_{\overline{n+1}|} = 12.0685$$

We can find the accumulated value of an $(n+1)$–year annuity-immediate:

$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

$$17.3958 = s_{\overline{n+1}|} - 1$$

$$s_{\overline{n+1}|} = 18.3958$$

We can now solve for the $(n+1)$–year discount factor:

$$a_{\overline{n+1}|} = v^{n+1} \times s_{\overline{n+1}|}$$

$$12.0685 = v^{n+1} \times 18.3958$$

$$v^{n+1} = 0.6560$$

We can use the present value of the $(n+1)$–year annuity-immediate to solve for i:

$$a_{\overline{n+1}|} = 12.0685$$

$$\frac{1 - v^{n+1}}{i} = 12.0685$$

$$\frac{1 - 0.6560}{i} = 12.0685$$

$$i = 0.02850$$

The present value of the 1-year annuity paid continuously is:

$$\bar{a}_{\overline{1}|} = \frac{1 - v}{r} = \frac{1 - \frac{1}{1.02850}}{\ln(1.02850)} = \mathbf{0.9861}$$

Chapter 8: Arithmetic Progression Annuities

Solution 8.01

D Section 8.01, Increasing Annuity

The annuity can be broken down into a level annuity-immediate and an increasing annuity-immediate:

$$PV_0 = 45a_{\overline{21}|} + 5(Ia)_{\overline{21}|} = 45 \times \frac{1-v^{21}}{0.05} + 5 \times \frac{\ddot{a}_{\overline{21}|} - 21v^{21}}{0.05}$$

$$= 45 \times 12.8212 + 5 \times \frac{13.4622 - 7.5378}{0.05} = 576.9519 + 5 \times 118.4884$$

$$= \mathbf{1,169.39}$$

Alternatively, we can use the PIn method with the following parameters:

$P_1 = 50$ $I = 5$ $n = 21$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(50 + \frac{5}{0.05}\right) \times \frac{1-(1.05)^{-21}}{0.05} - \frac{5 \times 21}{0.05}(1.05)^{-21}$$

$$= 150 \times 12.8212 - 753.7790 = \mathbf{1,169.39}$$

The BA-II Plus can be used to solve the problem as follows:

21 [*N*] 5 [*I/Y*] 50 [+] 5 [÷] 0.05 [=] [*PMT*]

5 [×] 21 [÷] 0.05 [=] [+/-] [*FV*] [CPT] [*PV*]

Result is –1,169.39. Answer is **1,169.39**.

Solution 8.02

C Section 8.01, Increasing Annuity

If the payments occurred at the end of each year, then the present value would be:

$$45a_{\overline{21}|} + 5(Ia)_{\overline{21}|}$$

Since each of the payments occur 6 months earlier than the end of each year, the present value factors implied in the expression above all discount the cash flows by 6 months too much. To fix this we multiply by a 6-month accumulation factor. The present value is:

$$\left(45a_{\overline{21}|} + 5(Ia)_{\overline{21}|}\right) \times 1.05^{0.5}$$

The portion in parentheses above can be found using the PIn method:

$P_1 = 50$ $I = 5$ $n = 21$

The portion in parentheses is:

$$\left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(50 + \frac{5}{0.05}\right) \times \frac{1-(1.05)^{-21}}{0.05} - \frac{5 \times 21}{0.05}(1.05)^{-21}$$

$$= 150 \times 12.8212 - 753.7790 = 1,169.3939$$

The present value is:

$$PV_0 = 1,169.3939 \times 1.05^{0.5} = \mathbf{1,198.27}$$

The BA-II Plus can be used to solve the problem as follows:

21 [*N*] 5 [*I*/*Y*] 50 [+] 5 [÷] 0.05 [=] [*PMT*]

5 [×] 21 [÷] 0.05 [=] [+/-] [*FV*] [CPT] [*PV*]

Result is –1,169.3939.

[×] 1.05 [y^x] 0.5 [=]

Result is –1,198.27. Answer is **1,198.27**.

Solution 8.03

D Section 8.01, Increasing Annuity

The annuity can be broken down into a level annuity-immediate and an increasing annuity-immediate:

$$AV_{21} = 45s_{\overline{21}|} + 5(Is)_{\overline{21}|} = 45 \times \frac{1.05^{21} - 1}{0.05} + 5 \times \frac{\ddot{s}_{\overline{21}|} - 21}{0.05}$$

$$= 45 \times 35.7193 + 5 \times \frac{37.5052 - 21}{0.05} = 1,607.3663 + 5 \times 330.1043$$

$$= \mathbf{3,257.89}$$

Alternatively, we can use the PIn method with the following parameters:

$P_1 = 50$ $I = 5$ $n = 21$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(50 + \frac{5}{0.05}\right) \times \frac{1 - (1.05)^{-21}}{0.05} - \frac{5 \times 21}{0.05}(1.05)^{-21}$$

$$= 150 \times 12.8212 - 753.7790 = 1,169.39$$

The accumulated value is:

$$AV_{21} = 1,169.39 \times 1.05^{21} = \mathbf{3,257.89}$$

The BA-II Plus can be used to solve the problem as follows:

21 [*N*] 5 [*I*/*Y*] 50 [+] 5 [÷] 0.05 [=] [*PMT*]

5 [×] 21 [÷] 0.05 [=] [+/-] [*FV*] [CPT] [*PV*]

Result is –1,169.39.

[×] 1.05 [y^x] 21 [=]

Result is –3,257.89. Answer is **3,257.89**.

Solution 8.04

E Section 8.01, Increasing Annuities

The increasing annuity is payable monthly, and the annual rate of payment in the first year is 60:

$$60 \times (Is)^{(12)}_{\overline{10}|} = 60 \times \frac{\ddot{s}_{\overline{10}|} - 10}{i^{(12)}}$$

The annual effective interest and discount rates are:

$$i = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.06168 \qquad d = \frac{0.06128}{1.06128} = 0.05809$$

The accumulated value of the annuity-immediate is:

$$60\times\frac{\ddot{s}_{\overline{10}|}-10}{i^{(12)}}=60\times\frac{\frac{(1.06168)^{10}-1}{0.05809}-10}{0.06}=60\times\frac{14.1045-10}{0.06}=60\times 68.4085$$
$$=\mathbf{4{,}104.51}$$

The BA-II Plus can be used to answer this question as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
1.005 [y^x] 12 [–] 1 [=] [×] 100 [=] [*I*/*Y*]
10 [*N*] 1 [*PMT*] [CPT] [*FV*]
Result is –14.1045.
[+/-] [–] 10 [=] [÷] 0.06 [×] 60 [=]
Answer is **4,104.51**.

Alternatively, we can use the PIn method to obtain the present value of an increasing annuity-immediate payable annually:

$$60\times(Ia)_{\overline{10}|}=60\times\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{i}$$

Then, we can multiply by the ratio $i\,/\,i^{(12)}$ to obtain the present value of the increasing annuity-immediate payable monthly:

$$\frac{i}{i^{(m)}}\times 60\times\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{i}=60\times\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{i^{(m)}}=60\times(I\ddot{a})_{\overline{10}|}$$

Finally, we can accumulate the annuity for 10 years to obtain the accumulated value.

Using the PIn method, we have:

$P_1=60 \qquad I=60 \qquad n=10 \qquad i=1.005^{12}-1=0.06168$

We use the BA II Plus in the END mode

(0.06 [÷] 12 + 1) [y^x] 12 [–] 1 [=] [STO] 1
10 [*N*] [RCL] 1 [×] 100 [=] [*I*/*Y*] 60 [+] 60 [÷] [RCL] 1 [=] [*PMT*]
60 [×] 10 [÷] [RCL] 1 [=] [+/-] [*FV*]
[CPT] [*PV*]
Result is *PV* = –2,194.6043. (We have $60\times(Ia)_{\overline{10}|}=2{,}194.6043$)
[×] [RCL] 1 [÷] 0.06 [=]
Result is –2,255.9732. (We have $60\times(I\ddot{a})_{\overline{10}|}=2{,}255.9732$)
[×] (1 [+] [RCL] 1) [y^x] 10 [=]
Result is –4,104.5103. Answer is **4,104.51**.

Solution 8.05

D Section 8.01, Increasing Annuities

The increasing annuity is payable monthly, and the annual rate of payment in the first year is 24:

$$24\times(I\ddot{a})^{(12)}_{\overline{20}|}=24\times\frac{\ddot{a}_{\overline{20}|}-20v^{20}}{d^{(12)}}$$

The annual effective interest and discount rates are:

$$i=\left(1+\frac{0.12}{12}\right)^{12}-1=0.1268 \qquad d=\frac{0.1268}{1.1268}=0.1125$$

The nominal discount rate compounded monthly is:

$$d^{(12)}=\frac{0.01}{1.01}\times 12=0.1188$$

The present value of the annuity-due is:

$$24\times\frac{\ddot{a}_{\overline{20}|}-20v^{20}}{d^{(12)}} \qquad =24\times\frac{\frac{1-1.1268^{-20}}{0.1126}-\frac{20}{1.1268^{20}}}{0.1188}=24\times\frac{8.0692-1.8361}{0.1188}$$

$$=24\times 52.4617=\mathbf{1{,}259.08}$$

The BA-II Plus can be used to answer this question as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

1.01 [y^x] 12 [–] 1 [=] [×] 100 [=] [*I*/*Y*]

20 [*N*] 1 [+/-] [*PMT*] 20 [*FV*] [CPT] [*PV*]

Result is 6.2331.

[÷] 0.01 [×] 1.01 [÷] 12 [=]

Result is 52.4617

[×] 24 [=]

Answer is **1,259.08**.

Solution 8.06

A Section 8.01, Increasing Annuity & Reinvested Funds

There are two funds, one earning 6% and one earning 3%. The amount of each level deposit is denoted by X. The deposits into the funds are described below:

Time	6%	3%
0	X	
1	X	$X\times 0.06$
2	X	$2X\times 0.06$
3	X	$3X\times 0.06$
14	X	$4X\times 0.06$
15		$5X\times 0.06$

Since the interest from the 6% fund is paid into the 3% fund, the value of the 6% fund at the end of 5 years is $5X$. The accumulated value of the 3% fund can be found using an increasing annuity immediate:

$$5X + (Is)_{\overline{5|}0.03} \times 0.06X = 800$$

$$5X + \frac{\ddot{s}_{\overline{5|}0.03} - 15}{0.03} \times 0.06X = 800$$

$$5X + \frac{\frac{1.03^5 - 1}{0.03/1.03} - 5}{0.03} \times 0.06X = 800$$

$$5X + \frac{5.4684 - 5}{0.03} \times 0.06X = 800$$

$$5X + 15.6137 \times 0.06X = 800$$

$$X = \mathbf{134.75}$$

We can use the BA II Plus to obtain the value of $(Is)_{\overline{5|}0.03}$:

5 [*N*] 3 [*I*/*Y*] 1 [*PMT*] [CPT] [*FV*] [×] 1.03 [+/-]
[–] 5 [=] [÷] 0.03 [=]
Result is 15.6137.

