Chapter 14: Duration and Convexity

Solution 14.01

D Section 14.01, Macaulay Duration

Since the bond is priced at par, its Macaulay duration is:

$$MacD = \ddot{a}_{\overline{n}|}^{(m)} = \ddot{a}_{\overline{9}|}^{(2)} = \frac{1-1.0325^{-18}}{0.065} \times 1.0325 = \mathbf{6.95}$$

Solution 14.02

A Section 14.01, Portfolio Duration

The duration of a portfolio is the weighted average duration of its components:

$$MacD_{Port} = \sum_{j=1}^{k} w_j \times MacD_j = \frac{100 \times 8 + 75.50 \times 13 + 60.41 \times 12}{100 + 75.50 + 60.41}$$

$$= \frac{2,506.42}{235.91} = \mathbf{10.62}$$

Solution 14.03

E Section 14.01, Macaulay Duration

The easiest way to approach this question is to keep in mind that the Macaulay duration is the weighted average of the times that the cash flows occur.

The formula for Macaulay Duration is:

$$MacD = \frac{\sum_{t>0} \left[t \times PV_0\left(CF_t\right) \right]}{\sum_{t>0} PV_0\left(CF_t\right)}$$

The expression above illustrates that the Macaulay duration is the weighted average of the times that the cash flows occur.

After 6 months, the bond's cash flows are still the same cash flows as before, but they now occur 6 months earlier. The weights are unchanged, because both the numerator and the denominator of each weight is multiplied by $(1.08)^{0.5}$. Therefore, the weighted average of the times that the cash flows occur is 0.5 less than the original weighted average:

$$X - Y = \mathbf{0.50}$$

Alternatively, we can calculate the Macaulay duration of the 3-year bond and the 2.5-year bond separately and then find the difference.

The price and the duration of the 3-year bond are found below:

$$P_{3-year} = \sum_{t>0} PV_0\left(CF_t\right) = \frac{6}{1.08} + \frac{6}{1.08^2} + \frac{106}{1.08^3} = 94.8458$$

$$X = \frac{\sum_{t>0} \left[t \times PV_0\left(CF_t\right) \right]}{\sum_{t>0} PV_0\left(CF_t\right)} = 1 \times \frac{\frac{6}{1.08^1}}{94.8458} + 2 \times \frac{\frac{6}{1.08^2}}{94.8458} + 3 \times \frac{\frac{106}{1.08^3}}{94.8458}$$

$$= 1 \times 0.0586 + 2 \times 0.0542 + 3 \times 0.8872 = 2.8286$$

The price and the duration of the 2.5-year bond are found below:

$$P_{2.5-year} = \sum_{t>0} PV_0\left(CF_t\right) = \frac{6}{1.08^{0.5}} + \frac{6}{1.08^{1.5}} + \frac{106}{1.08^{2.5}} = 98.5667$$

$$Y = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = 0.5 \times \frac{\frac{6}{1.08^{0.5}}}{98.5667} + 1.5 \times \frac{\frac{6}{1.08^{1.5}}}{98.5667} + 2.5 \times \frac{\frac{106}{1.08^{2.5}}}{98.5667}$$

$$= 0.5 \times 0.0586 + 1.5 \times 0.0542 + 2.5 \times 0.8872 = 2.3286$$

The times used to calculate Y are 0.5 less than the times used to calculate X, but the weights of 0.0586, 0.0542, and 0.8872 have not changed.

The difference is:

$$X - Y = 2.8286 - 2.3286 = \mathbf{0.50}$$

Solution 14.04

C Section 14.01, Macaulay Duration

The formula for Macaulay Duration is:

$$MacD = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)}$$

The expression above illustrates that the Macaulay duration is the weighted average of the times that the cash flows occur.

After 6 months, the bond's cash flows are still the same cash flows as before, but they now occur 6 months earlier. The weights are unchanged, because both the numerator and the denominator of each weight is multiplied by $(1+y)^{0.5}$. Therefore, the weighted average of the times that the cash flows occur is 0.5 less than the original weighted average:

$$9.8 - 0.5 = \mathbf{9.3}$$

Solution 14.05

D Section 14.01, Macaulay Duration

The easiest way to approach this question is to keep in mind that the Macaulay duration is the weighted average of the times that the cash flows occur, and each of the cash flows is moved back by one year when we go from an annuity-due to an annuity-immediate.

The formula for Macaulay Duration is:

$$MacD = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)}$$

The expression above illustrates that the Macaulay duration is the weighted average of the times that the cash flows occur.

When we go from an annuity-due to an annuity immediate, the cash flows are unchanged, but each cash flow occurs one year later. The weights have not changed, because both the numerator and the denominator of each weight is multiplied by v. Therefore, the weighted average of the times that the cash flows occur is just one year more than the weighted average for the annuity-due:

$$3 + 1 = \mathbf{4}$$

Alternatively, let's use Pmt_A to indicate the level payment of the annuity-due and Pmt_B to indicate the level payment of the annuity-immediate.

The Macaulay duration of the annuity-due is:

$$3 = \frac{Pmt_A\left(0 + v + 2v^2 + \cdots + (n-1)v^{n-1}\right)}{Pmt_A\left(1 + v + v^2 + \cdots + v^{n-1}\right)}$$

$$3 = \frac{v + 2v^2 + \cdots + (n-1)v^{n-1}}{1 + v + v^2 + \cdots + v^{n-1}}$$

The Macaulay duration of the second annuity is:

$$MacD_B = \frac{Pmt_B\left(v + 2v^2 + \cdots + nv^n\right)}{Pmt_B\left(v + v^2 + \cdots + v^n\right)} = \frac{v + 2v^2 + \cdots + nv^n}{v + v^2 + \cdots + v^n}$$

$$= \frac{1 + 2v + \cdots + nv^{n-1}}{1 + v + \cdots + v^{n-1}} = \frac{(1 + v + \cdots + v^{n-1}) + v + 2v^2 + \cdots + (n-1)v^{n-1}}{1 + v + \cdots + v^{n-1}}$$

$$= \frac{1 + v + \cdots + v^{n-1}}{1 + v + \cdots + v^{n-1}} + \frac{v + 2v^2 + \cdots + (n-1)v^{n-1}}{1 + v + \cdots + v^{n-1}} = 1 + 3 = \mathbf{4}$$