Solution 8.07

C Section 8.01, Increasing Annuity & Reinvested Funds

There are two funds, one earning 7% and one earning 4%. The amount of each level deposit is denoted by X. The deposits into the funds are described below:

Time	7%	4%
0	X	
1	X	$X \times 0.07$
2	X	$2X \times 0.07$
…	…	…
14	X	$14X \times 0.07$
15		$15X \times 0.07$

Since the interest from the 7% fund is paid into the 4% fund, the value of the 7% fund at the end of 15 years is $15X$. The accumulated value of the 4% fund can be found using an increasing annuity immediate:

$$15X + (Is)_{\overline{15|}0.04} \times 0.07X = 25{,}000$$

$$15X + \frac{\ddot{s}_{\overline{15|}0.04} - 15}{0.04} \times 0.07X = 25{,}000$$

$$15X + \frac{\frac{1.04^{15} - 1}{0.04/1.04} - 15}{0.04} \times 0.07X = 25{,}000$$

$$15X + \frac{20.8245 - 15}{0.04} \times 0.07X = 25{,}000$$

$$15X + 145.6133 \times 0.07X = 25{,}000$$

$$X = \mathbf{992.34}$$

We can use the BA II Plus to obtain the value of $(Is)_{\overline{15|}0.04}$:

15 [*N*] 4 [*I*/*Y*] 1 [*PMT*] [CPT] [*FV*] [×] 1.04 [+/-]

[–] 15 [=] [÷] 0.04 [=]
Result is 145.6133.

Solution 8.08

B Section 8.01, Increasing Coupons

The 6-month effective interest rate is:

$$\frac{0.08}{2} = 0.04$$

Since the last coupon payment was 30, then next coupon payment is $X+30$. The present value of the bond is 1,300. The time 0 equation of value can be used to solve for X:

$$\frac{30+X}{1.04}+\frac{30+2X}{1.04^2}+\cdots+\frac{30+18X}{1.04^{18}}+\frac{1,000}{1.03^{18}} = 1,300$$

$$30\times a_{\overline{18}|0.04} + X(Ia)_{\overline{18}|0.04} + \frac{1,000}{1.04^{18}} = 1,300$$

$$30\times\frac{1-1.04^{-18}}{0.04} + X\times\frac{\ddot{a}_{\overline{18}|} - 18(1.04)^{-18}}{0.04} + 493.6281 = 1,300$$

$$30\times 12.6593 + X\times\frac{13.1657-18(1.04)^{-18}}{0.04} + 493.6281 = 1,300$$

$$379.7789 + X\times 107.0091 + 493.6281 = 1,300$$

$$X = \mathbf{3.99}$$

We can use the BA II Plus to answer this question:

18 [*N*] 4 [*I*/*Y*] 1 [*PMT*] [CPT] [*PV*]
[×] 1.04 [=] [+/-] [–] 18 [÷] 1.04 [y^x] 18 [=] [÷] 0.04 [=] [STO] 1
30 [*PMT*] 1,000 [*FV*] [CPT] [*PV*] + 1,300 [=]
[÷] [RCL] 1 [=]
Answer is **3.99**.

Solution 8.09

C Section 8.02, Increasing Annuity

Using the PIn method, we have:

$P_1 = 200$ $I = 4$ $n = 75$

The present value is:

$$PV_0 = \left(P_1+\frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(200+\frac{4}{0.08}\right)\times\frac{1-(1.08)^{-75}}{0.08} - \frac{4\times 75}{0.08}(1.08)^{-75}$$
$$= 250\times 12.4611 - 11.6748$$
$$= \mathbf{3,103.60}$$

Using the BA II Plus, we have:

75 [*N*] 8 [*I*/*Y*] 200 [+] 4 [÷] 0.08 [=] [*PMT*]
4 [×] 75 [÷] 0.08 [=] [+/-] [*FV*]
[CPT] [*PV*]
Result is –3,103.60. Answer is **3,103.60**.

Solution 8.10

B Section 8.02, Varying Annuities

The time-0 equation of value shows that the amount of the loan is equal to the present value of the payments that repay the loan:

$$15,000 = 100(Ia)_{\overline{10|}} + Xv^{10}a_{\overline{15|}}$$

Let's use the PIn method to find the present value of the increasing annuity:

$$P_1 = 100 \qquad I = 100 \qquad n = 10$$

$$\begin{aligned}100(Ia)_{\overline{10|}} &= \left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n \\ &= \left(100 + \frac{100}{0.03}\right) \times \frac{1-(1.03)^{-10}}{0.03} - \frac{100 \times 10}{0.03}(1.03)^{-10} \\ &= 3,433.3333 \times 8.5302 - 24,803.1305 = 4,483.8992\end{aligned}$$

We can now use the equation of value to solve for X:

$$15,000 = 100(Ia)_{\overline{10|}} + Xv^{10}a_{\overline{15|}}$$

$$15,000 = 4,483.8992 + Xv^{10}a_{\overline{15|}}$$

$$15,000 = 4,483.8992 + X \times 1.03^{-10} \times \frac{1-1.03^{-15}}{0.03}$$

$$15,000 = 4,483.8992 + X \times 1.03^{-10} \times 11.9379$$

$$10,516.1008 = 8.8829X$$

$X =$ **1,183.85**

We can use the BA II Plus to answer this question:

10 [*N*] 3 [*I/Y*]
100 [+] 100 [÷] 0.03 [=] [*PMT*]
100 [×] 10 [÷] 0.03 [=] [+/-] [*FV*]
[CPT] [*PV*]
(Result is –4,483.8992.)
[+] 15,000 [=] [×] 1.03 [y^x] 10 [=] [*PV*]
15 [*N*] 0 [*FV*] [CPT] [*PMT*]
Result is –1,183.85. Answer is **1,183.85**.

Solution 8.11

D Section 8.02, Increasing Annuity

We use one month as the unit of time. The monthly effective interest rate is:

$$\left(1 + \frac{0.07}{4}\right)^{1/3} - 1 = 0.005800$$

Using the PIn method, we have:

$$P_1 = 3 \qquad I = 3 \qquad n = 72 \qquad i = 0.005800$$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

$$= \left(3 + \frac{3}{0.005800}\right) \times \frac{1-(1.005800)^{-72}}{0.005800} - \frac{3 \times 72}{0.005800}(1.005800)^{-72}$$

$$= 520.2741 \times 58.7213 - 24{,}559.9366 = \mathbf{5{,}991.24}$$

Using the BA II Plus, we have:

0.07 [÷] 4 [+] 1 [=] [y^x] 3 [1/*x*] [=] [–] 1 [=] [STO] 1

72 [*N*] [RCL] 1 [×] 100 [=] [*I*/*Y*] 3 [+] 3 [÷] [RCL] 1 [=] [*PMT*]

3 [×] 72 [÷] [RCL] 1 [=] [+/-] [*FV*]

[CPT] [*PV*]

Result is *PV* = –5,991.24. Answer is **5,991.24**.

Solution 8.12

C Section 8.02, Increasing Annuities

The value of Joel's perpetuity-due can be used to find the interest rate:

$$3{,}150 = 150 + \frac{150}{i}$$

$$i = 0.05$$

Using the PIn method, we have the following values for Ellen's annuity:

$P_1 = P$ $I = 20$ $n = 15$

The formula for the present value can be used to find the value of the first payment:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

$$3{,}150 = \left(P + \frac{20}{0.05}\right) \times \frac{1-(1.05)^{-15}}{0.05} - \frac{20 \times 15}{0.05}(1.05)^{-15}$$

$$3{,}150 = (P + 400) \times 10.3797 - 2{,}886.1026$$

$$P = \mathbf{181.53}$$

Using the BA II Plus, we have:

15 [*N*] 5 [*I*/*Y*] 3,150 [+/-] [*PV*] 20 [×] 15 [÷] 0.05 [=] [+/-] [*FV*]

[CPT] [*PMT*]

(Result is 581.5319)

[–] 20 [÷] 0.05 [=]

Answer is **181.53**.

Solution 8.13

E Section 8.02, Increasing Annuity

Let's use the PIn method find the value of the annuity at the end of 4 years. We have:

$P_1 = 500$ $I = 250$ $n = 20$

The present value is:

$$PV_4 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(500 + \frac{250}{0.06}\right) \times \frac{1-(1.06)^{-20}}{0.06} - \frac{250 \times 20}{0.06}(1.06)^{-20}$$

$$= 4,666.6667 \times 11.4699 - 25,983.7272 = 27,542.5718$$

To find the present value at time zero, we discount for 4 years:

$$\frac{27,542.5718}{1.06^4} = \mathbf{21,816.30}$$

Using the BA II Plus, we have:

20 [N] 6 [I/Y] 500 [+] 250 [÷] 0.06 [=] [PMT]

250 [×] 20 [÷] 0.06 [=] [+/-] [FV]

[CPT] [PV]

[÷] 1.06 [y^x] 4

Result is –21,816.30. Answer is **21,816.30**.

Solution 8.14

B Section 8.02, Increasing Annuities

We use one month as the unit of time. The monthly effective interest rate is:

$$\left(1 + \frac{0.07}{4}\right)^{1/3} - 1 = 0.005800$$

Using the PIn method, we have:

$P_1 = 5$ $I = 5$ $n = 72$ $i = 0.005800$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

$$= \left(5 + \frac{5}{0.005800}\right) \times \frac{1-(1.005800)^{-72}}{0.005800} - \frac{5 \times 72}{0.005800}(1.005800)^{-72}$$

$$= 867.1234 \times 58.7213 - 40,933.2277 = \mathbf{9,985.40}$$

Using the BA II Plus, we have:

0.07 [÷] 4 [+] 1 [=] [y^x] 3 [$1/x$] [=] [–] 1 [=] [STO] 1

72 [N] [RCL] 1 [×] 100 [=] [I/Y] 5 [+] 5 [÷] [RCL] 1 [=] [PMT]

5 [×] 72 [÷] [RCL] 1 [=] [+/-] [FV]

[CPT] [PV]

Result is PV = –9,985.40. Answer is 9,985.40.

Alternatively, we can use the formula for an increasing annuity-immediate with one month as the unit of time:

$$5 \times (Ia)_{\overline{72}|0.005800} = 5 \times \frac{\ddot{a}_{\overline{72}|} - 72v^{72}}{i} = 5 \times \frac{\ddot{a}_{\overline{72}|} - 72v^{72}}{0.005800}$$

$$= 5 \times \frac{59.0619 - 47.4795}{0.005800} = 5 \times 1,997.0803 = \mathbf{9,985.40}$$

Solution 8.15

E Section 8.02, Continuous Increasing Annuity

The present value of the annuity is:

$$50\times(I\bar{a})_{\overline{n}|} = 50\times\frac{\bar{a}_{\overline{n}|}-nv^n}{r} = 50\times\frac{\dfrac{1-v^n}{d}-nv^n}{r}$$

$$= 50\times\frac{\dfrac{1-e^{-10\times0.07}}{1-e^{-0.07}}-10e^{-10\times0.07}}{0.07} = 50\times\frac{7.4463-4.9659}{0.07}$$

$$= 50\times35.4347 = \mathbf{1{,}771.74}$$

The BA-II Plus can be used to answer this question as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

0.07 [2nd] [e^x] [–] 1 [=] [×] 100 [=] [*I*/*Y*]

10 [*N*] 1 [+/-] [*PMT*] 10 [*FV*] [CPT] [*PV*]

Result is 2.4804.

[÷] 0.07 [=]

Result is 35.4347

[×] 50 [=]

Answer is **1,771.74**.

Alternatively, we can use the PIn method to obtain the present value of an increasing annuity-immediate:

$$50\times(Ia)_{\overline{10}|} = 50\times\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{i}$$

Then, we can multiply by the ratio i/r to obtain the present value of the continuously payable annuity:

$$\frac{i}{r}\times50\times\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{i} = 50\times\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{r} = 50\times(I\bar{a})_{\overline{10}|}$$

Using the PIn method, we have:

$$P_1 = 50 \qquad I = 50 \qquad n = 10 \qquad i = e^{0.07}-1 = 0.0725$$

We use the BA II Plus in the END mode

0.07 [2nd] [e^x] [–] 1 [=] [STO] 1

10 [*N*] [RCL] 1 [×] 100 [=] [*I*/*Y*] 50 [+] 50 [÷] [RCL] 1 [=] [*PMT*]

50 [×] 10 [÷] [RCL] 1 [=] [+/-] [*FV*]

[CPT] [*PV*]

Result is *PV* = –1,710.4478.

[×] [RCL] 1 [÷] 0.07 [=]

Result is –1,771.74. Answer is **1,771.74**.

Solution 8.16

C Section 8.02, Increasing Annuities

We use one month as the unit of time. The quarterly effective interest rate is:

$$(1-0.08)^{-0.25}-1 = 0.02106$$

Using the PIn method, we have:

$$P_1 = 100 \qquad I = 25 \qquad n = 28 \qquad i = 0.02041$$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

$$= \left(100 + \frac{25}{0.02106}\right) \times \frac{1-(1.02106)^{-28}}{0.02106} - \frac{25 \times 28}{0.02106}(1.02106)^{-28}$$

$$= 1,286.8487 \times 20.9908 - 18,538.2258 = 8,473.7144$$

The future value at the end of 7 years is:

$$AV_7 = \frac{8,473.7144}{0.92^7} = \mathbf{15,190.04}$$

Using the BA II Plus, we have:

1 [–] 0.08 [=] [1/x] [y^x] 0.25 [=] [–] 1 [=] [STO] 1

28 [N] [RCL] 1 [×] 100 [=] [I/Y] 100 [+] 25 [÷] [RCL] 1 [=] [PMT]

25 [×] 28 [÷] [RCL] 1 [=] [+/-] [FV]

[CPT] [PV]

Result is PV = –8,4737144.

0 [PMT] [CPT] [FV]

Answer is **15,190.04**.

Alternatively, we can use one quarter as the unit of time in conjunction with a level annuity and an increasing annuity:

$$AV_7 = 75 \times s_{\overline{28}|0.02106} + 25(Is)_{\overline{28}|0.02106} = 75 \times \frac{1.02106^{28}-1}{0.02106} + 25 \times \frac{\ddot{s}_{\overline{28}|} - 28}{0.02106}$$

$$= 75 \times 37.6282 + 25 \times \frac{38.4208 - 28}{0.02106} = 2,822.1154 + 25 \times 494.7171$$

$$= \mathbf{15,190.04}$$

Solution 8.17

E Section 8.02, Increasing Annuities

Let's define the time unit to be 6 months. The 6-month effective interest rate and discount rate are:

$$i = \left(1 + \frac{0.06}{12}\right)^6 - 1 = 0.03038 \qquad d = \frac{0.03038}{1.03038} = 0.02948$$

The increasing annuity is payable monthly, and there are 6 months in a 6-month time unit, so the rate of payment during the first 6 months is 60 per 6-month time unit:

$$60 \times (Ia)^{(6)}_{\overline{10}|0.03038} = 60 \times \frac{\ddot{a}_{\overline{10}|0.03038} - 10v^{10}}{i^{(6)}}$$

$$= 60 \times \frac{\dfrac{1-1.005^{-60}}{0.02948} - 10(1.005)^{-60}}{0.005 \times 6}$$

$$= 60 \times \frac{8.7724 - 10(1.005)^{-60}}{0.03}$$

$$= 60 \times 45.2899 = \mathbf{2,717.40}$$

The BA-II Plus can be used to answer this question as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
0.06 [÷] 12 [+] 1 [=] [y^x] 6 [–] 1 [=] [×] 100 [=] [*I*/*Y*]
10 [*N*] 1 [+/-] [*PMT*] 10 [*FV*] [CPT] [*PV*]
Result is 1.3587.
[÷] 0.005 [÷] 6 [=]
Result is 45.2899
[×] 60 [=]
Answer is **2,717.40**.

Alternatively, we can use the PIn method with the time unit set to six months:

$$P_1 = 60 \qquad I = 60 \qquad n = 10 \qquad i = 0.03038$$

If the payments were made at the end of each six-month interval, then the present value would be:

$$\left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

$$= \left(600 + \frac{60}{0.03038}\right) \times \frac{1-(1.03038)^{-10}}{0.03038} - \frac{60 \times 10}{0.03038}(1.03038)^{-10}$$

$$= 2,035.1455 \times 8.5138 - 14,643.1793 = 2,683.6264$$

Since the payments are made monthly, we multiply by an adjustment factor that accumulates each of the payments to the end of each 6-month interval.

The unit of time continues to be six months:

$$PV_0 = s_{\overline{1}|}^{(6)} \times 2,683.6264 = \frac{1.005^6 - 1}{0.005 \times 6} \times 2,683.6264 = \frac{0.03038}{0.03} \times 2,683.6264$$

$$= \mathbf{2,717.40}$$

Using the BA II Plus, we have:

1 [+] 0.06 [÷] 12 [=] [y^x] 6 [–] 1 [=] [–] [STO] 1
10 [*N*] [RCL] 1 [×] 100 [=] [*I*/*Y*] 60 [+] 60 [÷] [RCL] 1 [=] [*PMT*]
60 [×] 10 [÷] [RCL] 1 [=] [+/-] [*FV*]
[CPT] [*PV*] Result is –2,683.6264.
[×] [RCL] 1 [÷] 0.03 [=]
Result is *PV* = –2,717.3962. Answer is **2,717.40**.

Solution 8.18

A Section 8.03, Increasing Perpetuity

The value of the level perpetuity-due can be used to find *d*:

$$26 = \frac{1}{d}$$

$$d = 0.03846$$

The value of the increasing perpetuity-due can be used to find *X*:

$$4,732 = \frac{X}{d^2}$$

$$4,732 = \frac{X}{0.03846^2}$$

$$X = 7$$

The 8th payment is:

$$8X = 8 \times 7 = \mathbf{56}$$

Solution 8.19

D Section 8.03, Increasing Perpetuity

We can use the formula for the present value of an increasing perpetuity-immediate to solve for i:

$$PV_0 = \frac{P_1}{i} + \frac{I}{i^2}$$

$$21,315 = \frac{300}{i} + \frac{25}{i^2}$$

$$21,315i^2 = 300i + 25$$

$$21,315i^2 - 300i - 25 = 0$$

The quadratic formula gives us two solutions for i, and we use the positive one:

$$i = \frac{300 \pm \sqrt{300^2 + 4 \times 21,315 \times 25}}{2 \times 21,315}$$

$$i = \mathbf{0.0420}$$

Solution 8.20

E Section 8.03, Increasing Perpetuity

We can use the formula for the present value of an increasing perpetuity-due to solve for i:

$$PV_0 = P_0 + \frac{P_1}{i} + \frac{I}{i^2}$$

$$21,590 = 275 + \frac{300}{i} + \frac{25}{i^2}$$

$$21,315 = \frac{300}{i} + \frac{25}{i^2}$$

$$21,315i^2 = 300i + 25$$

$$21,315i^2 - 300i - 25 = 0$$

The quadratic formula gives us two solutions for i, and we use the positive one:

$$i = \frac{300 \pm \sqrt{300^2 + 4 \times 21,315 \times 25}}{2 \times 21,315}$$

$$i = \mathbf{0.0420}$$

Solution 8.21

B Chapter 8.03, Perpetuities

The equation of value at the end of 1 year can be used to solve for $a_{\overline{n}|}$:

$$144.09 \times 1.075 = (Ia)_{\overline{n}|} + \frac{n}{i} v^n$$

$$144.09 \times 1.075 \times 0.075 = \ddot{a}_{\overline{n}|} - nv^n + nv^n$$

$$144.09 \times 0.075 = \frac{\ddot{a}_{\overline{n}|}}{1.075}$$

$$a_{\overline{n}|} = 10.8068$$

We can now solve for n:

$$a_{\overline{n}|} = 10.8068$$

$$\frac{1 - 1.075^{-n}}{0.075} = 10.8068$$

$$1.075^{-n} = 0.1895$$

$$-n \times \ln(1.075) = \ln(0.1895)$$

$$n = \mathbf{23.00}$$

Alternatively, the BA-II Plus can be used to solve for n:

144.09 [×] 0.075 [=] [+/-] [*PV*]

7.5 [*I*/*Y*] 1 [*PMT*] [CPT] [*N*]

Result is **23.00**.