Solution 14.06

C Section 14.01, Macaulay Duration

The price of the bond is:

$$\sum_{t>0} PV_0\left(CF_t\right) = \frac{80}{1.15} + \frac{80}{1.15^2} + \frac{1,080}{1.15^3} = 840.1742$$

The numerator of the formula for the Macaulay duration is:

$$\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right] = \frac{80 \times 1}{1.15} + \frac{80 \times 2}{1.15^2} + \frac{1,080 \times 3}{1.15^3} = 2,320.9008$$

The Macaulay duration is:

$$MacD = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{2,320.9008}{840.1742} = \mathbf{2.7624}$$

Solution 14.07

D Section 14.01, Macaulay Duration

Since the cash flows appear in both the numerator and the denominator of the formula for Macaulay duration, we can factor out the factor of 1,000 when using the cash flows in the formula. That is, 50,000 can be written as 50 in both the numerator and denominator below.

The Macaulay duration is:

$$MacD = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{\frac{50 \times 2}{1.06^2} + \frac{35 \times 3}{1.06^3} + \frac{120 \times 4}{1.06^4}}{\frac{50}{1.06^2} + \frac{35}{1.06^3} + \frac{120}{1.06^4}} = \frac{557.3646}{168.9377} = \mathbf{3.30}$$

Solution 14.08

D Section 14.01, Macaulay Duration

The Macaulay duration is:

$$MacD = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{10v + 20v^2 + \cdots + 60v^6 + 70v^7 + 700v^7}{10v + 10v^2 + \cdots + 10v^7 + 100v^7}$$

$$= \frac{10(Ia)_{\overline{7}|0.07} + \dfrac{700}{1.07^7}}{10a_{\overline{7}|0.07} + \dfrac{100}{1.07^7}} = \frac{10 \times \dfrac{5.7665 - 7v^7}{0.07} + 435.9248}{10 \times 5.3893 + 62.2750}$$

$$= \frac{10 \times 20.1042 + 435.9248}{53.8929 + 62.2750} = \frac{636.9665}{116.1679} = \mathbf{5.4832}$$

Alternatively, we can use the BA II Plus cash flow worksheet to answer this question:

7 [*N*] 7 [*I*/*Y*] 10 [*PMT*] 100 [*FV*]

[CPT] [*PV*]

(Result is −116.1679)

[CF] 0 [ENTER] ↓

10 [ENTER] ↓↓

20 [ENTER] ↓↓

30 [ENTER] ↓↓

40 [ENTER] ↓↓

50 [ENTER] ↓↓

60 [ENTER] ↓↓

770 [ENTER]

[NPV] 7 [ENTER] ↓ [CPT]

(Result is 636.9665)

[÷] [RCL] [*PV*] [=]

(Result is −5.4832)

Answer is **5.4832**.

Solution 14.09

B Section 14.01, Macaulay Duration

After the first coupon is paid, the bond is still priced at par, so the remaining 8-year bond has a price of 6,000.

The amount of each coupon payment is:

$$6{,}000 \times 0.04 = 240$$

The duration of an immediate payment of 240 is 0, and the duration of the 8-year bond is D_A. A portfolio consisting of an immediate payment of 240 and the 8-year bond has a duration of D_B.

$$D_B = MacD_{Port} = \sum_{j=1}^{k} w_j \times MacD_j = \frac{240}{240 + 6{,}000} \times 0 + \frac{6{,}000}{240 + 6{,}000} \times D_A$$

$$= 0.9615 \times D_A$$

The ratio is:

$$\frac{D_B}{D_A} = \frac{0.9615 \times D_A}{D_A} = \mathbf{0.9615}$$

Alternatively, we can make use of the fact that the bond is priced at par. At the outset, the Macaulay duration of the 9-year bond is $\ddot{a}_{\overline{9}|}$. As shown below, we can also write the Macaulay duration as the weighted average of the timing of the cash flows:

$$MacD \text{ at outset} = \frac{\sum_{t=1}^{9}\left[t \times 240 \times v^t\right] + 9 \times 6,000 \times v^9}{\sum_{t=1}^{9}\left[240 \times v^t\right] + 9 \times 6,000 \times v^9} = \ddot{a}_{\overline{9}|}$$

One year later, just before the first coupon is paid, the weighted average of the timing of the cash flows is one year less:

$$D_B = \frac{\sum_{t=1}^{9}\left[(t-1) \times 240 \times v^{t-1}\right] + (9-1) \times 6,000 \times v^{9-1}}{\sum_{t=1}^{9}\left[240 \times v^{t-1}\right] + 9 \times 6,000 \times v^{9-1}}$$

$$= \frac{\sum_{t=1}^{9}\left[(t-1) \times 240 \times v^{t}\right] + (9-1) \times 6,000 \times v^{9}}{\sum_{t=1}^{9}\left[240 \times v^{t}\right] + 9 \times 6,000 \times v^{9}} = \ddot{a}_{\overline{9}|} - 1$$

After the first coupon is paid, there are 8 coupon payments remaining, so the duration of the bond is then:

$$D_A = \ddot{a}_{\overline{8}|}$$

The ratio is:

$$\frac{D_B}{D_A} = \frac{\ddot{a}_{\overline{9}|} - 1}{\ddot{a}_{\overline{8}|}} = \frac{a_{\overline{8}|}}{\ddot{a}_{\overline{8}|}} = \frac{a_{\overline{8}|}}{1.04 \times a_{\overline{8}|}} = \mathbf{0.9615}$$

Solution 14.10

B Section 14.01, Macaulay Duration

After the 3rd coupon is paid, the bond is still priced at par, so the remaining (n – 3)-year bond has a price of par.