Solution 8.22

D Section 8.04, Decreasing Annuity

The present value is:

$$5 \times (Da)_{\overline{10}|} = 5 \times \frac{10 - a_{\overline{10}|}}{0.05} = 5 \times \frac{10 - 7.7217}{0.05} = 5 \times 45.5653 = \mathbf{227.83}$$

Alternatively, using the PIn method, we have:

$P_1 = 50$ $I = -5$ $n = 20$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right) a_{\overline{n}|} - \frac{In}{i} v^n = \left(50 + \frac{-5}{0.05}\right) \times \frac{1 - (1.05)^{-10}}{0.05} - \frac{-5 \times 10}{0.05} (1.05)^{-10}$$

$$= -50 \times 7.7217 + 613.9233$$

$$= \mathbf{227.83}$$

Using the BA II Plus, we have:

10 [*N*] 5 [*I*/*Y*] 50 [–] 5 [÷] 0.05 [=] [*PMT*]

5 [×] 10 [÷] 0.05 [=] [*FV*]

[CPT] [*PV*]

Result is –227.83. Answer is **227.83**.

Solution 8.23

E	Section 8.04, Decreasing Annuity

The accumulated value is:

$$5 \times (Ds)_{\overline{10|}} = 5 \times \frac{10(1.05)^{10} - s_{\overline{10|}}}{0.05} = 5 \times \frac{10(1.05)^{10} - 12.5779}{0.05} = 5 \times 74.2211$$
$$= \mathbf{371.11}$$

Alternatively, we can use the PIn method to find the present value and then accumulate the present value for 10 years to obtain the accumulated value:

$$P_1 = 50 \qquad I = -5 \qquad n = 10$$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n = \left(50 + \frac{-5}{0.05}\right) \times \frac{1-(1.05)^{-10}}{0.05} - \frac{-5 \times 10}{0.05}(1.05)^{-10}$$
$$= -50 \times 7.7217 + 613.9233$$
$$= 227.8265$$

The accumulated value is:

$$AV_{10} = 227.8265 \times 1.05^{10} = \mathbf{371.11}$$

Using the BA II Plus, we have:

10 [*N*]   5 [*I*/*Y*]   50 [–] 5 [÷] 0.05 [=] [*PMT*]

5 [×] 10 [÷] 0.05 [=]   [*FV*]

[CPT] [*PV*]

Result is –227.83.

0 [*PMT*]   [CPT] [*FV*]

Answer is **371.11**.

Solution 8.24

C	Section 8.04, Decreasing Annuities

We first find the present value of the otherwise equivalent annuity-immediate and then accumulate for one year to obtain the value of the annuity-due.

Using the PIn method, we have:

$$P_1 = 500 \qquad I = -25 \qquad n = 20$$

The present value of the annuity-immediate is:

$$\left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n = \left(500 - \frac{25}{0.06}\right) \times \frac{1-(1.06)^{-20}}{0.06} - \frac{-25 \times 20}{0.06}(1.06)^{-20}$$
$$= 83.3333 \times 11.4699 + 2{,}598.3727$$
$$= 3{,}554.1995$$

The annuity is actually an annuity-due, however, so we multiply by 1.06 to obtain the value of an annuity that makes each payment one year earlier than the annuity-immediate:

$$1.06 \times 3{,}554.1995 = \mathbf{3{,}767.45}$$

Using the BA II Plus, we have:

20 [*N*]   6 [*I*/*Y*]   500 [–] 25 [÷] 0.06 [=] [*PMT*]

25 [×] 20 [÷] 0.06 [=] [*FV*]

[CPT] [*PV*]

(Result is –3,554.1995)

[×] 1.06 [=]

Result is –3,767.45. Answer is **3,767.45**.

Solution 8.25

A Section 8.04, Decreasing Annuity

The value of the level perpetuity-due can be used to find the annual effective interest rate:

$$21 = 1 + \frac{1}{i}$$

$$i = 0.05$$

The value of the decreasing annuity-immediate can be used to find X. Let's use the PIn method:

$P_1 = X \qquad I = -1 \qquad n = 10$

We have:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n$$

$$68.73 = \left(X + \frac{-1}{0.05}\right) \times \frac{1-(1.05)^{-10}}{0.05} - \frac{-1 \times 10}{0.05}(1.05)^{-10}$$

$$68.73 = (X - 20) \times 7.7217 + 122.7827$$

$$X = 13$$

The first payment is 13, and after 7 decreases of 1, the 8th payment is:

$$13 - 7 \times 1 = \mathbf{6}$$

Solution 8.26

E Section 8.04, Decreasing Annuities

At time 20, the present value of the remaining payments can be found using the PIn method:

Using the PIn method, we have:

$P_1 = 29 \qquad I = -1 \qquad n = 29$

The present value is:

$$PV_{20} = \left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n = \left(29 - \frac{1}{0.09}\right) \times \frac{1-(1.09)^{-29}}{0.09} - \frac{-1 \times 29}{0.09}(1.09)^{-29}$$

$$= 17.8889 \times 10.1983 + 26.4720 = 208.9080$$

Discounting the present value found above for 20 years and adding the present value of the first 20 payments gives us the present value of the annuity-immediate:

$$208.9080v^{20} + 30a_{\overline{20|}} = \frac{208.9080}{1.09^{20}} + 30 \times \frac{1-1.09^{-20}}{0.09} = \mathbf{311.13}$$

We can use the BA II Plus to answer this question:

29 [*N*] 9 [*I/Y*] 29 [–] 1 [÷] 0.09 [=] [*PMT*]

1 [×] 29 [÷] 0.09 [=] [*FV*]

[CPT] [*PV*]

(Result is –208.9080)

[+/-] [*FV*]

20 [*N*]   30 [*PMT*]

[CPT] [*PV*]

Result is –311.13.   Answer is **311.13**.

Solution 8.27

E   Section 8.04, Decreasing Annuities

The decreasing annuity is payable quarterly, and the annual rate of payment in the final year is 20:

$$20 \times (Da)_{\overline{n}|}^{(m)} = 20 \times \frac{n - a_{\overline{n}|}}{i^{(m)}} = 20 \times \frac{5 - a_{\overline{5}|}}{i^{(4)}}$$

The annual effective interest rate is:

$$i = \left(1 + \frac{0.08}{4}\right)^4 - 1 = 0.08243$$

The present value of the annuity-immediate is:

$$20 \times \frac{5 - a_{\overline{5}|}}{i^{(4)}} = 20 \times \frac{5 - \frac{1 - 1.02^{-20}}{0.08243}}{0.08} = 20 \times \frac{5 - 3.9672}{0.08} = 20 \times 12.9094$$

$$= \mathbf{258.19}$$

The BA-II Plus can be used to answer this question as follows:

1.02 [y^x] 4 [–] 1 [=] [×] 100 [=] [*I/Y*]

5 [*N*] 1 [*PMT*] [CPT] [*PV*]

Result is –3.9672.

[+] 5 [=] [÷] 0.08 [×] 20 [=]

Answer is 258.19.

Alternatively, we can use the PIn method. The annual effective interest rate is:

$$\left(1 + \frac{0.08}{4}\right)^4 - 1 = 0.082432$$

The first payments are made at a rate of 100 per year, and the payments subsequently decrease by 20 per year:

$P_1 = 100$ $I = -20$ $n = 5$ $i = 0.082432$

If the annuity paid at the end of each year, the present value would be:

$$\left(P_1 + \frac{I}{i}\right) a_{\overline{n}|} - \frac{In}{i} v^n$$

$$= \left(100 + \frac{-20}{0.082432}\right) \times \frac{1 - (1.082432)^{-5}}{0.082432} - \frac{(-20) \times 5}{0.082432} (1.082432)^{-5}$$

$$= -142.6238 \times 3.9672 + 816.3942 = 250.5706$$

Since the annuity pays at the end of each quarter, we multiply by an adjustment factor:

$$PV_0 = s_{\overline{1}|}^{(4)} \times 250.5706 = \frac{i}{i^{(4)}} \times 250.5706 = \frac{0.082432}{0.08} \times 250.5706 = \mathbf{258.19}$$

Using the BA II Plus, we have:

0.08 [÷] 4 [+] 1 [=] [y^x] 4 [–] 1 [=] [STO] 1

5 [*N*]   [RCL] 1 [×] 100 [=] [*I/Y*]   100 [–] 20 [÷] [RCL] 1 [=] [*PMT*]

20 [×] 5 [÷] [RCL] 1 [=] [*FV*]

[CPT] [*PV*] Result is –250.5706.
[÷] 0.08 [×] [RCL] 1 [=]
Result is -258.1885. Answer is **258.19**.

Solution 8.28

D Section 8.04, Decreasing Annuities

We use one month as the unit of time. The monthly effective interest rate is:

$$\frac{0.03}{12} = 0.0025$$

Using the PIn method, we have:

$$P_1 = 600 \qquad I = -10 \qquad n = 60 \qquad i = 0.0025$$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$
$$= \left(600 + \frac{-10}{0.0025}\right)\times\frac{1-(1.0025)^{-60}}{0.0025} - \frac{-10\times 60}{0.0025}(1.0025)^{-60}$$
$$= -3,400\times 55.6524 + 206,608.5854 = \mathbf{17,390.57}$$

Using the BA II Plus, we have:
0.03 [÷] 12 [=] [STO] 1
60 [*N*] [RCL] 1 [×] 100 [=] [*I/Y*] 600 [–] 10 [÷] [RCL] 1 [=] [*PMT*]
10 [×] 60 [÷] [RCL] 1 [=] [*FV*]
[CPT] [*PV*]
Result is *PV* = –17,390.57. Answer is 17,390.57.

Alternatively, we can answer this question using the formula for a decreasing annuity-immediate by setting the time unit to one month:

$$10\times(Da)_{\overline{60}|} = 10\times\frac{n-a_{\overline{n}|}}{i} = 10\times\frac{60-a_{\overline{60}|0.0025}}{0.0025} = 10\times\frac{60-55.6524}{0.0025}$$
$$= 10\times 1,739.05693 = \mathbf{17,390.67}$$

Solution 8.29

E Section 8.04, Continuous Decreasing Annuity

The present value of an annuity that pays continuously at a rate of n during the first year, (n – 1) during the second year, and so on until paying at a rate of 1 in the n^{th} year is:

$$(D\bar{a})_{\overline{n}|} = \frac{n-a_{\overline{n}|}}{r}$$

The present value of the annuity described in this question is:

$$50\times(D\bar{a})_{\overline{n}|} = 50\times\frac{n-a_{\overline{n}|}}{r} = 50\times\frac{10-a_{\overline{10}|}}{0.07} = 50\times\frac{10-\frac{1-e^{-10\times 0.07}}{e^{0.07}-1}}{0.07}$$
$$= 50\times\frac{10-6.9429}{0.07} = 50\times 43.6733 = \mathbf{2,183.67}$$

The BA-II Plus can be used to answer this question as follows:
0.07 [2nd] [e^x] [–] 1 [=] [×] 100 [=] [*I/Y*]
10 [*N*] 1 [*PMT*] [CPT] [*PV*]

[+] 10 [=] [÷] 0.07 [=]

Result is 43.6733.

[×] 50 [=]

Answer is **2,183.67**.

Alternatively, we can use the PIn method to obtain the present value of a decreasing annuity-immediate:

$$50 \times (Da)_{\overline{10|}} = 50 \times \frac{10 - a_{\overline{10|}}}{i}$$

Then, we can multiply by the ratio i/r to obtain the present value of the continuously payable annuity:

$$\frac{i}{r} \times 50 \times \frac{10 - a_{\overline{10|}}}{i} = 50 \times \frac{10 - a_{\overline{10|}}}{r} = 50 \times (D\bar{a})_{\overline{n|}}$$

Using the PIn method, we have:

$P_1 = 500 \qquad I = -50 \qquad n = 10 \qquad i = e^{0.07} - 1 = 0.0725$

We use the BA II Plus in the END mode

0.07 [2nd] [e^x] [–] 1 [=] [STO] 1

10 [*N*] [RCL] 1 [×] 100 [=] [*I*/*Y*] 500 [–] 50 [÷] [RCL] 1 [=] [*PMT*]

50 [×] 10 [÷] [RCL] 1 [=] [*FV*]

[CPT] [*PV*]

Result is *PV* = –2,108.1293.

[×] [RCL] 1 [÷] 0.07 [=]

Result is –2,183.67. Answer is **2,183.67**.

Solution 8.30

C Section 8.04, Decreasing Annuities

At time 20, the present value of the remaining payments can be found using the PIn method:

Using the PIn method, we have:

$P_1 = 28 \qquad I = -2 \qquad n = 14$

The present value is:

$$PV_{20} = \left(P_1 + \frac{I}{i}\right) a_{\overline{n|}} - \frac{In}{i} v^n = \left(28 - \frac{2}{0.09}\right) \times \frac{1 - (1.09)^{-14}}{0.09} - \frac{-2 \times 14}{0.09} (1.09)^{-14}$$
$$= 5.7778 \times 7.7862 + 93.0989 = 138.0855$$

Discounting the present value found above for 20 years and adding the present value of the first 20 payments gives us the present value of the annuity-immediate:

$$138.08550 v^{20} + 30 a_{\overline{20|}} = \frac{138.0855}{1.09^{20}} + 30 \times \frac{1 - 1.09^{-20}}{0.09} = \mathbf{298.50}$$

We can use the BA II Plus to answer this question:

14 [*N*] 9 [*I*/*Y*] 28 [–] 2 [÷] 0.09 [=] [*PMT*]

2 [×] 14 [÷] 0.09 [=] [*FV*]

[CPT] [*PV*]

(Result is –138.0855)

[+/-] [*FV*]

20 [*N*] 30 [*PMT*]
[CPT] [*PV*]
Result is –298.50. Answer is **298.50**.

Solution 8.31

C Section 8.04, Decreasing Annuities

The accumulated value is:

$$\left[1,000(0.07)+100\right]1.05^9+\left[900(0.07)+100\right]1.05^8+\cdots+\left[100(0.07)+100\right]$$
$$=170\times1.05^9+163\times1.09^8+\cdots+107$$

Using the PIn method, we have:

$P_1=170$ $I=-7$ $n=10$

The present value is:

$$PV_0=\left(P_1+\frac{I}{i}\right)a_{\overline{n}|}-\frac{In}{i}v^n=\left(170+\frac{-7}{0.05}\right)\times\frac{1-(1.05)^{-10}}{0.05}-\frac{-7\times10}{0.05}(1.05)^{-10}$$
$$=30\times7.7217+859.4786=1,091.1306$$

The accumulated value at the end of 10 years is:

$$1,091.1306\times1.05^{10}=\mathbf{1,777.34}$$

Using the BA II Plus, we have:

10 [*N*] 5 [*I/Y*] 170 [–] 7 [÷] 0.05 [=] [*PMT*]
7 [×] 10 [÷] 0.05 [=] [*FV*]
[CPT] [*PV*]
Result is –1,091.1306.
0 [*PMT*]
[CPT] [*FV*] Answer = **1,777.34**

Chapter 9: Continuously Payable Varying Payments

Solution 9.01

B Section 9.02, Continuous Annuities, Decreasing Continuously

In the formula for a continuously increasing annuity, we replace the continuously compounded interest rate with $1/n$:

$$(\bar{I}\bar{s})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{r} = \frac{\frac{e^{nr} - 1}{r} - n}{r} = \frac{\frac{e^{n(1/n)} - 1}{(1/n)} - n}{(1/n)} = \frac{n(e-1) - n}{1/n} = \frac{ne - n - n}{1/n} = n(ne - 2n)$$

$$= \boldsymbol{n^2(e-2)}$$

Solution 9.02

E Section 9.01, Continuously Varying Payment Stream

The equation of value at time 12 is:

$$AV_b = \int_a^b Pmt_t e^{\int_t^b r_s ds} dt$$

$$22,344 = \int_0^{12} k(7+t) e^{\int_t^{12} (7+s)^{-1} ds} dt$$

We begin by evaluating the integral in the exponent:

$$\int_t^{12} (7+s)^{-1} ds = \ln(7+s)\Big|_t^{12} = \ln(19) - \ln(7+t) = \ln\left(\frac{19}{7+t}\right)$$

The equation of value can now be used to solve for k:

$$22,344 = \int_0^{12} k(7+t)\left(\frac{19}{7+t}\right) dt$$

$$22,344 = \int_0^{12} 19k\,dt$$

$$22,344 = k(228 - 0)$$

$$k = \mathbf{98}$$

Solution 9.03

D Section 9.02, Continuous Annuities, Increasing Continuously

The present value of the perpetuity is:

$$\lim_{n\to\infty}\left[3 \times (\bar{I}\bar{a})_{\overline{n}|}\right] = \lim_{n\to\infty}\left[3 \times \frac{\bar{a}_{\overline{n}|} - nv^n}{r}\right] = \lim_{n\to\infty}\left[3 \times \frac{\frac{1-v^n}{r} - nv^n}{r}\right] = \frac{3}{r^2}$$

$$= \frac{3}{\left[\ln(1.05)\right]^2} = \mathbf{1{,}260.25}$$

Solution 9.04

D Section 9.02, Continuous Annuities, Increasing Continuously

The accumulated value of the annuity is:

$$3\times(\bar{I}\bar{s})_{\overline{n}|} = 3\times\frac{\bar{s}_{\overline{n}|}-n}{r} = 3\times\frac{\dfrac{1.05^{10}-1}{\ln(1.05)}-10}{\ln(1.05)} = 3\times\frac{12.8898-10}{\ln(1.05)}$$

$$= 3\times 59.2288 = \mathbf{177.69}$$

Alternatively, we can find the accumulated value using integration:

$$AV_{10} = \int_0^{10} 3t(1.05)^{10-t}\,dt$$

Let's use integration by parts:

$$\int 3t(1.05)^{10-t}\,dt = \int u\,dv$$

$$\text{where:}\quad u = 3t \qquad dv = (1.05)^{10-t}\,dt$$

We have:

$$u = 3t \qquad v = -\frac{(1.05)^{10-t}}{\ln(1.05)}$$

$$du = 3dt \qquad dv = (1.05)^{10-t}\,dt$$

We can now find the integral:

$$\int 3t(1.05)^{10-t}\,dt = \int u\,dv = uv - \int v\,du$$

$$= -3t\,\frac{(1.05)^{10-t}}{\ln(1.05)} + \int 3\,\frac{(1.05)^{10-t}}{\ln(1.05)}dt = -3t\,\frac{(1.05)^{10-t}}{\ln(1.05)} - \frac{3(1.05)^{10-t}}{[\ln(1.05)]^2}$$

The accumulated value is:

$$AV_{10} = \int_0^{10} 3t(1.05)^{10-t}\,dt = \left[-3t\,\frac{(1.05)^{10-t}}{\ln(1.05)} - \frac{3(1.05)^{10-t}}{[\ln(1.05)]^2}\right]_0^{10}$$

$$= \left[-\frac{30}{\ln(1.05)} - \frac{3}{[\ln(1.05)]^2}\right] - \left[0 - \frac{3(1.05)^{10}}{[\ln(1.05)]^2}\right]$$

$$= -1{,}875.1280 + 2{,}052.8144 = \mathbf{177.69}$$

Solution 9.05

A Section 9.02, Continuous Annuities, Decreasing Continuously

The present value of an annuity that pays continuously at a rate of (30 – 3*t*) at time *t* for 10 years is equal to the value of a level annuity minus the value of an increasing annuity:

$$PV_0 = 30\bar{a}_{\overline{10}|} - 3(\bar{I}\bar{a})_{\overline{10}|} = 30\times\frac{1-v^{10}}{r} - 3\times\frac{\bar{a}_{\overline{10}|}-10v^{10}}{r}$$

$$= 30\times\frac{1-1.05^{-10}}{\ln(1.05)} - 3\times\frac{\bar{a}_{\overline{10}|}-10\times 1.05^{-10}}{\ln(1.05)}$$

$$= 30\times 7.9132 - 3\times\frac{7.9132-10\times 1.05^{-10}}{\ln(1.05)} = 237.3963 - 3\times 36.3613$$

$$= \mathbf{128.3122}$$

Alternatively, we can find the present value of the annuity using integration:

$$PV_0 = \int_0^{10} (30-3t)(1.05)^{-t}\,dt$$

Let's use integration by parts:

$$\int (30-3t)(1.05)^{-t}\,dt = \int u\,dv$$

$$\text{where:}\quad u = 30-3t \qquad dv = (1.05)^{-t}\,dt$$

We have:

$$u = 30-3t \qquad v = -\frac{(1.05)^{-t}}{\ln(1.05)}$$

$$du = -3dt \qquad dv = (1.05)^{-t}\,dt$$

We can now find the integral:

$$\int (30-3t)(1.05)^{-t}\,dt = \int u\,dv = uv - \int v\,du$$

$$= -(30-3t)\frac{(1.05)^{-t}}{\ln(1.05)} - \int 3\frac{(1.05)^{-t}}{\ln(1.05)}dt = (3t-30)\frac{(1.05)^{-t}}{\ln(1.05)} + \frac{3(1.05)^{-t}}{[\ln(1.05)]^2}$$