Let F be the par value of the bond. The amount of each coupon payment is:

$0.04F$

The duration of an immediate payment of $0.04F$ is 0, and the duration of the (n – 3)-year bond is D_A. A portfolio consisting of an immediate payment of $0.04F$ and the (n – 3)-year bond has a duration of D_B.

$$D_B = MacD_{Port} = \sum_{j=1}^{k} w_j \times MacD_j = \frac{0.04F}{0.04F + F} \times 0 + \frac{F}{0.04F + F} \times D_A$$

$$= \frac{1}{1.04} \times D_A$$

The ratio is:

$$\frac{D_B}{D_A} = \frac{\frac{1}{1.04} \times D_A}{D_A} = \mathbf{0.9615}$$

Alternatively, we can make use of the fact that the bond is priced at par. At the outset, the Macaulay duration of the n-year bond is $\ddot{a}_{\overline{n}|}$.

Immediately after the second coupon is paid, there are $n-2$ coupon payments remaining, so the Macaulay duration of the bond is $\ddot{a}_{\overline{n-2}|}$. One year later, just before the 3rd coupon is paid, the time until each cash flow is one year less, so the weighted average of the times is one year less:

$$D_B = \ddot{a}_{\overline{n-2}|} - 1$$

Just after the third coupon is paid, there are $n-3$ coupon payments remaining, so the duration of the bond is then:

$$D_A = \ddot{a}_{\overline{n-3}|}$$

The ratio is:

$$\frac{D_B}{D_A} = \frac{\ddot{a}_{\overline{n-2}|} - 1}{\ddot{a}_{\overline{n-3}|}} = \frac{a_{\overline{n-3}|}}{\ddot{a}_{\overline{n-3}|}} = \frac{a_{\overline{n-3}|}}{1.04 \times a_{\overline{n-3}|}} = \mathbf{0.9615}$$

Solution 14.11

C Section 14.01, Macaulay Duration

The formula for Macaulay Duration is:

$$MacD = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)}$$

We can use the duration of the 3-year annuity to solve for v. Let's use Pmt_3 to indicate the level payment of the 3-year annuity. We don't need to know the amount of the level payment, because it can be factored out of both the numerator and the denominator when calculating duration:

$$0.96 = \frac{Pmt_3\left(0 + v + 2v^2\right)}{Pmt_3\left(1 + v + v^2\right)}$$

$$0.96 + 0.96v + 0.96v^2 = v + 2v^2$$

$$1.04v^2 + 0.04v - 0.96 = 0$$

$$v = \frac{-0.04 \pm \sqrt{0.04^2 - 4(1.04)(-0.96)}}{2 \times 1.04}$$

$$v = 0.9417 \quad \text{or} \quad v = -0.9802$$

We use the positive discount factor to find the duration of Annuity B.

Once again, we do not need to know the amount of the level payment, because it can be factored out of both the numerator and the denominator:

$$\frac{Pmt_4\left(0+v+2v^2+3v^3\right)}{Pmt_4\left(1+v+v^2+v^3\right)}=\frac{0.9417+2\times 0.9417^2+3\times 0.9417^3}{1+0.9417+0.9417^2+0.9417^3}=\frac{5.2210}{3.6638}$$
$$=\mathbf{1.4250}$$

Solution 14.12

B Section 14.01, Macaulay Duration

The amount of each coupon payment is:

$$0.06\times 1,000=60$$

After the 4th coupon is paid, the price of the bond is:

$$P=60\times a_{\overline{5}|0.04}+\frac{1,000}{1.04^5}=1,089.0364$$

The duration of an immediate payment of 60 is 0, and the duration of the 5-year bond is D_A. A portfolio consisting of an immediate payment of 60 and the 5-year bond has a duration of D_B.

$$D_B=MacD_{Port}=\sum_{j=1}^{k}w_j\times MacD_j$$
$$=\frac{60}{60+1,089.0364}\times 0+\frac{1,089.0364}{60+1,089.0364}\times D_A=0.9478\times D_A$$

The ratio is:

$$\frac{D_B}{D_A}=\frac{0.9478\times D_A}{D_A}=\mathbf{0.9478}$$

Solution 14.13

E Section 14.02, Modified Duration

Modified duration can be expressed in terms of the derivative of the bond's price or in terms of the Macaulay duration:

$$ModD=\frac{MacD}{1+\frac{y^{(m)}}{m}}$$
$$-\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}=\frac{MacD}{1+\frac{y^{(m)}}{m}}$$
$$-\frac{-1,000}{100}=\frac{MacD}{1+0.05}$$
$$MacD=\mathbf{10.50}$$

Solution 14.14

D Section 14.02, Modified Duration

The price of the stock and the derivative of its price are:

$$P(y) = \frac{Div}{y} = Div \times y^{-1}$$

$$P'(y) = -Div \times y^{-2}$$

The modified duration is:

$$ModD = -\frac{P'(y)}{P(y)} = -\frac{-Div \times y^{-2}}{Div \times y^{-1}} = \frac{1}{y} = \frac{1}{0.08} = 12.5$$

The Macaulay duration is:

$$MacD = ModD \times (1+y) = 12.5 \times 1.08 = \mathbf{13.5}$$

Solution 14.15

A Section 14.02, Modified Duration

Since the bond is priced at par, its yield is equal to its coupon rate.

The modified duration is:

$$ModD = \frac{MacD}{1+\frac{y^{(m)}}{m}} = \frac{6.88}{1.074} = 6.4060$$

The estimated percentage change in the price is:

$$\%\Delta P \simeq -ModD \times \Delta y^{(m)} = -6.4060 \times (0.068 - 0.074) = 0.03844$$

The estimate for the new price is:

$$100(1+\%\Delta P) = 100(1+0.03844) = \mathbf{103.84}$$

Solution 14.16

E Section 14.02, Duration

The Macaulay duration of Bond A is:

$$MacD_A = \ddot{a}_{\overline{4}|} = \frac{1-1.08^{-4}}{0.08} \times 1.08 = 3.5771$$

The modified duration of Bond A is:

$$ModD_A = \frac{MacD_A}{1+\frac{y^{(m)}}{m}} = \frac{3.5771}{1.08} = 3.3121$$

The modified duration of Bond B is equal to the modified duration of Bond A:

$$ModD_B = 3.3121$$

The Macaulay duration of Bond B is found below:

$$ModD_B = \frac{MacD_B}{1+\frac{y^{(m)}}{m}}$$

$$3.3121 = \frac{MacD_B}{1+\frac{0.08}{2}}$$

$$MacD_B = 3.3121 \times 1.04$$

$$MacD_B = \mathbf{3.4446}$$

Solution 14.17

C Section 14.02, Modified Duration

The price of the stock and the derivative of its price are:

$$P(y) = \frac{Div_1}{y-g} = Div_1 \times (y-g)^{-1}$$

$$P'(y) = -Div_1 \times (y-g)^{-2}$$

The modified duration is:

$$ModD = -\frac{P'(y)}{P(y)} = -\frac{-Div_1 \times (y-g)^{-2}}{Div_1 \times (y-g)^{-1}} = \frac{1}{y-g} = \frac{1}{0.04-0.01} = \frac{1}{0.03}$$

$$= 33.3333$$

The Macaulay duration is:

$$MacD = ModD \times (1+y) = 33.3333 \times 1.04 = \mathbf{34.67}$$

Solution 14.18

A Section 14.02, Modified Duration

The price and the derivative of the price can be used to obtain an expression for the modified duration:

$$P = 1 + \frac{1}{y}$$

$$P' = -y^{-2}$$

$$ModD = -\frac{P'}{P} = -\frac{-y^{-2}}{1+y^{-1}} = \frac{1}{y^2+y} = \frac{1}{y(1+y)}$$

The relationship between the modified duration and the Macaulay duration can be used to find the yield:

$$ModD = \frac{MacD}{1+y}$$

$$\frac{1}{y(1+y)} = \frac{28}{(1+y)}$$

$$\frac{1}{y} = 28$$

$$y = \frac{1}{28}$$

The modified duration is:

$$ModD = \frac{1}{y(1+y)} = \frac{1}{\frac{1}{28}\left(1+\frac{1}{28}\right)} = \mathbf{27.03}$$

Solution 14.19

D Section 14.02, Modified Duration

Modified duration can be expressed in terms of the derivative of the bond's price or in terms of the Macaulay duration:

$$ModD = -\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} = \frac{MacD}{1+\dfrac{y^{(m)}}{m}}$$

$$-\frac{-1{,}000}{100} = \frac{MacD}{1+0.025}$$

$$MacD = 10.25$$

Since the bond is priced at par, its Macaulay duration is equal to the present value of an annuity-due:

$$MacD = \ddot{a}_{\overline{n}|}^{(2)}$$

$$MacD = \frac{1-(1.025)^{-2n}}{0.05} \times 1.025$$

$$10.25 = \frac{1-(1.025)^{-2n}}{0.05} \times 1.025$$

$$0.5 = 1.025^{-2n}$$

$$-2n = \frac{\ln(0.5)}{\ln(1.025)}$$

$$n = \mathbf{14.0355}$$

We can use the BA II Plus to answer this question:

1,000 [÷] 100 [×] 1.025 [=]

Result is 10.25.

[+/-] [*PV*] 0.5 [*PMT*] 2.5 [*I*/*Y*]

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

[CPT] [*N*]

[÷] 2 [=]

Answer is **14.0355**.

Solution 14.20

D Section 14.03, Modified Convexity

The estimated percentage change in price is:

$$\%\Delta P \simeq -ModD \times \Delta y^{(m)} + 0.5 \times ModC \times \left(\Delta y^{(m)}\right)^2$$

$$= -11.904 \times (-0.01) + 0.5 \times 197.238 \times (-0.01)^2$$

$$= \mathbf{0.1289}$$

Solution 14.21

B Section 14.03, Modified Convexity

The estimated percentage change in price is:

$$\%\Delta P \simeq -ModD \times \Delta y^{(m)} + 0.5 \times ModC \times \left(\Delta y^{(m)}\right)^2$$
$$= -11.904 \times (-0.01) + 0.5 \times 197.238 \times (-0.01)^2$$
$$= 0.1289$$

The estimate for the new price is:

$$X = 885.30(1 + \%\Delta P) = 885.30 \times 1.1289 = 999.4169$$

If the yield decreases to 5%, then the yield will be equal to the coupon rate, and the bond will be a par bond:

$$Y = 1{,}000$$

The difference is:

$$X - Y = 999.4169 - 1{,}000 = \mathbf{-0.5831}$$

Solution 14.22

D Section 14.03, Macaulay Convexity

The dispersion of a zero-coupon bond is zero, so the Macaulay convexity of a zero-coupon bond is equal to the square of its time until maturity:

$$MacC_{10} = MacD^2 + Dispersion = 10^2 + 0 = 100$$
$$MacC_{20} = MacD^2 + Dispersion = 20^2 + 0 = 400$$

The Macaulay convexity of the portfolio is the weighted average of the Macaulay convexities of its components:

$$MacC_{Port} = \sum_{j=1}^{2} w_j \times MacC_j = 0.50 \times 10^2 + 0.50 \times 20^2 = \mathbf{250}$$

Solution 14.23

C Section 14.03, Macaulay Convexity

The Macaulay duration of a zero-coupon bond is equal to its time until maturity, so the durations of the bonds are:

$$MacD_{10} = 10 \qquad MacD_{20} = 20$$

The two bonds have the same price, so their weights in Mike's portfolio are equal:

$$P_{10} = \frac{895.42}{1.06^{10}} = 500.00$$
$$P_{20} = \frac{1{,}603.57}{1.06^{20}} = 500.00$$

The Macaulay duration of the portfolio is the weighted average of the Macaulay durations of its components:

$$MacD_{Port} = \sum_{j=1}^{2} w_j \times MacD_j = \frac{500}{500+500} \times 10 + \frac{500}{500+500} \times 20 = 0.50 \times 10 + 0.50 \times 20$$
$$= 15$$