The present value is:

$$PV_0 = \int_0^{10} (30-3t)(1.05)^{-t}\,dt = \left[(3t-30)\frac{(1.05)^{-t}}{\ln(1.05)} + \frac{3(1.05)^{-t}}{[\ln(1.05)]^2}\right]_0^{10}$$

$$= \left[0 + \frac{3(1.05)^{-10}}{[\ln(1.05)]^2}\right] - \left[\frac{-30}{\ln(1.05)} + \frac{3}{[\ln(1.05)]^2}\right]$$

$$= 773.6842 - 645.3719 = \mathbf{128.3122}$$

Solution 9.06

C Section 9.01, Continuously Varying Payment Stream

The present value is:

$$PV_0 = \int_0^8 (30t+20)e^{-\int_0^t (0.03s+0.02)ds}dt = \int_0^8 (30t+20)e^{-(0.015t^2+0.02t)}dt$$

Let's use substitution:

$$u = 0.015t^2 + 0.02t \qquad du = (0.03t+0.02)dt$$

We have:

$$\int (30t+20)e^{-(0.015t^2+0.02t)}dt = 1{,}000\int (0.03t+0.02)e^{-(0.015t^2+0.02t)}dt$$

$$= 1{,}000\int e^{-u}du = -1{,}000e^{-u}$$

$$= -1{,}000e^{-(0.015t^2+0.02t)}$$

The present value is:

$$PV_0 = \int_0^8 (30t+20)e^{-\int_0^t (0.03s+0.02)ds}dt = \left[-1{,}000e^{-(0.015t^2+0.02t)}\right]_0^8$$

$$= \left[-1{,}000e^{-(0.015\times 64+0.02\times 8)}\right] - \left[-1{,}000e^0\right]$$

$$= -326.280 + 1{,}000 = \mathbf{673.7202}$$

Solution 9.07

A Section 9.01, Continuously Varying Payment Stream

The present value at time 5:

$$PV_5 = \int_5^{10} (3t^2 + 20t) e^{-\int_5^t (0.006s^2 + 0.04s)ds} dt$$

$$= \int_5^{10} (3t^2 + 20t) e^{-\left[0.002s^3 + 0.02s^2\right]_5^t} dt$$

$$= \int_5^{10} (3t^2 + 20t) e^{-(0.002t^3 0.02t^2 - 0.002\times 125 - 0.02\times 25)} dt$$

$$= \int_5^{10} (3t^2 + 20t) e^{(0.75 - 0.002t^3 - 0.02t^2)} dt$$

Let's use substitution:

$$u = 0.75 - 0.002t^3 - 0.02t^2 \quad du = -(0.006t^2 + 0.04t)dt$$

We have:

$$\int (3t^2 + 20t) e^{(0.75 - 0.002t^3 - 0.02t^2)} dt$$

$$= 500 \int (0.006t^2 + 0.04t) e^{(0.75 - 0.002t^3 - 0.02t^2)} dt$$

$$= -500 \int e^u du = -500 e^u$$

$$= -500 e^{(0.75 - 0.002t^3 - 0.02t^2)}$$

The present value at time 5 is:

$$PV_5 = -500 e^{(0.75 - 0.002t^3 - 0.02t^2)} \Big|_5^{10} = -500\left[e^{-3.25} - e^0\right] = -500\left[1 - e^{-3.25}\right]$$

$$= 480.6129$$

The present value at time 0 is:

$$PV_0 = 480.6129 e^{-5\times 0.05} = \mathbf{374.30}$$

Solution 9.08

C Section 9.01, Continuously Payable Annuity

Let's break the perpetuity into two parts.

The first part consists of the payments made in the first 20 years. The present value of the first part is:

$$\bar{a}_{\overline{20|}} = \frac{1 - v^n}{r} = \frac{1 - 1.08^{-20}}{\ln(1.08)} = 10.2058$$

The second part consists of the payments made after 20 years. The formula for the present value of a continuously payable annuity is:

$$PV_a = \int_a^b Pmt_t e^{-\int_a^t r_s ds} dt$$

The present value at time 20 of the payments made after 20 years is:

$$PV_{20} = \int_{20}^{\infty} 1.05^{t-20} e^{-\int_{20}^{t} \ln(1.08)ds} dt = \int_{20}^{\infty} 1.05^{t-20} e^{-\ln(1.08)(t-20)} dt$$

$$= \int_{20}^{\infty} 1.05^{t-20} \times 1.08^{20-t}\, dt$$

$$= \left(\frac{1.08}{1.05}\right)^{20} \int_{20}^{\infty} \left(\frac{1.05}{1.08}\right)^{t} dt = \left(\frac{1.08}{1.05}\right)^{20} \left(\frac{1.05}{1.08}\right)^{t} \frac{1}{\ln\left(\frac{1.05}{1.08}\right)}\Bigg|_{20}^{\infty}$$

$$= 0 - \left(\frac{1.08}{1.05}\right)^{20} \left(\frac{1.05}{1.08}\right)^{20} \frac{1}{\ln\left(\frac{1.05}{1.08}\right)} = \frac{-1}{\ln\left(\frac{1.05}{1.08}\right)} = 35.4977$$

The present value at time 0 of the payments made after 20 years is the present value at time 20, discounted for 20 years:

$$PV_0 = \left(\frac{1}{1.08}\right)^{20} PV_{20} = \left(\frac{1}{1.08}\right)^{20} 35.4977 = 7.6160$$

The present value of the perpetuity is equal to the sum of the present values of the two parts of the perpetuity:

$$10.2058 + 7.6160 = \mathbf{17.8218}$$

Solution 9.09

B Section 9.02, Continuously Payable Annuity

The payments begin at an annual rate of 0 at time 0, increase to an annual rate of 50 at time 5, and then decrease back to 0 at time 10.

The present of the payments over the first 5 years is:

$$10(\bar{I}\bar{a})_{\overline{5|}} = 10 \times \frac{\bar{a}_{\overline{5|}} - 5v^5}{r} = 10 \times \frac{\frac{1-v^5}{\ln(1.05)} - 5v^5}{r} = 10 \times \frac{4.4368 - \frac{5}{1.05^5}}{\ln(1.05)} = 10 \times 10.6415$$

$$= 106.4154$$

After 5 years, the rate of payment is 50 and it steadily decreases thereafter. This can be valued at time 5 with an annuity that pays at a constant rate of 50 minus an increasing annuity that increases at a rate of 10 per year. To find the value at time 0, we discount for 5 years. The present value of the payments over the final 5 years is therefore:

$$\left(50\bar{a}_{\overline{5|}} - 10(\bar{I}\bar{a})_{\overline{5|}}\right)v^5 = \left(50 \times 4.4368 - 10 \times 10.6415\right)1.05^{-5} = \frac{115.4262}{1.05^5} = 90.4395$$

The present value of the entire payment stream is:

$$PV_0 = 10(\bar{I}\bar{a})_{\overline{5|}} + \left(50\bar{a}_{\overline{5|}} - 10(\bar{I}\bar{a})_{\overline{5|}}\right)v^5 = 106.4154 + 90.4395 = \mathbf{196.8549}$$

Solution 9.10

A Section 9.01, Continuously Varying Payment Stream

The present value is:

$$PV_0 = \int_0^4 3t^3 e^{-\int_0^t 0.001s^3 ds} dt$$

The integral in the exponent is:

$$-\int_0^t 0.001s^3 ds = -0.001\left.\frac{s^4}{4}\right|_0^t = -0.001\frac{t^4}{4}$$

Let’s use substitution:

$$u = -0.001\frac{t^4}{4} \qquad du = -0.001t^3 dt$$

We have:

$$PV_0 = \int_0^4 3t^3 e^{-\int_0^t 0.001s^3 ds} dt = \int_0^4 3t^3 e^{-0.001\frac{t^4}{4}} dt = \int_{t=0}^{t=4} -\frac{3}{0.001} e^u du = -\left.\frac{3}{0.001} e^u\right|_{t=0}^{t=4}$$

$$= -\left.\frac{3}{0.001} e^{-0.001\frac{t^4}{4}}\right|_{t=0}^{t=4} = -\frac{3}{0.001} e^{-0.064} + \frac{3}{0.001} = 3,000\left(1 - e^{-0.064}\right) = \mathbf{185.99}$$

Chapter 10: Geometric Progression Annuities

Solution 10.01

D Section 10.02, Geometric Annuity Formulas

We begin by finding j:

$$j = \frac{1+i}{1+g} - 1 = \frac{1.04}{1.12} - 1 = -0.07143$$

The present value of the annuity-due is:

$$PV_0 = 100\ddot{a}_{\overline{n}|j} = 100 \times \frac{1-(1-0.07143)^{-10}}{\frac{-0.07143}{1-0.07143}} = 100 \times 14.2766 = 1,427.6597$$

The accumulated value of the annuity is:

$$AV_{10} = (1+i)^{10} \times PV_0 = (1.04)^{10} \times 1,427.6597 = \mathbf{2,113.29}$$

The BA-II Plus can be used to answer this question as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

1.04 [÷] 1.12 [–] 1 [=] [×] 100 [=] [*I/Y*]

10 [*N*] 100 [*PMT*] [CPT] [*PV*]

Result is –1,427.6597.

0 [*PMT*] 4 [*I/Y*] [CPT] [*FV*]

Answer is **2,113.29**.

Alternatively, we can find the accumulated value using the formula for the sum of a geometric series:

$$\begin{aligned} AV_{10} &= 100(1.04)^{10} + 100(1.04)^{9}(1.12) + \cdots + 100(1.04)(1.12)^{9} \\ &= \frac{\text{First term} - \text{Term that would come next}}{1 - \text{Ratio}} \\ &= \frac{100(1.04)^{10} - 100(1.12)^{10}}{1 - \frac{1.12}{1.04}} = 100 \times 21.1329 = \mathbf{2,113.29} \end{aligned}$$

Solution 10.02

A Section 10.02, Geometric Annuity Formulas

Let's use one quarter as our unit of time and use i to represent the effective interest rate for one unit of time:

$$i = 1.10^{0.25} - 1 = 0.02411$$

We can now determine j, using the growth rate and the interest rate applicable to one quarter:

$$j = \frac{1+i}{1+g} - 1 = \frac{1.02411}{1.02} - 1 = 0.004033$$

The present value of the annuity-immediate is:

$$PV_0 = \frac{100\ddot{a}_{\overline{n|}j}}{1+i} = \frac{100}{1.02411} \times \frac{1-(1.004033)^{-40}}{\frac{0.004033}{1.004033}} = \frac{100}{1.02411} \times 37.0205$$

$$= \mathbf{3,614.88}$$

The BA-II Plus can be used to answer this question as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

1.10 [y^x] 0.25 [=] [STO] 1

[÷] 1.02 [–] 1 [=] [×] 100 [=] [*I*/*Y*]

40 [*N*] 100 [*PMT*] [CPT] [*PV*]

Result is –3,702.0531.