The Macaulay convexity of a zero-coupon bond is equal to the square of its time until maturity, so the Macaulay convexities of the two bonds are:

$$MacC_{10} = MacD^2 + Dispersion = 10^2 + 0 = 100$$

$$MacC_{20} = MacD^2 + Dispersion = 20^2 + 0 = 400$$

The Macaulay convexity of the portfolio is the weighted average of the Macaulay convexities of its components:

$$MacC_{Port} = \sum_{j=1}^{2} w_j \times MacC_j = 0.50 \times 10^2 + 0.50 \times 20^2 = 250$$

We can now find the modified convexity of the portfolio:

$$ModC = \frac{MacC + \dfrac{MacD}{m}}{\left(1 + \dfrac{y^{(m)}}{m}\right)^2} = \frac{250 + \dfrac{15}{1}}{1.06^2} = \mathbf{235.85}$$

Solution 14.24

E Section 14.03, Modified Convexity

The price of the stock and the derivatives of its price are:

$$P(y) = \frac{Div}{y} = Div \times y^{-1}$$

$$P'(y) = -Div \times y^{-2}$$

$$P''(y) = 2 \times Div \times y^{-3}$$

The modified convexity is:

$$ModC = \frac{P''(y)}{P(y)} = -\frac{2 \times Div \times y^{-3}}{Div \times y^{-1}} = \frac{2}{y^2} = \frac{2}{0.08^2} = \mathbf{312.50}$$

Solution 14.25

C Section 14.03, Macaulay Convexity

The price of the stock and the derivatives of its price are:

$$P(y) = \frac{Div}{y} = Div \times y^{-1}$$

$$P'(y) = -Div \times y^{-2}$$

$$P''(y) = 2 \times Div \times y^{-3}$$

The modified duration and convexity are:

$$ModD = -\frac{P'(y)}{P(y)} = \frac{Div \times y^{-2}}{Div \times y^{-1}} = \frac{1}{y} = \frac{1}{0.08} = 12.5$$

$$ModC = \frac{P''(y)}{P(y)} = -\frac{2 \times Div \times y^{-3}}{Div \times y^{-1}} = \frac{2}{y^2} = \frac{2}{0.08^2} = 312.50$$

The Macaulay duration is:

$$MacD = ModD \times (1 + y) = 12.5 \times 1.08 = 13.5$$

We can use the formula that expresses modified convexity in terms of Macaulay convexity:

$$ModC = \frac{MacC + \frac{MacD}{m}}{\left(1 + \frac{y^{(m)}}{m}\right)^2}$$

$$312.50 = \frac{MacC + \frac{13.5}{1}}{\left(1 + \frac{0.08}{1}\right)^2}$$

$$MacC = \mathbf{351}$$

Solution 14.26

D Section 14.05, First-Order Macaulay Approximation

The first-order Macaulay approximation of the new price is:

$$P(y + \Delta y) \approx P(y) \times \left(\frac{1+y}{1+y+\Delta y}\right)^{MacD}$$

The approximate percentage change in price, using the first-order Macaulay approximation, is:

$$\frac{P(y+\Delta y)}{P(y)} - 1 \approx \left(\frac{1+y}{1+y+\Delta y}\right)^{MacD} - 1 = \left(\frac{1+0.08}{1+0.08-0.0040}\right)^{14} - 1$$

$$= \left(\frac{1.0800}{1.0760}\right)^{14} - 1 = 1.053321 - 1 = \mathbf{0.053321}$$

Solution 14.27

C Section 14.05, First-Order Macaulay Approximation

The first-order Macaulay approximation of the new price is:

$$P(y + \Delta y) \approx P(y) \times \left(\frac{1+y}{1+y+\Delta y}\right)^{MacD} = 1{,}000 \times \left(\frac{1.074}{1.070}\right)^{8.776} = \mathbf{1{,}033.29}$$

Solution 14.28

A Section 14.05, First-Order Macaulay Approximation

The Macaulay duration is the modified duration multiplied by the one-period accumulation factor:

$$MacD = ModD \times (1+y) = 9 \times 1.074 = 9.666$$

The first-order Macaulay approximation of the new price is:

$$E_{Mac} \approx P(y) \times \left(\frac{1+y}{1+y+\Delta y}\right)^{MacD} = 850.46 \times \left(\frac{1.074}{1.080}\right)^{9.666} = 805.874$$

The first-order modified approximation of the new price is:

$$E_{Mod} \approx P(y) \times \left[1 - ModD \times \Delta y\right] = 850.46 \times \left[1 - 9 \times 0.006\right] = 804.535$$

The difference in the estimates is:

$$E_{Mac} - E_{Mod} = 805.874 - 804.535 = \mathbf{1.339}$$

Solution 14.29

D Section 14.05, First-Order Macaulay Approximation

The Macaulay duration of the portfolio is the weighted average of the durations of Bond A and Bond B:

$$MacD = \frac{5.4 \times 70,000 + 11.9 \times 30,000}{70,000 + 30,000} = 5.4 \times 0.70 + 11.9 \times 0.30 = 7.35$$

The first-order Macaulay approximation of the new price is 96,000:

$$P(y + \Delta y) \approx P(y) \times \left(\frac{1 + y}{1 + y + \Delta y} \right)^{MacD}$$

$$96,000 = (70,000 + 30,000) \times \left(\frac{1.065}{1 + i} \right)^{7.35}$$

$$0.96 = \left(\frac{1.065}{1 + i} \right)^{7.35}$$

$$0.96^{1/7.35} = \frac{1.065}{1 + i}$$

$$i = \mathbf{0.070931}$$

Solution 14.30

B Section 14.05, First-Order Macaulay Approximation

The first-order Macaulay approximation of the new price can be used to solve for the Macaulay Duration:

$$P(y + \Delta y) \approx P(y) \times \left(\frac{1 + y}{1 + y + \Delta y} \right)^{MacD}$$

$$57,021.04 = 59,776.39 \times \left(\frac{1.074}{1.082} \right)^{MacD}$$

$$\ln\left(\frac{57,021.04}{59,776.39} \right) = MacD \times \ln\left(\frac{1.074}{1.082} \right)$$

$$MacD = 6.3589$$

The modified duration is the Macaulay duration divided by the one-period accumulation factor:

$$ModD = \frac{MacD}{1 + y} = \frac{6.3589}{1.074} = \mathbf{5.9207}$$

Chapter 15: Asset-Liability Matching

Solution 15.01

B Chapter 15.02, Dedication

The liability cash flows can be matched by purchasing 6 of the 1-year bonds and 15 of the 2-year bonds:

$$6 \times \frac{100}{1.03} + 15 \times \frac{100}{1.04^2} = \mathbf{1,969.36}$$

Solution 15.02

C Section 15.02, Dedication

The present value of the asset cash flows is equal to the present value of the liability cash flows, so the cost of the bonds is equal to the present value of the liability cash flows. Since the yield is the same for both bonds, that yield can be used to calculate the present value of the liability cash flows:

$$\frac{5,000}{1.04} + \frac{7,000}{1.04^2} = \mathbf{11,279.59}$$

Solution 15.03

B Section 15.02, Dedication

The liability of 1,200 due in two years is met by arranging a payment of 1,200 in two years from Loan B. Since Loan B makes an equal-sized payment at time 1, it also pays 1,200 at time 1. That leaves 1,800 of net liability at time 1 to be met by Loan A. The amount lent is:

$$X + Y = \frac{3,000 - 1,200}{1.05} + \frac{1,200}{1.06} + \frac{1,200}{1.06^2} = \mathbf{3,914.36}$$

Solution 15.04

B Section 15.02, Dedication

We do not use the two-year 4% bond, because:

1. It has the lowest yield, and
2. It produces cash flow at time 1, reducing our ability the use the highest-yielding bond, which is the 7% bond.

The smallest cost of assets that provide the necessary cash flows is:

$$\frac{10,000}{1.07} + \frac{15,000}{1.055^2} = \mathbf{22,822.58}$$

Solution 15.05

B Section 15.02, Dedication

The quantity of Bond B to purchase is the quantity that produces a cash flow of $1,000 at time 1:

$$Q_B = \frac{1,000}{1,030} = 0.9709$$

The quantity of Bond A to purchase is the quantity that produces the net liability remaining after the payment from Bond B is received:

$$Q_A = \frac{1,000 - 30 \times 0.9709}{1,025} = 0.9472$$

Choice **B** is the correct answer.

Solution 15.06

A Section 15.02, Dedication

The quantity of Bond C that is purchased is:

$$Q_C = \frac{107}{104}$$

The quantity of Bond A that is purchased is the amount needed to cover the remaining liability after the cash flow from Bond C is received:

$$Q_A = \frac{98 - \frac{107}{104} \times 4}{105} = \mathbf{0.894}$$

Solution 15.07

D Section 15.02, Asset-Liability Management

The liability cash flows occur at time 1 and time 2 in the following amounts:

Time 1: $10,000 \times 1.04 \times 0.5 = 5,200$

Time 2: $(10,000 \times 1.04 \times 0.5) \times 1.04 = 5,408$

The strategy that costs the least to implement produces the highest profit. To answer this question, we:

1. Determine the cost of each strategy.
2. Find the lowest cost strategy that provides the necessary cash flows.

The cost of each strategy is listed below:

A: $4,500 + 4,598 + 320 = 9,418$

B: $5,000 + 4,905 = 9,905$

C: $5,000 + 5,000 = 10,000$

D: $4,706 + 5,102 = 9,808$

E: $4,723 + 290 + 4,800 = 9,813$

Choice A has the lowest cost, so we check to see if it provides the necessary cash flows:

Time 1: $4,500 \times 1.04 + 320 \times 0.06 = 4,699.2000$

Time 2: $4,598 \times 1.05^2 + 320 \times 1.06 = 5,408.4950$

Since Choice A does not provide at least 5,200 in year 1, it is not correct.

Since Choice D has the next lowest cost, we check to see if provides the necessary cash flows:

Time 1: $4,706 \times 1.04 + 5,102 \times 0.06 = 5,200.3600$

Time 2: $5,102 \times 1.06 = 5,408.1200$

Choice D provides the necessary cash flows to make the liability payments, so Choice **D** is the correct answer.

Solution 15.08

B Section 15.03, Redington Immunization

We don't need to know the amount of the liability payment to answer this question.

The duration of the asset portfolio must be equal to duration of the liability. Let X be the percentage of the asset portfolio that is invested in the asset that pays at time 3:

$$3X + 10(1 - X) = 6$$

$$-7X = -4$$

$$X = \frac{4}{7}$$

Since the present value of the asset portfolio is equal to the present value of the liability, the present values of the asset cash flows at time 0 are:

$$PV_A = \frac{4}{7} \times PV_L$$

$$PV_B = \frac{3}{7} \times PV_L$$

The amounts of the cash flows are found by accumulating their present values:

$$\frac{A}{B} = \frac{PV_A \times 1.05^3}{PV_B \times 1.05^{10}} = \frac{\frac{4}{7} \times PV_L \times 1.05^3}{\frac{3}{7} \times PV_L \times 1.05^{10}} = \frac{4}{3 \times 1.05^7} = \mathbf{0.9476}$$

Solution 15.09

E Section 15.03, Redington Immunization

We don't need to know the amount of the liability payment to answer this question.