[÷] [RCL] 1 [=]

Result is –3,614.8849. Answer is 3,614.88.

Alternatively, we can find the present value using the formula for the sum of a geometric series. Below we define *v* to be the discount factor for one quarter:

$$PV_0 = 100v + 100v^2(1.02) + \cdots + 100v^{40}(1.02)^{39}$$

$$= \frac{\text{First term} - \text{Term that would come next}}{1 - \text{Ratio}}$$

$$= \frac{100v - 100v^{41}(1.02)^{40}}{1-1.02v} = \frac{\frac{100}{1.02411} - \frac{100}{1.02411}\left(\frac{1.02}{1.02411}\right)^{40}}{1-\frac{1.02}{1.02411}}$$

$$= \frac{100}{1.02411} \times \frac{1-\left(\frac{1.02}{1.02411}\right)^{40}}{1-\frac{1.02}{1.02411}} = \frac{100}{1.02411} \times 37.0205$$

$$= \mathbf{3,614.88}$$

Solution 10.03

D Section 10.02, Geometric Annuity Formulas

Let's use one quarter as our unit of time and use *i* to represent the effective interest rate for one unit of time:

$$i = 1.10^{0.25} - 1 = 0.02411$$

We can now determine *j*, using the growth rate and the interest rate applicable to one quarter:

$$j = \frac{1+i}{1+g} - 1 = \frac{1.02411}{1-0.02} - 1 = 0.04501$$

The present value of the annuity-immediate is:

$$PV_0 = \frac{100\ddot{a}_{\overline{n|}j}}{1+i} = \frac{100}{1.02411} \times \frac{1-(1.04501)^{-40}}{\frac{0.04501}{1.04501}} = \frac{100}{1.02411} \times 19.2261$$

$$= 1,877.3383$$

The accumulated value of the annuity is:

$$AV_{10} = (1.10)^{10} \times PV_0 = (1.10)^{10} \times 1,877.3383 = \mathbf{4,869.33}$$

The BA-II Plus can be used to answer this question as follows:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
1.10 [y^x] 0.25 [=] [STO] 1
[÷] 0.98 [–] 1 [=] [×] 100 [=] [I/Y]
40 [N] 100 [PMT] [CPT] [PV]
Result is –1,922.6079.
[÷] [RCL] 1 [=]
Result is –1,877.3383.
[×] [RCL] 1 [y^x] 40 [=]
Result is –4,869.3322. Answer is **4,869.33**.

Alternatively, we can find the accumulated value using the formula for the sum of a geometric series:

$$\begin{aligned} AV_{10} &= 100(1.02411)^{39} + 100(1.02411)^{38}(0.98) + \cdots + 100(0.98)^{39} \\ &= \frac{\text{First term} - \text{Term that would come next}}{1 - \text{Ratio}} \\ &= \frac{100(1.02411)^{39} - \dfrac{100}{1.02411}(0.98)^{40}}{1 - \dfrac{0.98}{1.02411}} = 100 \times 48.6933 \\ &= \mathbf{4{,}869.33} \end{aligned}$$

Solution 10.04

A Section 10.02, Geometric Annuity Formulas

The annual effective interest rate is:

$$i = e^r - 1 = e^{0.07} - 1 = 0.07251$$

We find j:

$$j = \frac{1+i}{1+g} - 1 = \frac{1.07251}{1.03} - 1 = 0.04127$$

The present value of the annuity-due is:

$$PV_0 = \lim_{n\to\infty} 36\ddot{a}_{\overline{n}|j} = 36 \times \lim_{n\to\infty} \frac{1-(1.04127)^{-n}}{\frac{0.04127}{1.04127}} = 36 \times \frac{1}{\frac{0.04127}{1.04127}} = \mathbf{908.30}$$

Alternatively, we can find the present value using the formula for the sum of a geometric series:

$$\begin{aligned} PV_0 &= 36 + 36\left(\frac{1.03}{e^{0.07}}\right) + 36\left(\frac{1.03}{e^{0.07}}\right)^2 + \cdots \\ &= \frac{\text{First term} - \text{Term that would come next}}{1 - \text{Ratio}} \\ &= \frac{36 - 0}{1 - \dfrac{1.03}{e^{0.07}}} = \mathbf{908.30} \end{aligned}$$

Solution 10.05

B Section 10.02, Geometric Varying Annuities

The 20 payments are described below:

Time	Payment
0	100
1	100×1.10
2	100×1.10^2
$\cdots$	$\cdots$
9	100×1.10^9
10	$100 \times 1.10^9 \times 0.90$
11	$100 \times 1.10^9 \times 0.90^2$
$\cdots$	$\cdots$
19	$100 \times 1.10^9 \times 0.90^{10}$

Let's begin by finding the present value of the first 10 payments:

$$100 + 100(1.10)v + \cdots + 100(1.10)^9 v^9 = 100 \times \frac{1 - \left(\frac{1.10}{1.04}\right)^{10}}{1 - \frac{1.10}{1.04}} = 1,303.8817$$

The present value of the second set of 10 payments is:

$$100(1.10)^9 v^{10}(0.90) + 100(1.10)^9 v^{11}(0.90)^2 + \cdots + 100(1.10)^9 v^{19}(0.90)^{10}$$

$$= 100(1.10)^9 v^{10}(0.90) \times \frac{1 - \left(\frac{0.90}{1.04}\right)^{10}}{1 - \frac{0.90}{1.04}} = 814.1323$$

The present value of the payments is the sum of the present value of the first 10 payments and the second 10 payments:

$1,303.8817 + 814.1323 =$ **2,118.01**

Solution 10.06

E Section 10.02, Geometric Progression Annuities

The 15 payments are described below:

Time	Payment
1	$4,000 \times 1.08$
2	$4,000 \times 1.08^2$
3	$4,000 \times 1.08^3$
$\cdots$	$\cdots$
15	$4,000 \times 1.08^{15}$

The present value of the payments is:

$$4,000 \times 1.08v + 4,000 \times 1.08^2 v^2 + \cdots + 4,000 \times 1.08^{15} v^{15}$$

$$= 4,000 \times \frac{\frac{1.08}{1.06} - \left(\frac{1.08}{1.06}\right)^{16}}{1 - \frac{1.08}{1.06}} = \mathbf{69,905.23}$$

Solution 10.07

C Section 10.02, Geometric Varying Annuities

At the end of 10 years, the value of the perpetuity's remaining payments is equal to the value of the 20-year annuity-immediate:

$$\frac{200}{0.04} = X\left(v + 1.04v^2 + 1.04^2 v^3 + \cdots + 1.04^{19} v^{20}\right)$$

$$\frac{200}{0.04} = X\left(\frac{1}{1.04} + 1.04\left(\frac{1}{1.04}\right)^2 + 1.04^2\left(\frac{1}{1.04}\right)^3 + \cdots + 1.04^{19}\left(\frac{1}{1.04}\right)^{20}\right)$$

$$\frac{200}{0.04} = X\left(\frac{20}{1.04}\right)$$

$$X = \frac{200}{0.04} \times \frac{1.04}{20}$$

$$X = \mathbf{260}$$

Alternatively, we can write the equation of value at the end of 10 years as follows:

$$\frac{200}{0.04} = X\frac{\ddot{a}_{\overline{20}|j}}{1.04} \text{ where } j = \frac{1+i}{1+g} - 1 = \frac{1.04}{1.04} - 1 = 0$$

$$\frac{200}{0.04} = X\left(\frac{20}{1.04}\right)$$

$$X = \frac{200}{0.04} \times \frac{1.04}{20}$$

$$X = \mathbf{260}$$

Solution 10.08

B Section 10.02, Geometric Progression Annuities

The present value of the perpetuity-immediate is 486.26:

$$20v + 20v^2 + 20v^3 + 20v^4 + 20\left[v^5 + (1+K)v^6 + (1+K)^2 v^7 + \cdots\right] = 486.26$$

$$20a_{\overline{4}|} + 20 \times \frac{v^5 - 0}{1 - \frac{1+K}{1.08}} = 486.26$$

$$20 \times \frac{1 - 1.08^{-4}}{0.08} + 20 \times \frac{v^5}{1 - \frac{1+K}{1.08}} = 486.26$$

$$20 \times 3.3121 + 20 \times \frac{v^5}{1 - \frac{1+K}{1.08}} = 486.26$$

$$\frac{1}{1 - \frac{1+K}{1.08}} = 30.8572$$

$$K = 0.0450$$

The question referred to K% instead of K, so we multiply the value of K found above by 100:

$$0.0450 \times 100 = \mathbf{4.50}$$

Solution 10.09

B Section 10.02, Geometric Progression Annuities

There are 48 monthly payments. We use one month as the unit of time:

$$i = \frac{0.06}{12} = 0.005$$

$$v = \frac{1}{1.005}$$

The 48 payments are described below:

Time	Payment
1	$10,000$
2	$10,000 \times 0.99$
3	$10,000 \times 0.99^2$
$\cdots$	$\cdots$
38	$10,000 \times 0.99^{37}$
39	$1,000 \times 0.99^{38}$
$\cdots$	$\cdots$
48	$1,000 \times 0.99^{47}$

The outstanding balance after the 38th payment is the present value of the payments after the 38th payment:

$$PV_{38} = 10,000\times 0.99^{38}v + 10,000\times 0.99^{39}v^2 + \cdots + 10,000\times 0.99^{47}v^{10}$$

$$= 10,000\times 0.99^{38}v\times\frac{1-0.99^{10}v^{10}}{1-0.99v} = 10,000\times 0.99^{38}v\times\frac{1-\left(\frac{0.99}{1.005}\right)^{10}}{1-\frac{0.99}{1.005}}$$

$$= \mathbf{63,531.26}$$

Solution 10.10

A Section 10.02, Geometric Progression Annuities

The 18 payments are described below:

Time	Payment
1	$3,000$
2	$3,000\times 1.04$
3	$3,000\times 1.04^2$
$\cdots$	$\cdots$
8	$3,000\times 1.04^7$
9	$3,000\times 1.04^7\times 0.96$
10	$3,000\times 1.04^7\times 0.96^2$
$\cdots$	$\cdots$
18	$3,000\times 1.04^7\times 0.96^{10}$