The duration of the asset portfolio must be equal to duration of the liability. Let X be the percentage of the asset portfolio that is invested in the asset that pays at time 5:

$$5X + 10(1 - X) = 9$$

$$-5X = -1$$

$$X = \frac{1}{5}$$

Since the present value of the asset portfolio is equal to the present value of the liability, the present values of the asset cash flows at time 0 are:

$$PV_A = \frac{1}{5} \times PV_L$$

$$PV_B = \frac{4}{5} \times PV_L$$

The amounts of the cash flows are found by accumulating their present values:

$$\frac{B}{A} = \frac{PV_B \times 1.05^{10}}{PV_A \times 1.05^5} = \frac{\frac{4}{5} \times PV_L \times 1.05^{10}}{\frac{1}{5} \times PV_L \times 1.05^5} = 4 \times 1.05^5 = \mathbf{5.1051}$$

Solution 15.10

E Section 15.03, Immunization

The Macaulay duration of the liabilities is:

$$MacDur_L = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{\frac{374.11}{1.06}\times 1 + \frac{374.11}{1.06^2}\times 2 + \frac{374.11}{1.06^3}\times 3}{1,000} = 1.9612$$

The Macaulay duration of the assets must be equal to the Macaulay duration of the liabilities. Let w be weight of the one-year bond:

$$w + 3(1-w) = 1.9612$$

$$w = 0.51941$$

The amounts to invest in the one-year bond and the 3-year bond are:

One-year bond: $1,000w = 1,000 \times 0.51941 =$ **519.41**

Three-year bond: $1,000(1-w) = 1,000(1-0.51941) =$ **480.59**

Solution 15.11

C Section 15.03, Redington Immunization

All of the answer choices have the same present value as the present value of the liabilities, so we'll focus on the requirements that the Macaulay duration of the assets be equal to that of the liabilities and that the Macaulay convexity of the assets be greater than that of the liabilities.

For Choice A, the Macaulay duration of the assets is not equal to the Macaulay duration of the liabilities:

$$MacD_A = \frac{1,000}{5,000}\times 5 + \frac{4,000}{5,000}\times 10 = 9$$

For Choice B, the Macaulay duration of the assets is not equal to the Macaulay duration of the liabilities:

$$MacD_B = \frac{2,000}{5,000}\times 5 + \frac{3,000}{5,000}\times 20 = 14$$

For Choice C, we have:

$$MacD_C = \frac{2,666.67}{5,000}\times 5 + \frac{2,333.33}{5,000}\times 20 = 12$$

$$MacC_C = \frac{2,666.67}{5,000}\times 5^2 + \frac{2,333.33}{5,000}\times 20^2 = 200$$

Since the Macaulay duration of the assets is equal (within rounding tolerance) to the Macaulay duration of the liabilities, and the Macaulay convexity of the assets is greater than the Macaulay convexity of the liabilities, Choice **C** is the correct answer.

For Choice D, the Macaulay duration of the assets is not equal to the Macaulay duration of the liabilities:

$$MacD_D = \frac{3,000}{5,000}\times 10 + \frac{2,000}{5,000}\times 20 = 14$$

For Choice E, the Macaulay convexity is not greater than the Macaulay convexity of the liabilities:

$$MacD_E = \frac{4,000}{5,000} \times 10 + \frac{1,000}{5,000} \times 20 = 12$$

$$MacC_E = \frac{4,000}{5,000} \times 10^2 + \frac{1,000}{5,000} \times 20^2 = 160$$

Solution 15.12

C Section 15.01, Interest Rate Risk

On 12/31/2020, the company will receive the par value of 821,972.11, leaving the following net liability:

$$1,000,000 - 821,927.11 = 178,072.89$$

Under Scenario A, the profit is:

$$0.04 \times 821,927.11 \times s_{\overline{5}|0.037} - 178,072.89$$

$$= 0.04 \times 821,927.11 \times \frac{1.037^5 - 1}{0.037} - 178,072.89$$

$$= -1,064.47$$

Under Scenario B, the profit is:

$$0.04 \times 821,927.11 \times s_{\overline{5}|0.043} - 178,072.89$$

$$= 0.04 \times 821,927.11 \times \frac{1.043^5 - 1}{0.043} - 178,072.89$$

$$= 1,070.76$$

Choice **C** best describes the company's profit or loss.

Solution 15.13

A Section 15.03, Redington Immunization

The present value, Macaulay duration, and Macaulay convexity of the liabilities are:

$$PV_L = \frac{595.51}{1.06^3} + \frac{709.26}{1.06^6} = 1,000.0020$$

$$MacD_L = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{\frac{595.51}{1.06^3} \times 3 + \frac{709.26}{1.06^6} \times 6}{1,000.0020} = 4.5000$$

$$MacC_L = \frac{\sum_{t>0}\left[t^2 \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{\frac{595.51}{1.06^3} \times 3^2 + \frac{709.26}{1.06^6} \times 6^2}{1,000.0020} = 22.5000$$

The present value, Macaulay duration, and Macaulay convexity of Choice A are:

$$PV_A = 500 + 500 = 1,000$$

$$MacD_A = 0.5 \times 1 + 0.5 \times 8 = 4.5$$

$$MacC_A = 0.5 \times 1^2 + 0.5 \times 8^2 = 32.5$$

Since the present value and the Macaulay duration of Choice A are equal (within rounding tolerance) to the present value and Macaulay duration of the liabilities, and the Macaulay convexity of Choice A exceeds the Macaulay convexity of the liabilities, Choice **A** is the correct answer.

Choice B's Macaulay duration can be found as the weighted average of the timing of its cash flows:

$$MacD0_B = 0.400 \times 1 + 0.600 \times 8 = 5.2$$

Since 5.2 is not equal to the duration of the liabilities, Choice B is not correct.

Choices C and D have present values that are not equal to the present value of the liabilities, so neither is correct.

Choice E has a single cash flow, so its Macaulay convexity is equal to the square of the time of the cash flow:

$$MacC_E = 4.5^2 = 20.25$$

Since 20.25 is less than the Macaulay convexity of the liabilities, Choice E is not correct.

Solution 15.14

D Section 15.03, Redington Immunization

The Macaulay duration of the asset is:

$$MacD = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{700(Ia)_{\overline{15}|}}{700a_{\overline{15}|}} = \frac{\dfrac{\ddot{a}_{\overline{15}|} - 15v^{15}}{0.062}}{9.5866} = \frac{66.0721}{9.5866}$$

$$= 6.8921$$

Let's define X to be the weight of the liability cash flow that occurs at time 10. Since the position satisfies the conditions of Redington immunization, the duration of the liabilities is equal to the duration of the asset:

$$6.8921 = 10X + (1 - X)5$$

$$X = 0.3784$$

The expression above for X can now be used to find the value of B:

$$0.3784 = \frac{\dfrac{B}{1.062^{10}}}{700a_{\overline{15}|}}$$

$$0.3784 \times 6{,}710.6060 = \frac{B}{1.062^{10}}$$

$$B = \mathbf{4{,}634.3653}$$

The BA II Plus can be used to answer this question:

15 [*N*] 6.2 [*I*/*Y*] 1[*PMT*] [CPT] [*PV*] [+/-]

Result is 9.5865.

[STO] 1 [×] 1.062 [–] 15 [÷] 1.062 [y^x] 15 [=] [÷] 0.062 [=]

Result is 66.0721.

[÷] [RCL] 1 [=]

Result is 6.8921.

[–] 5 [=] [÷] 5 [=]

Result is 0.3784.

[×] [RCL] 1 [×] 700 [×] 1.062 [y^x] 10 [=]

Solution is **4,634.3653**.

Solution 15.15

A Section 15.03, Redington Immunization

The present value, Macaulay duration, and Macaulay convexity of the liabilities are:

$$PV_L = \frac{600}{1.05} + \frac{1,500}{1.05^5} = 1,746.7178$$

$$MacD_L = \frac{\sum_{t>0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{\frac{600}{1.05} \times 1 + \frac{1,500}{1.05^5} \times 5}{1,746.7178} = 3.6914$$

$$MacC_L = \frac{\sum_{t>0}\left[t^2 \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = \frac{\frac{600}{1.05} \times 1^2 + \frac{1,500}{1.05^5} \times 5^2}{1,746.7178} = 17.1485$$

Since the duration of the assets is equal to the duration of the liabilities, the average duration of the assets is 3.6914. Let the weight of the cash flow of X be w:

$$0 \times w + (1 - w) \times 6 = 3.6914$$

$$w = 0.3848$$

Since the cash flow of X occurs at time 0, the value of X is its weight times the present value of the liabilities:

$$X = 0.3848 \times 1,746.7178 = \mathbf{672.0720}$$

The Macaulay convexity of the assets is:

$$MacC_A = \frac{\sum_{t>0}\left[t^2 \times PV_0\left(CF_t\right)\right]}{\sum_{t>0} PV_0\left(CF_t\right)} = w \times 0^2 + (1 - w) \times 6^2$$

$$= 0.3848 \times 0 + (1 - 0.3848) \times 36 = 22.1485$$

Since the Macaulay convexity of the assets is greater than the Macaulay convexity of the liabilities, the conditions of Redington immunization are satisfied. Choice **A** is the correct answer.

Solution 15.16

A Section 15.04, Asset-Liability Management

A is false, because the modified duration is the Macaulay duration divided by the sum of one and the periodic effective yield, where the period is determined by the compounding frequency of the yield. Choice **A** is the correct answer.

B is true, because as the compounding frequency increases to infinity, the compounding periods become smaller, and the periodic effective yield approaches zero.

C is true because in a cash flow matched portfolio, the present value of the liabilities is equal to the present value of the assets, and the duration of the liabilities is equal to the duration of the assets.

D is true because the first two conditions of a fully immunized portfolio are the same as the first two conditions of a Redington immunized portfolio (see the preceding paragraph regarding Choice C).

E is true, because fully immunized portfolios are a subset of Redington immunized portfolios.

Solution 15.17

D Section 15.04, Full Immunization

The duration of the asset portfolio must be equal to duration of the liability. Let w be the percentage of the asset portfolio that is invested in the asset that pays at time 2:

$$2w + 8(1 - w) = 5$$
$$-6w = -3$$
$$w = 0.5$$

The present value of A is therefore one half of the present value of the liability. The present value of B is also one half of the present value of the liability:

$$\frac{A}{1.035^2} = 0.5 \times \frac{7{,}000}{1.035^5} \quad \Rightarrow \quad A = 3{,}156.7995$$
$$\frac{B}{1.035^8} = 0.5 \times \frac{7{,}000}{1.035^5} \quad \Rightarrow \quad B = 3{,}880.5126$$

The absolute value of the difference is:

$$|A - B| = |3{,}156.7995 - 3{,}880.5126| = \mathbf{723.7131}$$

Solution 15.18

C Section 15.04, Full Immunization

Let w be the weight of the first asset cash flow:

$$w = \frac{\frac{200}{1.03^5}}{\frac{1{,}000}{1.03^7}} = 0.2122$$

The Macaulay duration of the assets must be equal to the Macaulay duration of the liability:

$$5w + y(1 - w) = 7$$
$$5(0.2122) + y(1 - 0.2122) = 7$$
$$y = 7.5387$$

The present value of the assets must be equal to the present value of the liabilities:

$$\frac{200}{1.03^5} + \frac{B}{1.03^{7.5387}} = \frac{1{,}000}{1.03^7}$$
$$B = \mathbf{800.46}$$

Solution 15.19

C Section 15.04, Full Immunization

Let w be the weight of the first asset cash flow:

$$w = \frac{\frac{7{,}000}{1.04^4}}{\frac{15{,}000}{1.04^9}} = 0.5678$$

The Macaulay duration of the assets must be equal to the Macaulay duration of the liability:

$$4w + (9 + y)(1 - w) = 9$$
$$4(0.5678) + (9 + y)(1 - 0.5678) = 9$$
$$y = 6.5680$$

The present value of the assets must be equal to the present value of the liabilities:

$$\frac{7,000}{1.04^4} + \frac{Y}{1.04^{9+6.5680}} = \frac{15,000}{1.04^9}$$
$$Y = 8,388.3965$$

The ratio is:

$$\frac{Y}{y} = \frac{8,388.3965}{6.5680} = \mathbf{1,277.17}$$