The present value of the payments is:

$$PV_0 = 3,000v + 3,000v^2\times 1.04 + \cdots + 3,000v^8\times 1.04^7$$
$$+3,000\times 1.04^7v^8\left[0.96v + 0.96^2v^2 + \cdots + 0.96^{10}v^{10}\right]$$
$$= 3,000\times\frac{v-1.04^8v^9}{1-1.04v} + 3,000\times 1.04^7v^8\times\frac{0.96v-0.96^{11}v^{11}}{1-0.96v}$$
$$= 3,000\times 7.0671 + 2,476.8956\times 6.0361$$
$$= 21,201.4270 + 14,950.8019$$
$$= \mathbf{36,152.23}$$

Solution 10.11

C Section 10.02, Geometric Progression Annuities

The monthly effective interest rate is:

$$\frac{0.09}{12} = 0.0075$$

The annual effective interest rate is:

$$1.0075^{12} - 1 = 0.09381$$

The present value of the first year's payments is:

$$600a_{\overline{12}|0.0075} = 600 \times \frac{1 - 1.0075^{-12}}{0.0075} = 6,860.9476$$

The payments in the 14 subsequent years increase by 3% per year:

$$X = 6,860.9476\left[1 + 1.03v + \cdots + (1.03v)^{14}\right] = 6,860.9476 \times \frac{1 - \left(\frac{1.03}{1.09381}\right)^{15}}{1 - \frac{1.03}{1.09381}}$$

$$= 6,860.9476 \times 10.1839 = \mathbf{69,870.94}$$

Solution 10.12

E Section 10.02, Geometric Progression Annuity

The equation of value after 25 years can be used to solve for i:

$$2,000\left[(1+i)^{24} + (1+i)^{23}(1+g) + \cdots + (1+g)^{24}\right]$$

$$= 7,395.44\left[1 + \frac{1+g}{1+i} + \cdots + \left(\frac{1+g}{1+i}\right)^{24}\right]$$

$$2,000(1+i)^{24}\left[1 + \frac{1+g}{1+i} + \cdots + \left(\frac{1+g}{1+i}\right)^{24}\right] = 7,395.44\left[1 + \frac{1+g}{1+i} + \cdots + \left(\frac{1+g}{1+i}\right)^{24}\right]$$

$$2,000(1+i)^{24} = 7,395.44$$

$$(1+i) = \left(\frac{7,395.44}{2,000}\right)^{1/24}$$

$$i = \mathbf{0.05600}$$

Alternatively, we can use the formula for the present value of an annuity-immediate that increases geometrically:

$$PV_0 = \frac{\ddot{a}_{\overline{25}|j}}{1+i} \qquad \text{where: } 1 + j = \frac{1+i}{1+g}$$

The following time-25 equation of value can be used to solve for i:

$$2,000 \times \frac{\ddot{a}_{\overline{25}|j}}{1+i} \times (1+i)^{25} = 7,395.44 \times \ddot{a}_{\overline{25}|j}$$

$$2,000(1+i)^{24} = 7,395.44$$

$$(1+i) = \left(\frac{7,395.44}{2,000}\right)^{1/24}$$

$$i = \mathbf{0.05600}$$

Solution 10.13

D Section 10.02, Geometric Progression Annuity

We begin by noting that:

$$1.04^2 = 1.0816$$

The present value of the annuity is:

$$\begin{aligned}
&(v+v^2)+1.0816(v^3+v^4)+1.0816^2(v^5+v^6)+1.0816^3(v^7+v^8)+1.0816^4(v^9+v^{10}) \\
&= (v+v^2)\left[1+1.0816v^2+1.0816^2v^4+1.0816^3v^6+1.0816^4v^8\right] \\
&= a_{\overline{2}|}\left[1+1.0816v^2+1.0816^2v^4+1.0816^3v^6+1.0816^4v^8\right] \\
&= a_{\overline{2}|}\left[1+\frac{1.0816}{1.04^2}+\frac{1.0816^2}{1.04^4}+\frac{1.0816^3}{1.04^6}+\frac{1.0816^4}{1.04^8}\right] \\
&= a_{\overline{2}|}\left[1+1+1+1+1\right] \\
&= \mathbf{5}\boldsymbol{a_{\overline{2}|}}
\end{aligned}$$

Solution 10.14

A Section 10.02, Geometric Progression Annuity

We begin by noting that:

$$1.04^2 = 1.0816$$

The present value of the annuity is:

$$\begin{aligned}
&a_{\overline{4}|}\left[1+1.0816v^4+1.0816^2v^8+1.0816^3v^{12}+1.0816^4v^{16}\right] \\
&= a_{\overline{4}|}\left[1+\frac{1.0816}{1.04^4}+\frac{1.0816^2}{1.04^8}+\frac{1.0816^3}{1.04^{12}}+\frac{1.0816^4}{1.04^{16}}\right] \\
&= a_{\overline{4}|}\left[1+\frac{1.04^2}{1.04^4}+\frac{1.04^4}{1.04^8}+\frac{1.04^6}{1.04^{12}}+\frac{1.04^8}{1.04^{16}}\right] \\
&= a_{\overline{4}|}\left[1+v^2+v^4+v^6+v^8\right] \\
&= a_{\overline{4}|}\left[\frac{1-v^{10}}{1-v^2}\right]
\end{aligned}$$

If we divide both the numerator and the denominator of the fraction in the final expression above by the effective interest rate, then we obtain the ratio of two immediate annuities:

$$\begin{aligned}
&a_{\overline{4}|}\left[\frac{1-v^{10}}{1-v^2}\right] = a_{\overline{4}|}\left[\frac{\frac{1-v^{10}}{0.04}}{\frac{1-v^2}{0.04}}\right] a_{\overline{4}|}\left[\frac{a_{\overline{10}|}}{a_{\overline{2}|}}\right] = \frac{a_{\overline{4}|}}{a_{\overline{2}|}}\times a_{\overline{10}|} = \frac{\frac{1-v^4}{0.04}}{\frac{1-v^2}{0.04}}\times a_{\overline{10}|} = \frac{1-v^4}{1-v^2}\times a_{\overline{10}|} \\
&= \mathbf{(1+v^2)}\boldsymbol{a_{\overline{10}|}}
\end{aligned}$$

Solution 10.15

C Section 10.02, Geometric Progression Annuity

The equation of value after 25 years can be used to solve for i:

$$2,000\left[(1+i)^{24}+(1+i)^{23}(1.02)+\cdots+(1.02)^{24}\right]$$

$$=7,395.44\left[1+\frac{1.02}{1+i}+\cdots+\left(\frac{1.02}{1+i}\right)^{24}\right]$$

$$2,000(1+i)^{24}\left[1+\frac{1.02}{1+i}+\cdots+\left(\frac{1.02}{1+i}\right)^{24}\right]=7,395.44\left[1+\frac{1.02}{1+i}+\cdots+\left(\frac{1.02}{1+i}\right)^{24}\right]$$

$$2,000(1+i)^{24}=7,395.44$$

$$(1+i)=\left(\frac{7,395.44}{2,000}\right)^{1/24}$$

$$i=0.05600$$

The account balance after the final deposit is equal to the value of the future deposits at the time of retirement:

$$7,395.44\left[1+\frac{1.02}{1+i}+\cdots+\left(\frac{1.02}{1+i}\right)^{24}\right]=7,395.44\times\frac{1-\left(\frac{1.02}{1.05600}\right)^{25}}{1-\frac{1.02}{1.05600}}$$

$$=7,395.44\times 17.0089=\mathbf{125,788.13}$$

Solution 10.16

E Section 10.03, Dividend Discount Model

We can use the dividend discount growth model to find the return on the stock:

$$PV_0=\frac{Div_1}{i-g}$$

$$175=\frac{7}{i-0.02}$$

$$175i-3.5=7$$

$$i=\mathbf{0.06}$$

Solution 10.17

E Section 10.03, Dividend Discount Model

The equation of value at time 0 can be used to solve for i. To find the present value of the payments received by the company, we can use the dividend discount model:

$$1,000+\frac{130}{i}=\frac{100}{i-0.03}$$

$$1,000i(i-0.03)+130(i-0.03)=100i$$

$$1,000i^2-30i+130i-3.9-100i=0$$

$$1,000i^2=3.9$$

$$i=\sqrt{\frac{3.9}{1,000}}$$

$$i=\mathbf{0.0624}$$

Solution 10.18

B Section 10.03, Dividend Discount Model

Although this does not refer to the perpetuity as payments from a share of common stock, we can treat the payments as dividends and use the dividend discount model to find the present value of the payments.

The rate of growth of the quarterly payments is:

$$\frac{1,704.25}{1,700} - 1 = 0.0025$$

The present value of the payments can be used to solve for the quarterly effective interest rate:

$$316,965.95 = 1,700 + \frac{1,704.25}{\frac{i^{(4)}}{4} - 0.0025}$$

$$\frac{i^{(4)}}{4} = 0.007905$$

The annual effective interest rate is:

$$\left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = 1.007905^4 - 1 = \mathbf{0.0320}$$

Solution 10.19

C Section 10.03, Dividend Discount Model

At the end of 20 years, the present value of the remaining payments is found using the dividend discount model:

$$PV_{20} = \frac{4(1.01)}{0.05 - 0.01} = 101$$

The present value of the payments at time 0 is:

$$\begin{aligned} PV_0 &= 2a_{\overline{10}|} + 4a_{\overline{10}|} \times v^{10} + 101v^{20} \\ &= \left(2 + 4v^{10}\right)a_{\overline{10}|} + 101v^{20} \\ &= \left(2 + \frac{4}{1.05^{10}}\right)\frac{1 - 1.05^{-10}}{0.05} + \frac{101}{1.05^{20}} \\ &= (4.4557)7.7217 + 38.0658 \\ &= \mathbf{72.47} \end{aligned}$$

Solution 10.20

B Section 10.03, Dividend Discount Model

Let Div_N be the next dividend of Stock N. The value of Stock M is 3 times the value of Stock N:

$$\text{Value of Stock M} = 3 \times (\text{Value of Stock N})$$

$$\frac{(1/3)Div_N}{0.07 - g} = 3 \times \frac{Div_N}{0.07 + g}$$

$$\frac{1}{0.07 - g} = 9 \times \frac{1}{0.07 + g}$$

$$9(0.07 - g) = 0.07 + g$$

$$0.56 = 10g$$

$$g = \mathbf{0.0560}$$